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Research Article

Applying GG-Convex Function to Hermite-Hadamard Inequalities Involving Hadamard Fractional Integrals

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By virtue of fractional integral identities, incomplete beta function, useful series, and inequalities, we apply the concept of GG-convex function to derive new type Hermite-Hadamard inequalities involving Hadamard fractional integrals. Finally, some applications to special means of real numbers are demonstrated.

1. Introduction

Fractional calculus played an important role in various fields such as electricity, biology, economics, and signal and image processing [1–8]. The fractional Hermite-Hadamard inequality gives a lower and an upper estimation for both right-hand and left-hand integrals average of any convex function defined on a compact interval, involving the midpoint and the endpoints of the domain.

As we know, Set [9] firstly studied fractional Ostrowski inequalities involving Riemann-Liouville fractional integrals. Then, Sarikaya et al. [10] studied Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals. Further, our group go on studying fractional version Hermite-Hadamard inequality involving Riemann-Liouville and Hadamard fractional integrals for all kinds of functions [11–19].

Recently, Wang et al. [16, 17] established the following two powerful fractional integral identities involving Hadamard fractional integrals.

Lemma 1 (see [12, Lemma 3.1]). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b). If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}} \left[{}_HJ^{\alpha}_{a^+} f(b) + {}_HJ^{\alpha}_{b^-} f(a) \right] = \frac{\ln b - \ln a}{2} \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha} \right] a^t b^{1-t} f'\left(a^t b^{1-t}\right) dt, \tag{1}$$

where the symbols ${}_HJ^{\alpha}_{a^+} f$ and ${}_HJ^{\alpha}_{b^-} f$ are defined by

$$\left({}_HJ^{\alpha}_{a^+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad (0 < a < x \le b),$$
$$\left({}_HJ^{\alpha}_{b^-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad (0 < a \le x < b), \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function.

Lemma 2 (see [11, Lemma 2.1]). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 < a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_{H}J_{a^+}^{\alpha}f(b)+{}_{H}J_{b^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$=\frac{b-a}{2}\int_0^1 kf'(ta+(1-t)b)\,dt$$
$$-\frac{\ln b-\ln a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]a^tb^{1-t}f'\left(a^tb^{1-t}\right)dt, \tag{3}$$

where

$$k=\begin{cases}1, & 0\le t<\dfrac{1}{2},\\ -1, & \dfrac{1}{2}\le t<1.\end{cases} \tag{4}$$

Remark 3. It is remarkable that Professor Srivastava et al. [20] give some further refinements and extensions of the Hermite-Hadamard inequalities in n variables. In the forthcoming works, we will try to extend to study fractional version Hermite-Hadamard inequalities in n variables based on such fundamental results.

Next, we recall the following basic concepts and results in our previous papers.

Definition 4 (see [21, 22]). Let $f: I\subseteq\mathbb{R}^+\to\mathbb{R}^+$. A function f is said to be GG-convex on I if, for every $x,y\in I$ and $\lambda\in[0,1]$, one has

$$f\left(x^{\lambda}y^{1-\lambda}\right)\le[f(x)]^{\lambda}[f(y)]^{1-\lambda}. \tag{5}$$

Remark 5. By the arithmetic-geometric mean inequality, we have

$$[f(x)]^{\lambda}[f(y)]^{1-\lambda}\le\lambda f(x)+(1-\lambda)f(y). \tag{6}$$

Linking (5) and (6), we obtain

$$f\left(x^{\lambda}y^{1-\lambda}\right)\le\lambda f(x)+(1-\lambda)f(y), \tag{7}$$

which appears in the standard definition of GA-convex function [21]. So GG-convex function is GA-convex function.

Lemma 6 (see [19, Lemma 2.5]). *For $t\in[0,1]$, $x,y>0$, one has*

$$tx+(1-t)y\ge y^{1-t}x^t. \tag{8}$$

Lemma 7 (see [13, Lemma 2.1]). *For $\alpha>0$ and $k>0$, one has*

$$I(\alpha)=\int_0^1 t^{\alpha-1}k^t\,dt=k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(\alpha)_i}<+\infty, \tag{9}$$

where $(\alpha)_i=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1)$.

Lemma 8 (see [13, Lemma 2.2]). *For $\alpha>0$ and $k>0$, $z>0$, one has*

$$J(\alpha,k)=\int_0^1(1-t)^{\alpha-1}k^t\,dt=\sum_{i=1}^{\infty}\frac{(\ln k)^{i-1}}{(\alpha)_i}<+\infty,$$
$$H(\alpha,k,z)=\int_0^z t^{\alpha-1}k^t\,dt=z^{\alpha}k^z\sum_{i=1}^{\infty}\frac{(-z\ln k)^{i-1}}{(\alpha)_i}<+\infty. \tag{10}$$

Lemma 9 (see [13, Lemma 2.3]). *For $\alpha>0$ and $k>0$, $1>z>0$, one has*

$$R(\alpha,k,z)=\int_0^z(1-t)^{\alpha-1}k^t\,dt$$
$$=\sum_{i=0}^{\infty}\frac{(\ln k)^{i-1}}{(\alpha)_i}\left(1-k^z(1-z)^{\alpha+i-1}\right). \tag{11}$$

In the present paper, we will use the above concepts and lemmas to derive some new fractional Hermite-Hadamard inequalities involving Hadamard fractional integrals.

2. Main Results Based on Lemma 1

Now we are ready to state the following main results in this section.

Theorem 10. *Let $f:[0,b]\to\mathbb{R}^+$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[a,b]$ for some fixed $\alpha\in(0,\infty)$ and $t\in[0,1]$, $0\le a<b$, then the following integrals hold:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_{H}J_{a^+}^{\alpha}f(b)+{}_{H}J_{b^-}^{\alpha}f(a)\right]\right|$$
$$\le\frac{\ln b-\ln a}{2}$$
$$\times\left\{\left(1-\frac{1}{2^{\alpha+1}}\right)\right.$$
$$\times\left[\frac{b\left|f'(b)\right|}{\alpha+1}+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2b\left|f'(b)\right|}{\alpha+2}\right]$$
$$+\frac{a\left|f'(a)\right|-a\left|f'(b)\right|-b\left|f'(a)\right|}{\alpha+3}$$
$$-\frac{a\left|f'(a)\right|-b\left|f'(a)\right|}{(\alpha+3)2^{\alpha+3}}+\frac{2a\left|f'(a)\right|}{(\alpha+1)(\alpha+2)(\alpha+3)}$$
$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2a\left|f'(a)\right|}{(\alpha+1)(\alpha+3)}$$
$$+\frac{2a\left(\left|f'(a)\right|-\left|f'(b)\right|\right)-2b\left|f'(a)\right|}{2^{\alpha+3}(\alpha+2)(\alpha+3)}$$
$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|}{2^{\alpha+2}(\alpha+2)}$$
$$\left.+\frac{a\left|f'(a)\right|}{(\alpha+1)2^{\alpha+2}}-\frac{a\left|f'(a)\right|}{(\alpha+2)^2 2^{\alpha+2}}\right\}. \tag{12}$$

Proof. Noting Definition 4 and Lemmas 1 and 6, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_{H}J^{\alpha}_{a^+}f(b)+{}_{H}J^{\alpha}_{b^-}f(a)\right]\right|$$
$$=\left|\frac{\ln b-\ln a}{2}\int_0^1 a^t b^{1-t}\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(a^t b^{1-t}\right)dt\right|$$
$$\le\frac{\ln b-\ln a}{2}\int_0^1 a^t b^{1-t}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left(a^t b^{1-t}\right)\right|dt$$
$$\le\frac{\ln b-\ln a}{2}\int_{1/2}^1 a^t b^{1-t}\left[t^{\alpha}-(1-t)^{\alpha}\right]\left|f'\left(a^t b^{1-t}\right)\right|dt+\frac{\ln b-\ln a}{2}\int_0^{1/2} a^t b^{1-t}\left[(1-t)^{\alpha}-t^{\alpha}\right]\times\left|f'\left(a^t b^{1-t}\right)\right|dt$$
$$\le\frac{\ln b-\ln a}{2}\int_{1/2}^1\left[at+b(1-t)\right]\left[t^{\alpha}-(1-t)^{\alpha}\right]\times\left|f'(a)\right|^t\left|f'(b)\right|^{1-t}dt+\frac{\ln b-\ln a}{2}\int_0^{1/2}\left[at+b(1-t)\right]\left[(1-t)^{\alpha}-t^{\alpha}\right]\times\left|f'(a)\right|^t\left|f'(b)\right|^{1-t}dt$$
$$\le\frac{\ln b-\ln a}{2}\int_{1/2}^1\left[at+b(1-t)\right]\left[t^{\alpha}-(1-t)^{\alpha}\right]\times\left[\left|f'(a)\right|t+\left|f'(b)\right|(1-t)\right]dt+\frac{\ln b-\ln a}{2}\int_0^{1/2}\left[at+b(1-t)\right]\left[(1-t)^{\alpha}-t^{\alpha}\right]\times\left[\left|f'(a)\right|t+\left|f'(b)\right|(1-t)\right]dt$$
$$\le\frac{\ln b-\ln a}{2}\left\{\frac{a\left|f'(a)\right|}{\alpha+3}\int_{1/2}^1(\alpha+3)t^{\alpha+2}dt+\frac{b\left|f'(a)\right|}{\alpha+2}\int_{1/2}^1(\alpha+2)t^{\alpha+1}dt-\frac{b\left|f'(a)\right|}{\alpha+3}\int_{1/2}^1(\alpha+3)t^{\alpha+2}dt+\frac{a\left|f'(b)\right|}{\alpha+2}\int_{1/2}^1(\alpha+2)t^{\alpha+1}dt-\frac{a\left|f'(b)\right|}{\alpha+3}\int_{1/2}^1(\alpha+3)t^{\alpha+2}dt+\frac{b\left|f'(b)\right|}{\alpha+1}\int_{1/2}^1(\alpha+1)t^{\alpha}dt-\frac{2b\left|f'(b)\right|}{\alpha+2}\int_{1/2}^1(\alpha+2)t^{\alpha+1}dt+\frac{b\left|f'(b)\right|}{\alpha+3}\int_{1/2}^1(\alpha+3)t^{\alpha+2}dt-a\left|f'(a)\right|\int_{1/2}^1(1-t)^{\alpha}t^2dt-b\left|f'(a)\right|\int_{1/2}^1 t(1-t)^{\alpha+1}dt-a\left|f'(b)\right|\int_{1/2}^1 t(1-t)^{\alpha+1}dt-\frac{b\left|f'(b)\right|}{\alpha+3}\int_{1/2}^1(\alpha+3)(1-t)^{\alpha+2}dt\right\}$$
$$+\frac{\ln b-\ln a}{2}\left\{a\left|f'(a)\right|\int_0^{1/2}t^2(1-t)^{\alpha}dt+b\left|f'(a)\right|\int_0^{1/2}t(1-t)^{\alpha+1}dt+a\left|f'(b)\right|\int_0^{1/2}t(1-t)^{\alpha+1}dt+\frac{b\left|f'(b)\right|}{\alpha+3}\int_0^{1/2}(\alpha+3)(1-t)^{\alpha+2}dt-\frac{a\left|f'(a)\right|}{\alpha+3}\int_0^{1/2}(\alpha+3)t^{\alpha+2}dt-\frac{b\left|f'(a)\right|}{\alpha+2}\int_0^{1/2}(\alpha+2)t^{\alpha+1}dt+\frac{b\left|f'(a)\right|}{\alpha+3}\int_0^{1/2}(\alpha+3)t^{\alpha+2}dt-\frac{a\left|f'(b)\right|}{\alpha+2}\int_0^{1/2}(\alpha+2)t^{\alpha+1}dt+\frac{a\left|f'(b)\right|}{\alpha+3}\int_0^{1/2}(\alpha+3)t^{\alpha+2}dt-\frac{b\left|f'(b)\right|}{\alpha+1}\int_0^{1/2}(\alpha+1)t^{\alpha}dt+\frac{2b\left|f'(b)\right|}{\alpha+2}\int_0^{1/2}(\alpha+2)t^{\alpha+1}dt-\frac{b\left|f'(b)\right|}{\alpha+3}\int_0^{1/2}(\alpha+3)t^{\alpha+2}dt\right\}$$
$$\le\frac{\ln b-\ln a}{2}\times\left\{\frac{a\left|f'(a)\right|}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)\right.$$

$$+\frac{b\left|f'(a)\right|}{\alpha+2}\left(1-\frac{1}{2^{\alpha+2}}\right)-\frac{b\left|f'(a)\right|}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)$$
$$+\frac{a\left|f'(b)\right|}{\alpha+2}\left(1-\frac{1}{2^{\alpha+2}}\right)-\frac{a\left|f'(b)\right|}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)$$
$$+\frac{b\left|f'(b)\right|}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right)-\frac{2b\left|f'(b)\right|}{\alpha+2}\left(1-\frac{1}{2^{\alpha+2}}\right)$$
$$+\frac{b\left|f'(b)\right|}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)-a\left|f'(a)\right|\int_{1/2}^{1}(1-t)^{\alpha}t^{2}dt$$
$$-b\left|f'(a)\right|\int_{1/2}^{1}t(1-t)^{\alpha+1}dt$$
$$-a\left|f'(b)\right|\int_{1/2}^{1}t(1-t)^{\alpha+1}dt$$
$$-\frac{b\left|f'(b)\right|}{\alpha+3}\int_{1/2}^{1}(\alpha+3)(1-t)^{\alpha+2}dt\Bigg\}$$
$$+\frac{\ln b-\ln a}{2}$$
$$\times\Bigg\{a\left|f'(a)\right|\int_{0}^{1/2}t^{2}(1-t)^{\alpha}dt$$
$$+b\left|f'(a)\right|\int_{0}^{1/2}t(1-t)^{\alpha+1}dt$$
$$+a\left|f'(b)\right|\int_{0}^{1/2}t(1-t)^{\alpha+1}dt$$
$$+\frac{b\left|f'(b)\right|}{\alpha+3}\left(\frac{1}{2^{\alpha+3}}-1\right)-\frac{a\left|f'(a)\right|}{\alpha+3}\frac{1}{2^{\alpha+3}}$$
$$-\frac{b\left|f'(a)\right|}{\alpha+2}\frac{1}{2^{\alpha+2}}+\frac{b\left|f'(a)\right|}{\alpha+3}\frac{1}{2^{\alpha+3}}$$
$$-\frac{a\left|f'(b)\right|}{\alpha+2}\frac{1}{2^{\alpha+2}}+\frac{a\left|f'(b)\right|}{\alpha+3}\frac{1}{2^{\alpha+3}}$$
$$-\frac{b\left|f'(b)\right|}{\alpha+1}\frac{1}{2^{\alpha+1}}+\frac{2b\left|f'(b)\right|}{\alpha+2}\frac{1}{2^{\alpha+2}}$$
$$-\frac{b\left|f'(a)\right|}{\alpha+3}\frac{1}{2^{\alpha+3}}\Bigg\}$$
$$\leq\frac{\ln b-\ln a}{2}\Bigg\{\frac{(a-b)\left[\left|f'(a)\right|-\left|f'(b)\right|\right]}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)$$
$$+\frac{b\left|f'(b)\right|}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right)$$
$$+\frac{b\left[\left|f'(a)\right|-\left|f'(b)\right|\right]+(a-b)\left|f'(b)\right|}{\alpha+2}$$
$$\times\left(1-\frac{1}{2^{\alpha+2}}\right)-a\left|f'(a)\right|\int_{0}^{1}(1-t)^{\alpha}t^{2}dt$$
$$+a\left|f'(a)\right|\int_{0}^{1/2}(1-t)^{\alpha}t^{2}dt$$
$$-b\left|f'(a)\right|\int_{0}^{1}t(1-t)^{\alpha+1}dt$$
$$+b\left|f'(a)\right|\int_{0}^{1/2}t(1-t)^{\alpha+1}dt$$
$$-a\left|f'(b)\right|\int_{0}^{1}t(1-t)^{\alpha+1}dt$$
$$+a\left|f'(b)\right|\int_{0}^{1/2}t(1-t)^{\alpha+1}dt\Bigg\}$$
$$+\frac{\ln b-\ln a}{2}\Bigg\{a\left|f'(a)\right|\int_{0}^{1/2}t^{2}(1-t)^{\alpha}dt$$
$$+b\left|f'(a)\right|\int_{0}^{1/2}t(1-t)^{\alpha+1}dt$$
$$+a\left|f'(b)\right|\int_{0}^{1/2}t(1-t)^{\alpha+1}dt$$
$$+\frac{b\left|f'(b)\right|}{\alpha+3}\left(\frac{1}{2^{\alpha+3}}-1\right)-\frac{a\left|f'(b)\right|}{(\alpha+3)\,2^{\alpha+3}}$$
$$-\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2b\left|f'(b)\right|}{(\alpha+2)\,2^{\alpha+2}}$$
$$-\frac{b\left|f'(b)\right|}{(\alpha+1)\,2^{\alpha+1}}\Bigg\}$$
$$\leq\frac{\ln b-\ln a}{2}$$
$$\times\Bigg\{\left(1-\frac{1}{2^{\alpha+1}}\right)$$
$$\times\Bigg[\frac{b\left|f'(b)\right|}{\alpha+1}$$
$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2b\left|f'(b)\right|}{\alpha+2}\Bigg]$$
$$+\frac{a\left|f'(a)\right|-a\left|f'(b)\right|-b\left|f'(a)\right|}{\alpha+3}$$
$$-\frac{a\left|f'(a)\right|-b\left|f'(a)\right|}{(\alpha+3)\,2^{\alpha+3}}$$
$$+\frac{2a\left|f'(a)\right|}{(\alpha+1)(\alpha+2)(\alpha+3)}$$
$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2a\left|f'(a)\right|}{(\alpha+1)(\alpha+3)}$$

$$+\frac{2a\left(\left|f'(a)\right|-\left|f'(b)\right|\right)-2b\left|f'(a)\right|}{2^{\alpha+3}(\alpha+2)(\alpha+3)}$$
$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|}{2^{\alpha+2}(\alpha+2)}+\frac{a\left|f'(a)\right|}{(\alpha+1)2^{\alpha+2}}$$
$$\left.-\frac{a\left|f'(a)\right|}{(\alpha+2)^2 2^{\alpha+2}}\right\}. \quad (13)$$

The proof is done. □

Theorem 11. *Let $f:[0,b]\to\mathbb{R}^+$ be a differentiable mapping. If $|f'|^q$ is measurable and $|f'|^q$, $(q>1)$ is GG-convex on $[a,b]$ for some fired $\alpha\in(0,\infty)$ and $t\in[0,1]$, $0\le a<b$, then the following integrals hold:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\left[{}_HJ^\alpha_{a^+}f(b)+{}_HJ^\alpha_{b^-}f(a)\right]\right|$$
$$\le\frac{\ln b-\ln a}{2}\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q-\left|f'(b)\right|^q}{2}\right)^{1/q}$$
$$\times\left(\frac{a^p-2b^p}{p\alpha+2}+\frac{b^p}{p\alpha+1}-\frac{2b^p+a^p}{2^{p\alpha+2}(p\alpha+1)}\right.$$
$$+\frac{2b^p-a^p}{2^{p\alpha+2}(p\alpha+2)}-\frac{a^p}{p\alpha(p\alpha+1)}$$
$$-\frac{a^p+b^p}{2^{p\alpha+2}(p\alpha+1)(p\alpha+2)}$$
$$\left.+\frac{2a^p}{(p\alpha+1)(p\alpha+2)}\right)^{1/p}, \quad (14)$$

where $1/p+1/q=1$.

Proof. By using Definition 4 and Lemmas 1 and 6, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\left[{}_HJ^\alpha_{a^+}f(b)+{}_HJ^\alpha_{b^-}f(a)\right]\right|$$
$$\le\frac{\ln b-\ln a}{2}\left(\int_0^1 a^{pt}b^{p(1-t)}\left|(1-t)^\alpha-t^\alpha\right|^p dt\right)^{1/p}$$
$$\times\left(\int_0^1\left|f'(a)\right|^{qt}\left|f'(b)\right|^{q(1-t)}dt\right)^{1/q}$$
$$\le\frac{\ln b-\ln a}{2}$$
$$\times\left(\int_0^1\left[ta^p+(1-t)b^p\right]\left|(1-t)^\alpha-t^\alpha\right|^p dt\right)^{1/p}$$
$$\times\left(\int_0^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{1/q}$$
$$\le\frac{\ln b-\ln a}{2}$$
$$\times\left(\int_{1/2}^1\left[ta^p+(1-t)b^p\right]\left(t^{pa}-(1-t)^{pa}\right)dt\right.$$
$$\left.+\int_0^{1/2}\left[ta^p+(1-t)b^p\right]\left[(1-t)^{p\alpha}-t^{p\alpha}\right]dt\right)^{1/p}$$
$$\times\left(\frac{\left|f'(a)\right|^q}{2}+\left|f'(b)\right|^q-\frac{\left|f'(b)\right|^q}{2}\right)^{1/q}$$
$$\le\frac{\ln b-\ln a}{2}\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q-\left|f'(b)\right|^q}{2}\right)^{1/q}$$
$$\times\left(a^p\int_{1/2}^1 t^{p\alpha+1}dt-a^p\int_{1/2}^1 t(1-t)^{p\alpha}dt\right.$$
$$+b^p\int_{1/2}^1 t^{p\alpha}(1-t)\,dt-b^p\int_{1/2}^1(1-t)^{p\alpha+1}dt$$
$$+a^p\int_0^{1/2}t(1-t)^{p\alpha}dt-a^p\int_0^{1/2}t^{p\alpha+1}dt$$
$$+b^p\int_0^{1/2}(1-t)^{p\alpha+1}dt$$
$$\left.-b^p\int_0^{1/2}t^{p\alpha}(1-t)\,dt\right)^{1/p}$$
$$\le\frac{\ln b-\ln a}{2}\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q-\left|f'(b)\right|^q}{2}\right)^{1/q}$$
$$\times\left(\frac{a^p}{p\alpha+2}-a^p\int_0^1 t(1-t)^{p\alpha-1}dt\right.$$
$$+a^p\int_0^{1/2}t(1-t)^{p\alpha}dt+\frac{b^p}{p\alpha+1}$$
$$-\frac{1}{2^{p\alpha+1}}\frac{b^p}{p\alpha+1}-\frac{b^p}{p\alpha+2}+\frac{1}{2^{p\alpha+2}}\frac{b^p}{p\alpha+2}$$
$$+a^p\int_0^{1/2}t(1-t)^{p\alpha}dt-\frac{1}{2^{p\alpha+2}}\frac{a^p}{p\alpha+2}$$
$$+\frac{1}{2^{p\alpha+2}}\frac{b^p}{p\alpha+2}-\frac{b^p}{p\alpha+2}$$
$$\left.-b^p\int_0^{1/2}t^{p\alpha}(1-t)\,dt\right)^{1/p}$$
$$\le\frac{\ln b-\ln a}{2}\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q-\left|f'(b)\right|^q}{2}\right)^{1/q}$$
$$\times\left(\frac{a^p-2b^p}{p\alpha+2}+\frac{b^p}{p\alpha+1}-\frac{2b^p+a^p}{2^{p\alpha+2}(p\alpha+1)}\right.$$

$$+\frac{2b^p - a^p}{2^{p\alpha+2}(p\alpha+2)} - \frac{a^p}{p\alpha(p\alpha+1)} - \frac{a^p + b^p}{2^{p\alpha+2}(p\alpha+1)(p\alpha+2)} + \frac{2a^p}{(p\alpha+1)(p\alpha+2)}\Bigg)^{1/p}. \quad (15)$$

The proof is done. □

Theorem 12. *Let $f : [0, b] \to \mathbb{R}^+$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[a, b]$ for some fired $\alpha \in (0, \infty)$, $t \in [0, 1]$ and $k > 0$, $0 \le a < b$, then the following integrals hold:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\left[{}_H J^\alpha_{a^+} f(b) + {}_H J^\alpha_{b^-} f(a)\right]\right| \le b\left|f'(b)\right|\frac{\ln b - \ln a}{2} \times \left\{\frac{a|f'(a)|}{b|f'(b)|} \times \sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln\left(a|f'(a)|/b|f'(b)|\right)\right]^{i-1}}{(\alpha+1)_i} + \sum_{i=1}^{\infty}\frac{\left[\ln\left(a|f'(a)|/b|f'(b)|\right)\right]^{i-1}}{(\alpha+1)_i} - \frac{(1/2)^{\alpha-1}\left(a|f'(a)|/b|f'(b)|\right)^{1/2}}{\alpha+1} + 2\frac{1-(1/2)^\alpha\left(a|f'(a)|/b|f'(b)|\right)^{1/2}}{\ln\left(a|f'(a)|/b|f'(b)|\right)}\right\}. \quad (16)$$

Proof. By using Definition 4 and Lemmas 1, 7, 8, and 9, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\left[{}_H J^\alpha_{a^+} f(b) + {}_H J^\alpha_{b^-} f(a)\right]\right| \\
&\le \frac{\ln b - \ln a}{2}\int_{1/2}^{1} a^t b^{1-t}\left[t^\alpha - (1-t)^\alpha\right]\left|f'\left(a^t b^{1-t}\right)\right| dt \\
&\quad + \frac{\ln b - \ln a}{2}\int_0^{1/2} a^t b^{1-t}\left[(1-t)^\alpha - t^\alpha\right] \times \left|f'\left(a^t b^{1-t}\right)\right| dt \\
&\le \frac{\ln b - \ln a}{2}\int_{1/2}^{1} a^t b^{1-t}\left[t^\alpha - (1-t)^\alpha\right] \times \left|f'(a)\right|^t\left|f'(b)\right|^{1-t} dt \\
&\quad + \frac{\ln b - \ln a}{2}\int_0^{1/2} a^t b^{1-t}\left[(1-t)^\alpha - t^\alpha\right] \times \left|f'(a)\right|^t\left|f'(b)\right|^{1-t} dt \\
&\le b\left|f'(b)\right|\frac{\ln b - \ln a}{2} \times \int_{1/2}^{1}\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^t\left[t^\alpha - (1-t)^\alpha\right] dt \\
&\quad + b\left|f'(b)\right|\frac{\ln b - \ln a}{2} \times \int_0^{1/2}\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^t\left[(1-t)^\alpha - t^\alpha\right] dt \\
&\le b\left|f'(b)\right|\frac{\ln b - \ln a}{2} \times \left\{\int_0^1\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^t t^\alpha dt - 2\int_0^{1/2}\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^t t^\alpha dt - \int_0^1\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^t (1-t)^\alpha dt + 2\int_0^{1/2}\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^t (1-t)^\alpha dt\right\} \\
&\le b\left|f'(b)\right|\frac{\ln b - \ln a}{2} \times \left\{\frac{a|f'(a)|}{b|f'(b)|} \times \sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln\left(a|f'(a)|/b|f'(b)|\right)\right]^{i-1}}{(\alpha+1)_i} - \sum_{i=1}^{\infty}\frac{\left[\ln\left(a|f'(a)|/b|f'(b)|\right)\right]^{i-1}}{(\alpha+1)_i} - \left(\frac{1}{2}\right)^\alpha\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^{1/2} \times \sum_{i=1}^{\infty}\frac{\left[-(1/2)\ln\left(a|f'(a)|/b|f'(b)|\right)\right]^{i-1}}{(\alpha+1)_i}\right.
\end{aligned}$$

$$+2\sum_{i=0}^{\infty}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_i}\times\left[1-\left(\frac{1}{2}\right)^{\alpha+i}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{1/2}\right]\Bigg\}$$

$$\leq b\left|f'(b)\right|\frac{\ln b-\ln a}{2}\times\Bigg\{\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\times\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_i}-\sum_{i=1}^{\infty}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_i}-\left(\frac{1}{2}\right)^{\alpha}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{1/2}\times\sum_{i=1}^{\infty}\frac{\left[-(1/2)\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_i}+2\frac{1-(1/2)^{\alpha}\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)^{1/2}}{\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)}+2\sum_{i=1}^{\infty}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_i}-\left(\frac{1}{2}\right)^{\alpha}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{1/2}\times\sum_{i=1}^{\infty}\frac{\left[(1/2)\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_i}\Bigg\}$$

$$\leq b\left|f'(b)\right|\frac{\ln b-\ln a}{2}\times\Bigg\{\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\times\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_i}+\sum_{i=1}^{\infty}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_i}-\frac{(1/2)^{\alpha-1}\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)^{1/2}}{\alpha+1}+2\frac{1-(1/2)^{\alpha}\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)^{1/2}}{\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)}\Bigg\}. \tag{17}$$

The proof is done. □

Theorem 13. *Let $f:[0,b]\to\mathbb{R}^+$ be a differentiable mapping. If $|f'|^q$ is measurable and $|f'|^q$, $(q>1)$ is GG-convex on $[a,b]$, for some fired $\alpha\in(0,\infty)$, $t\in[0,1]$, and $k>0$, $0\leq a<b$, then the following integrals hold:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]\right|\leq\frac{\ln b-\ln a}{2}\times\left(a^p\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[(1/2)\ln(a/b)^p\right]^{i-1}}{(p\alpha+1)_i}+b^p\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^p\right]^{i-1}}{(p\alpha+1)_i}-\frac{(1/2)^{p\alpha-1}\left(ab^{p/2}\right)}{p\alpha+1}-\left[\ln\left(\frac{a}{b}\right)^p\right]^{-1}\left[\left(\frac{1}{2}\right)^{p\alpha-1}(ab)^{p/2}+2b^p\right]\right)^{1/p}\times\left(\frac{\left|f'(b)\right|^q}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^q}\left[\left|\frac{f'(a)}{f'(b)}\right|^q-1\right]\right)^{1/q}, \tag{18}$$

where $1/p+1/q=1$.

Proof. By using Definition 4 and Lemmas 1, 7, 8, and 9, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]\right|\leq\frac{\ln b-\ln a}{2}\left(\int_0^1\left(a^tb^{1-t}\left|(1-t)^{\alpha}-t^{\alpha}\right|\right)^pdt\right)^{1/p}\times\left(\int_0^1\left|f'\left(a^tb^{1-t}\right)\right|^qdt\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}\left(b^p\int_{1/2}^1\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt-b^p\int_{1/2}^1\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt+b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt-b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt\right)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\right.$$

$$\left.\times\int_{0}^{1}\ln\left(\frac{\left|f'(a)\right|}{\left|f'(b)\right|}\right)^{q}\left[\frac{\left|f'(a)\right|}{\left|f'(b)\right|}\right]^{qt}dt\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}\left(b^{p}\left(\frac{a}{b}\right)^{p}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}\right.$$

$$-b^{p}\sum_{i=0}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$-2b^{p}\int_{0}^{1/2}\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt$$

$$\left.+2b^{p}\int_{0}^{1/2}\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt\right)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\left[\left|\frac{f'(a)}{f'(b)}\right|^{q}-1\right]\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}$$

$$\times\left(b^{p}\left(\frac{a}{b}\right)^{p}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}\right.$$

$$-b^{p}\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$-\left(\frac{1}{2}\right)^{p\alpha}b^{p}\left(\frac{a}{b}\right)^{p/2}\sum_{i=1}^{\infty}\frac{\left[-(1/2)\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$+2b^{p}\sum_{i=0}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$\left.\times\left[1-\left(\frac{a}{b}\right)^{p/2}\left(1-\frac{1}{2}\right)^{p\alpha+i}\right]\right)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\left[\left|\frac{f'(a)}{f'(b)}\right|^{q}-1\right]\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}$$

$$\times\left(a^{p}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}\right.$$

$$-b^{p}\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$-\left(\frac{1}{2}\right)^{p\alpha}(ab)^{p/2}\sum_{i=1}^{\infty}\frac{\left[-(1/2)\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$+2b^{p}\sum_{i=0}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$\left.-2b^{p}\left(\frac{a}{b}\right)^{p/2}\sum_{i=0}^{\infty}\left(1-\frac{1}{2}\right)^{p\alpha+i}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}\right)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\left[\left|\frac{f'(a)}{f'(b)}\right|^{q}-1\right]\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}$$

$$\times\left(a^{p}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}-b^{p}\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}\right.$$

$$-\left(\frac{1}{2}\right)^{p\alpha}(ab)^{p/2}\sum_{i=1}^{\infty}\frac{\left[-(1/2)\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$+2b^{p}\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}+2b^{p}\left[\ln(a/b)^{p}\right]^{-1}$$

$$\left.-2b^{p}\left(\frac{a}{b}\right)^{p/2}\sum_{i=0}^{\infty}\left(1-\frac{1}{2}\right)^{p\alpha+i}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}\right)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\left[\left|\frac{f'(a)}{f'(b)}\right|^{q}-1\right]\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}$$

$$\times\left(a^{p}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}+b^{p}\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}\right.$$

$$-\left(\frac{1}{2}\right)^{p\alpha}(ab)^{p/2}\sum_{i=1}^{\infty}\frac{\left[-(1/2)\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$-\left(\frac{1}{2}\right)^{p\alpha}(ab)^{p/2}\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i-1}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$\left.+2b^{p}\left[\ln\left(\frac{a}{b}\right)^{p}\right]^{-1}\right)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\left[\left|\frac{f'(a)}{f'(b)}\right|^{q}-1\right]\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}$$

$$\times\left(a^{p}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}+b^{p}\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}\right.$$

$$-\left(\frac{1}{2}\right)^{p\alpha}(ab)^{p/2}\sum_{i=1}^{\infty}\frac{\left[-(1/2)\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$-\left(\frac{1}{2}\right)^{p\alpha}(ab)^{p/2}\sum_{i=1}^{\infty}\frac{\left[(1/2)\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$-\left(\frac{1}{2}\right)^{p\alpha-1}(ab)^{p/2}\left[\ln\left(\frac{a}{b}\right)^{p}\right]^{-1}$$

$$+2b^{p}\left[\ln\left(\frac{a}{b}\right)^{p}\right]^{-1}\Bigg)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\left[\left|\frac{f'(a)}{f'(b)}\right|^{q}-1\right]\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}$$

$$\times\Bigg(a^{p}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}+b^{p}\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$-2\left(\frac{1}{2}\right)^{p\alpha}(ab)^{p/2}\frac{1}{p\alpha+1}$$

$$-\left[\ln\left(\frac{a}{b}\right)^{p}\right]^{-1}\left[\left(\frac{1}{2}\right)^{p\alpha-1}(ab)^{p/2}+2b^{p}\right]\Bigg)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\left[\left|\frac{f'(a)}{f'(b)}\right|^{q}-1\right]\right)^{1/q}$$

$$\leq\frac{\ln b-\ln a}{2}$$

$$\times\Bigg(a^{p}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[(1/2)\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}$$

$$+b^{p}\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^{p}\right]^{i-1}}{(p\alpha+1)_{i}}-\frac{(1/2)^{p\alpha-1}\left(ab^{p/2}\right)}{p\alpha+1}$$

$$-\left[\ln\left(\frac{a}{b}\right)^{p}\right]^{-1}\left[\left(\frac{1}{2}\right)^{p\alpha-1}(ab)^{p/2}+2b^{p}\right]\Bigg)^{1/p}$$

$$\times\left(\frac{\left|f'(b)\right|^{q}}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^{q}}\left[\left|\frac{f'(a)}{f'(b)}\right|^{q}-1\right]\right)^{1/q}. \quad (19)$$

The proof is done. □

3. Main Results Based on Lemma 2

Theorem 14. *Let $f:[0,b]\to\mathbb{R}^{+}$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[a,b]$, for some fired $\alpha\in(0,\infty)$ and $t\in[0,1]$, $0\leq a<b$, then the following integrals hold:*

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_{H}J_{a^{+}}^{\alpha}f(b)+{}_{H}J_{b^{-}}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{\left|f'(b)\right|-\left|f'(a)\right|}{2}+\frac{\ln b-\ln a}{2}$$

$$\times\Bigg\{\left(1-\frac{1}{2^{\alpha+1}}\right)$$

$$\times\left[\frac{b\left|f'(b)\right|}{\alpha+1}+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2b\left|f'(b)\right|}{\alpha+2}\right]$$

$$+\frac{a\left|f'(a)\right|-a\left|f'(b)\right|-b\left|f'(a)\right|}{\alpha+3}$$

$$-\frac{a\left|f'(a)\right|-b\left|f'(a)\right|}{(\alpha+3)\,2^{\alpha+3}}+\frac{2a\left|f'(a)\right|}{(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2a\left|f'(a)\right|}{(\alpha+1)(\alpha+3)}$$

$$+\frac{2a\left(\left|f'(a)\right|-\left|f'(b)\right|\right)-2b\left|f'(a)\right|}{2^{\alpha+3}(\alpha+2)(\alpha+3)}$$

$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|}{2^{\alpha+2}(\alpha+2)}+\frac{a\left|f'(a)\right|}{(\alpha+1)\,2^{\alpha+2}}$$

$$-\frac{a\left|f'(a)\right|}{(\alpha+2)^{2}2^{\alpha+2}}\Bigg\}. \quad (20)$$

Proof. By using Definition 4 and Lemmas 2 and 6, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_{H}J_{a^{+}}^{\alpha}f(b)+{}_{H}J_{b^{-}}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{2}\int_{0}^{1}\left|kf'(ta+(1-t)b)\right|dt$$

$$+\frac{\ln b-\ln a}{2}\int_{0}^{1}a^{t}b^{1-t}\left|(1-t)^{\alpha}-t^{\alpha}\right|$$

$$\times\left|f'\left(a^{t}b^{1-t}\right)\right|dt$$

$$\leq\frac{-1}{2}\left(\left|f(a)\right|-\left|f(b)\right|\right)$$

$$+\frac{\ln b-\ln a}{2}$$

$$\times\left\{\int_{1/2}^{1}a^{t}b^{1-t}\left[t^{\alpha}-(1-t)^{\alpha}\right]\left|f'\left(a^{t}b^{1-t}\right)\right|dt\right.$$

$$\left.+\int_{0}^{1/2}a^{t}b^{1-t}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left|f'\left(a^{t}b^{1-t}\right)\right|dt\right\}$$

$$\begin{aligned}
&\leq \frac{-1}{2}\left(\left|f(a)\right|-\left|f(b)\right|\right)\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\int_{1/2}^{1} t^{\alpha}\left[at+b(1-t)\right]\right.\\
&\qquad\times\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\\
&\qquad-\int_{1/2}^{1}(1-t)^{\alpha}\left[at+b(1-t)\right]\\
&\qquad\left.\times\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right\}\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\int_{0}^{1/2}(1-t)^{\alpha}\left[at+b(1-t)\right]\right.\\
&\qquad\times\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\\
&\qquad-\int_{0}^{1/2} t^{\alpha}\left[at+b(1-t)\right]\\
&\qquad\left.\times\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right\}\\
&\leq \frac{-1}{2}\left(\left|f(a)\right|-\left|f(b)\right|\right)\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\left|f'(a)\right|\int_{1/2}^{1} t^{\alpha+1}\left[at+b(1-t)\right]dt\right.\\
&\qquad+\left|f'(b)\right|\int_{1/2}^{1}(1-t)\,t^{\alpha}\left[at+b(1-t)\right]dt\\
&\qquad-\left|f'(a)\right|\int_{1/2}^{1}(1-t)^{\alpha}t\left[at+b(1-t)\right]dt\\
&\qquad\left.-\left|f'(b)\right|\int_{1/2}^{1}(1-t)^{\alpha+1}\left[at+b(1-t)\right]dt\right\}\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\left|f'(a)\right|\int_{0}^{1/2} t(1-t)^{\alpha}\left[at+b(1-t)\right]dt\right.\\
&\qquad+\left|f'(b)\right|\int_{0}^{1/2}(1-t)^{\alpha+1}\left[at+b(1-t)\right]dt\\
&\qquad-\left|f'(a)\right|\int_{0}^{1/2} t^{\alpha+1}\left[at+b(1-t)\right]dt\\
&\qquad\left.-\left|f'(b)\right|\int_{0}^{1/2}(1-t)\,t^{\alpha}\left[at+b(1-t)\right]dt\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\left|f'(b)\right|-\left|f'(a)\right|}{2}\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\frac{a\left|f'(a)\right|}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)+\frac{b\left|f'(a)\right|}{\alpha+2}\left(1-\frac{1}{2^{\alpha+2}}\right)\right.\\
&\qquad-\frac{b\left|f'(a)\right|}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)+\frac{a\left|f'(b)\right|}{\alpha+2}\left(1-\frac{1}{2^{\alpha+2}}\right)\\
&\qquad-\frac{a\left|f'(b)\right|}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)+\frac{b\left|f'(b)\right|}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right)\\
&\qquad-\frac{2b\left|f'(b)\right|}{\alpha+2}\left(1-\frac{1}{2^{\alpha+2}}\right)+\frac{b\left|f'(b)\right|}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right)\\
&\qquad-a\left|f'(a)\right|\int_{1/2}^{1}(1-t)^{\alpha}t^{2}dt\\
&\qquad-b\left|f'(a)\right|\int_{1/2}^{1}t(1-t)^{\alpha+1}dt\\
&\qquad-a\left|f'(b)\right|\int_{1/2}^{1}t(1-t)^{\alpha+1}dt\\
&\qquad\left.-\frac{b\left|f'(b)\right|}{\alpha+3}\int_{1/2}^{1}(\alpha+3)(1-t)^{\alpha+2}dt\right\}\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{a\left|f'(a)\right|\int_{0}^{1/2}t^{2}(1-t)^{\alpha}dt\right.\\
&\qquad+b\left|f'(a)\right|\int_{0}^{1/2}t(1-t)^{\alpha+1}dt\\
&\qquad+a\left|f'(b)\right|\int_{0}^{1/2}t(1-t)^{\alpha+1}dt\\
&\qquad+\frac{b\left|f'(b)\right|}{\alpha+3}\left(\frac{1}{2^{\alpha+3}}-1\right)-\frac{a\left|f'(a)\right|}{\alpha+3}\frac{1}{2^{\alpha+3}}\\
&\qquad-\frac{b\left|f'(a)\right|}{\alpha+2}\frac{1}{2^{\alpha+2}}+\frac{b\left|f'(a)\right|}{\alpha+3}\frac{1}{2^{\alpha+3}}\\
&\qquad-\frac{a\left|f'(b)\right|}{\alpha+2}\frac{1}{2^{\alpha+2}}+\frac{a\left|f'(b)\right|}{\alpha+3}\frac{1}{2^{\alpha+3}}-\frac{b\left|f'(b)\right|}{\alpha+1}\\
&\qquad\left.\times\frac{1}{2^{\alpha+1}}+\frac{2b\left|f'(b)\right|}{\alpha+2}\frac{1}{2^{\alpha+2}}-\frac{b\left|f'(a)\right|}{\alpha+3}\frac{1}{2^{\alpha+3}}\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\left|f'(b)\right|-\left|f'(a)\right|}{2} \\
&\quad + \frac{\ln b-\ln a}{2} \\
&\quad \times \left\{ \frac{(a-b)\left[\left|f'(a)\right|-\left|f'(b)\right|\right]}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right) \right. \\
&\qquad + \frac{b\left|f'(b)\right|}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right) \\
&\qquad + \frac{b\left[\left|f'(a)\right|-\left|f'(b)\right|\right]+(a-b)\left|f'(b)\right|}{\alpha+2} \\
&\qquad \times\left(1-\frac{1}{2^{\alpha+2}}\right) - a\left|f'(a)\right|\int_0^1 (1-t)^{\alpha}t^2\,dt \\
&\qquad + a\left|f'(a)\right|\int_0^{1/2}(1-t)^{\alpha}t^2\,dt \\
&\qquad - b\left|f'(a)\right|\int_0^1 t(1-t)^{\alpha+1}\,dt \\
&\qquad + b\left|f'(a)\right|\int_0^{1/2} t(1-t)^{\alpha+1}\,dt \\
&\qquad - a\left|f'(b)\right|\int_0^1 t(1-t)^{\alpha+1}\,dt \\
&\qquad \left. + a\left|f'(b)\right|\int_0^{1/2} t(1-t)^{\alpha+1} \right\} \\
&\quad + \frac{\ln b-\ln a}{2} \\
&\quad \times \left\{ a\left|f'(a)\right|\int_0^{1/2} t^2(1-t)^{\alpha}\,dt \right. \\
&\qquad + b\left|f'(a)\right|\int_0^{1/2} t(1-t)^{\alpha+1}\,dt \\
&\qquad + a\left|f'(b)\right|\int_0^{1/2} t(1-t)^{\alpha+1}\,dt \\
&\qquad + \frac{b\left|f'(b)\right|}{\alpha+3}\left(\frac{1}{2^{\alpha+3}}-1\right) - \frac{a\left|f'(b)\right|}{(\alpha+3)\,2^{\alpha+3}} \\
&\qquad - \frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2b\left|f'(b)\right|}{(\alpha+2)\,2^{\alpha+2}} \\
&\qquad \left. -\frac{b\left|f'(b)\right|}{(\alpha+1)\,2^{\alpha+1}} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\left|f'(b)\right|-\left|f'(a)\right|}{2} \\
&\quad + \frac{\ln b-\ln a}{2} \\
&\quad \times \left\{ \frac{(a-b)\left[\left|f'(a)\right|-\left|f'(b)\right|\right]}{\alpha+3}\left(1-\frac{1}{2^{\alpha+3}}\right) \right. \\
&\qquad + \frac{b\left|f'(b)\right|}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right) \\
&\qquad + \frac{b\left[\left|f'(a)\right|-\left|f'(b)\right|\right]+(a-b)\left|f'(b)\right|}{\alpha+2} \\
&\qquad \times\left(1-\frac{1}{2^{\alpha+2}}\right) - a\left|f'(a)\right|\int_0^1 t^2(1-t)^{\alpha}\,dt \\
&\qquad + a\left|f'(a)\right|\int_0^{1/2} t^2(1-t)^{\alpha}\,dt \\
&\qquad - b\left|f'(a)\right|\int_0^1 t(1-t)^{\alpha+1}\,dt \\
&\qquad + b\left|f'(a)\right|\int_0^{1/2} t(1-t)^{\alpha+1}\,dt \\
&\qquad - a\left|f'(b)\right|\int_0^1 t(1-t)^{\alpha+1}\,dt \\
&\qquad \left. + a\left|f'(b)\right|\int_0^{1/2} t(1-t)^{\alpha+1}\,dt \right\} \\
&\quad + \frac{\ln b-\ln a}{2}\left\{ a\left|f'(a)\right|\int_0^{1/2} t^2(1-t)^{\alpha}\,dt \right. \\
&\qquad + b\left|f'(a)\right|\int_0^{1/2} t(1-t)^{\alpha+1}\,dt \\
&\qquad + a\left|f'(b)\right|\int_0^{1/2} t(1-t)^{\alpha+1}\,dt \\
&\qquad + \frac{b\left|f'(b)\right|}{\alpha+3}\left(\frac{1}{2^{\alpha+3}}-1\right) - \frac{a\left|f'(b)\right|}{(\alpha+3)\,2^{\alpha+3}} \\
&\qquad - \frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2b\left|f'(b)\right|}{(\alpha+2)\,2^{\alpha+2}} \\
&\qquad \left. -\frac{b\left|f'(b)\right|}{(\alpha+1)\,2^{\alpha+1}} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\left|f'(b)\right|-\left|f'(a)\right|}{2} \\
&\quad + \frac{\ln b-\ln a}{2} \\
&\quad \times \left\{ \left(1-\frac{1}{2^{\alpha+1}}\right) \right.
\end{aligned}$$

$$\times\left[\frac{b\left|f'(b)\right|}{\alpha+1}+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2b\left|f'(b)\right|}{\alpha+2}\right]$$
$$+\frac{a\left|f'(a)\right|-a\left|f'(b)\right|-b\left|f'(a)\right|}{\alpha+3}$$
$$-\frac{a\left|f'(a)\right|-b\left|f'(a)\right|}{(\alpha+3)2^{\alpha+3}}+\frac{2a\left|f'(a)\right|}{(\alpha+1)(\alpha+2)(\alpha+3)}$$
$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|-2a\left|f'(a)\right|}{(\alpha+1)(\alpha+3)}$$
$$+\frac{2a\left(\left|f'(a)\right|-\left|f'(b)\right|\right)-2b\left|f'(a)\right|}{2^{\alpha+3}(\alpha+2)(\alpha+3)}$$
$$+\frac{b\left|f'(a)\right|+a\left|f'(b)\right|}{2^{\alpha+2}(\alpha+2)}$$
$$\left.+\frac{a\left|f'(a)\right|}{(\alpha+1)2^{\alpha+2}}-\frac{a\left|f'(a)\right|}{(\alpha+2)^2 2^{\alpha+2}}\right\}. \tag{21}$$

The proof is done. □

Theorem 15. *Let $f:[0,b]\to\mathbb{R}^+$ be a differentiable mapping. If $|f'|^q$ is measurable and $|f'|^q$, $(q>1)$ is GG-convex on $[a,b]$, for some fired $\alpha\in(0,\infty)$ and $t\in[0,1]$, $0\le a<b$, then the following integrals hold:*

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\left[{}_HJ^\alpha_{a^+}f(b)+{}_HJ^\alpha_{b^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{\left|f'(b)\right|-\left|f'(a)\right|}{2}+\frac{\ln b-\ln a}{2}$$
$$\times\left(\frac{\left|f'(a)\right|^q+2\left|f'(a)\right|^q-\left|f'(a)\right|^q}{2}\right)^{1/q}$$
$$\times\left(\frac{a^p-2b^p}{p\alpha+2}+\frac{b^p}{p\alpha+1}-\frac{2b^p+a^p}{2^{p\alpha+2}(p\alpha+1)}\right.$$
$$+\frac{2b^p-a^p}{2^{p\alpha+2}(p\alpha+2)}-\frac{a^p}{p\alpha(p\alpha+1)}$$
$$-\frac{a^p+b^p}{2^{p\alpha+2}(p\alpha+1)(p\alpha+2)}$$
$$\left.+\frac{2a^p}{(p\alpha+1)(p\alpha+2)}\right)^{1/p}, \tag{22}$$

where $1/p+1/q=1$.

Proof. By using Definition 4 and Lemmas 2 and 6, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\left[{}_HJ^\alpha_{a^+}f(b)+{}_HJ^\alpha_{b^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{b-a}{2}\int_0^1\left|kf'(ta+(1-t)b)\right|dt$$
$$+\frac{\ln b-\ln a}{2}\int_0^1 a^tb^{1-t}\left|(1-t)^\alpha-t^\alpha\right|\left|f'\left(a^tb^{1-t}\right)\right|dt$$
$$\le\frac{b-a}{2}\int_0^1|k|\left|f'(at+b(1-t))\right|dt$$
$$+\frac{\ln b-\ln a}{2}\int_0^1 a^tb^{1-t}\left|(1-t)^\alpha-t^\alpha\right|\left|f'\left(a^tb^{1-t}\right)\right|dt$$
$$\le\frac{(b-a)}{2}\int_0^1\left|f'(at+b(1-t))\right|dt$$
$$+\frac{\ln b-\ln a}{2}\int_0^1 a^tb^{1-t}\left|(1-t)^\alpha-t^\alpha\right|\left|f'\left(a^tb^{1-t}\right)\right|dt$$
$$\le\frac{-1}{2}\left(\left|f'(a)\right|-\left|f'(b)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}\left(\int_0^1\left(a^tb^{1-t}\left|(1-t)^\alpha-t^\alpha\right|\right)^pdt\right)^{1/p}$$
$$\times\left(\int_0^1\left|f'\left(a^tb^{1-t}\right)\right|^qdt\right)^{1/q}$$
$$\le\frac{-1}{2}\left(\left|f'(a)\right|-\left|f'(b)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}\left(\int_0^1 a^{pt}b^{p(1-t)}\left|(1-t)^\alpha-t^\alpha\right|^pdt\right)^{1/p}$$
$$\times\left(\int_0^1\left|f'(a)\right|^{qt}\left|f'(b)\right|^{q(1-t)}dt\right)^{1/q}$$
$$\le\frac{-1}{2}\left(\left|f'(a)\right|-\left|f'(b)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}$$
$$\times\left(\int_0^1\left[ta^p+(1-t)b^p\right]\left|(1-t)^\alpha-t^\alpha\right|^pdt\right)^{1/p}$$
$$\times\left(\int_0^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{1/q}$$
$$\le\frac{-1}{2}\left(\left|f'(a)\right|-\left|f'(b)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}$$
$$\times\left(\int_{1/2}^1\left[ta^p+(1-t)b^p\right]\left(t^{p\alpha}-(1-t)^{p\alpha}\right)dt\right.$$

$$+\int_0^{1/2}\left[ta^p+(1-t)b^p\right]\left[(1-t)^{p\alpha}-t^{p\alpha}\right]dt\Bigg)^{1/p}$$

$$\times\left(\frac{|f'(a)|^q}{2}+|f'(b)|^q-\frac{|f'(b)|^q}{2}\right)^{1/q}$$

$$\le\frac{-1}{2}\left(|f'(a)|-|f'(b)|\right)$$

$$+\frac{\ln b-\ln a}{2}\left(\frac{|f'(a)|^q+2|f'(b)|^q-|f'(b)|^q}{2}\right)^{1/q}$$

$$\times\Bigg(a^p\int_{1/2}^1 t^{p\alpha+1}dt-a^p\int_{1/2}^1 t(1-t)^{p\alpha}dt$$

$$+b^p\int_{1/2}^1 t^{p\alpha}(1-t)\,dt-b^p\int_{1/2}^1(1-t)^{p\alpha+1}dt$$

$$+a^p\int_0^{1/2}t(1-t)^{p\alpha}dt-a^p\int_0^{1/2}t^{p\alpha+1}dt$$

$$+b^p\int_0^{1/2}(1-t)^{p\alpha+1}-b^p\int_0^{1/2}t^{p\alpha}(1-t)dt\Bigg)^{1/p}$$

$$\le\frac{-1}{2}\left(|f'(a)|-|f'(b)|\right)$$

$$+\frac{\ln b-\ln a}{2}\left(\frac{|f'(a)|^q+2|f'(b)|^q-|f'(b)|^q}{2}\right)^{1/q}$$

$$\times\Bigg(\frac{a^p}{p\alpha+2}-a^p\int_0^1 t(1-t)^{p\alpha-1}dt$$

$$+a^p\int_0^{1/2}t(1-t)^{p\alpha}dt+\frac{b^p}{p\alpha+1}-\frac{1}{2^{p\alpha+1}}\frac{b^p}{p\alpha+1}$$

$$-\frac{b^p}{p\alpha+2}+\frac{1}{2^{p\alpha+2}}\frac{b^p}{p\alpha+2}+a^p\int_0^{1/2}t(1-t)^{p\alpha}dt$$

$$-\frac{1}{2^{p\alpha+2}}\frac{a^p}{p\alpha+2}+\frac{1}{2^{p\alpha+2}}\frac{b^p}{p\alpha+2}-\frac{b^p}{p\alpha+2}$$

$$-b^p\int_0^{1/2}t^{p\alpha}(1-t)dt\Bigg)^{1/p}$$

$$\le\frac{|f'(b)|-|f'(a)|}{2}$$

$$+\frac{\ln b-\ln a}{2}\left(\frac{|f'(a)|^q+2|f'(b)|^q-|f'(b)|^q}{2}\right)^{1/q}$$

$$\times\Bigg(\frac{a^p-2b^p}{p\alpha+2}+\frac{b^p}{p\alpha+1}-\frac{2b^p+a^p}{2^{p\alpha+2}(p\alpha+1)}$$

$$+\frac{2b^p-a^p}{2^{p\alpha+2}(p\alpha+2)}-\frac{a^p}{p\alpha(p\alpha+1)}$$

$$-\frac{a^p+b^p}{2^{p\alpha+2}(p\alpha+1)(p\alpha+2)}$$

$$+\frac{2a^p}{(p\alpha+1)(p\alpha+2)}\Bigg)^{1/p}. \tag{23}$$

The proof is done. □

Theorem 16. *Let $f:[0,b]\to\mathbb{R}^+$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[a,b]$, for some fired $\alpha\in(0,\infty)$ and $t\in[0,1]$, $0\le a<b$, then the following integrals hold:*

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\left[{}_HJ_{a^+}^\alpha f(b)+{}_HJ_{b^-}^\alpha f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\le\frac{1}{2}\left(|f'(b)|-|f'(a)|\right)$$

$$+b|f'(b)|\frac{\ln b-\ln a}{2}$$

$$\times\Bigg\{\frac{a|f'(a)|}{b|f'(b)|}\sum_{i=1}^\infty(-1)^{i-1}\frac{\left[\ln\left(a|f'(a)|/b|f'(b)|\right)\right]^{i-1}}{(\alpha+1)_i}$$

$$+\sum_{i=1}^\infty\frac{\left[\ln\left(a|f'(a)|/b|f'(b)|\right)\right]^{i-1}}{(\alpha+1)_i}$$

$$-\frac{(1/2)^{\alpha-1}\left(a|f'(a)|/b|f'(b)|\right)^{1/2}}{\alpha+1}$$

$$+2\frac{1-(1/2)^\alpha\left(a|f'(a)|/b|f'(b)|\right)^{1/2}}{\ln\left(a|f'(a)|/b|f'(b)|\right)}\Bigg\}. \tag{24}$$

Proof. By using Definition 4 and Lemmas 2, 7, 8, and 9, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\left[{}_HJ_{a^+}^\alpha f(b)+{}_HJ_{b^-}^\alpha f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\le\left|\frac{b-a}{2}\int_0^1 kf'(ta+(1-t)b)\,dt\right|$$

$$+\frac{\ln b-\ln a}{2}\int_0^1 a^tb^{1-t}\left|(1-t)^\alpha-t^\alpha\right|$$

$$\times\left|f'\left(a^tb^{1-t}\right)\right|dt$$

$$\begin{aligned}
&\leq \frac{-1}{2}\left(|f(a)|-|f(b)|\right)\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\int_{1/2}^{1} a^{t}b^{1-t}\left[t^{\alpha}-(1-t)^{\alpha}\right]\left|f'\left(a^{t}b^{1-t}\right)\right|dt\right.\\
&\quad\quad\left.+\int_{0}^{1/2} a^{t}b^{1-t}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left|f'\left(a^{t}b^{1-t}\right)\right|dt\right\}\\
&\leq \frac{-1}{2}\left(|f(a)|-|f(b)|\right)\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\int_{1/2}^{1} a^{t}b^{1-t}\left[t^{\alpha}-(1-t)^{\alpha}\right]\left|f'(a)\right|^{t}\left|f'(b)\right|^{1-t}dt\\
&\quad+\frac{\ln b-\ln a}{2}\\
&\quad\times\int_{0}^{1/2} a^{t}b^{1-t}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left|f'(a)\right|^{t}\left|f'(b)\right|^{1-t}dt\\
&\leq \frac{-1}{2}\left(|f(a)|-|f(b)|\right)\\
&\quad+b\left|f'(b)\right|\frac{\ln b-\ln a}{2}\\
&\quad\times\int_{1/2}^{1}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{t}\left[t^{\alpha}-(1-t)^{\alpha}\right]dt\\
&\quad+b\left|f'(b)\right|\frac{\ln b-\ln a}{2}\\
&\quad\times\int_{0}^{1/2}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{t}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt\\
&\leq \frac{1}{2}\left(|f(b)|-|f(a)|\right)\\
&\quad+b\left|f'(b)\right|\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\int_{0}^{1}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{t}t^{\alpha}dt\right.\\
&\quad\quad-\int_{0}^{1}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{t}(1-t)^{\alpha}dt\\
&\quad\quad-2\int_{0}^{1/2}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{t}t^{\alpha}dt\\
&\quad\quad\left.+2\int_{0}^{1/2}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{t}(1-t)^{\alpha}dt\right\}\\
&\leq \frac{1}{2}\left(|f(b)|-|f(a)|\right)\\
&\quad+b\left|f'(b)\right|\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_{i}}\right.\\
&\quad\quad-\sum_{i=1}^{\infty}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_{i}}\\
&\quad\quad-\left(\frac{1}{2}\right)^{\alpha-1}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{1/2}\\
&\quad\quad\times\sum_{i=1}^{\infty}\frac{\left[-(1/2)\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_{i}}\\
&\quad\quad+2\sum_{i=0}^{\infty}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_{i}}\\
&\quad\quad\left.\times\left[1-\left(\frac{1}{2}\right)^{\alpha+i}\left(\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\right)^{1/2}\right]\right\}\\
&\leq \frac{1}{2}\left(\left|f'(b)\right|-\left|f'(a)\right|\right)\\
&\quad+b\left|f'(b)\right|\frac{\ln b-\ln a}{2}\\
&\quad\times\left\{\frac{a\left|f'(a)\right|}{b\left|f'(b)\right|}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_{i}}\right.\\
&\quad\quad+\sum_{i=1}^{\infty}\frac{\left[\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)\right]^{i-1}}{(\alpha+1)_{i}}\\
&\quad\quad-\frac{(1/2)^{\alpha-1}\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)^{1/2}}{\alpha+1}\\
&\quad\quad\left.+2\frac{1-(1/2)^{\alpha}\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)^{1/2}}{\ln\left(a\left|f'(a)\right|/b\left|f'(b)\right|\right)}\right\}.
\end{aligned}\tag{25}$$

The proof is done. □

Theorem 17. *Let $f:[0,b]\to\mathbb{R}^{+}$ be a differentiable mapping. If $|f'|^{q}$ is measurable and $|f'|^{q}$, $(q>1)$ is GG-convex on $[a,b]$,* *for some fired $\alpha \in (0,\infty)$ and $t \in [0,1]$, $0 \le a < b$, then the following integrals hold:*

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{1}{2}\left(\left|f(b)\right|-\left|f(a)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}$$
$$\times\left(a^p\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[(1/2)\ln(a/b)^p\right]^{i-1}}{(p\alpha+1)_i}\right.$$
$$+b^p\sum_{i=1}^{\infty}\frac{\left[\ln(a/b)^p\right]^{i-1}}{(p\alpha+1)_i}-\frac{(1/2)^{p\alpha-1}\left(ab^{p/2}\right)}{p\alpha+1}$$
$$\left.-\left[\ln\left(\frac{a}{b}\right)^p\right]^{-1}\left[\left(\frac{1}{2}\right)^{p\alpha-1}(ab)^{p/2}+2b^p\right]\right)^{1/p}$$
$$\times\left(\frac{\left|f'(b)\right|^q}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^q}\left[\left|\frac{f'(a)}{f'(b)}\right|^q-1\right]\right)^{1/q}, \tag{26}$$

where $1/p+1/q=1$.

Proof. By using Definition 4 and Lemmas 2, 7, 8, and 9, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{b-a}{2}\int_0^1\left|kf'(ta+(1-t)b)\right|dt$$
$$+\frac{\ln b-\ln a}{2}\int_0^1 a^tb^{1-t}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left(a^tb^{1-t}\right)\right|dt$$
$$\le \frac{b-a}{2}\int_0^1|k|\left|f'(ta+(1-t)b)\right|dt$$
$$+\frac{\ln b-\ln a}{2}\int_0^1 a^tb^{1-t}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left(a^tb^{1-t}\right)\right|dt$$
$$\le \frac{b-a}{2}\int_0^1|k|\left|f'(ta+(1-t)b)\right|dt$$
$$+\frac{\ln b-\ln a}{2}\left(\int_0^1\left(a^tb^{1-t}\left|(1-t)^{\alpha}-t^{\alpha}\right|\right)^p dt\right)^{1/p}$$
$$\times\left(\int_0^1\left|f'\left(a^tb^{1-t}\right)\right|^q dt\right)^{1/q}$$

$$\le \frac{-1}{2}\left(\left|f(a)\right|-\left|f(b)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}$$
$$\times\left(b^p\int_{1/2}^1\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt-b^p\int_{1/2}^1\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt\right.$$
$$\left.+b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt-b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt\right)^{1/p}$$
$$\times\left(\left|f'(b)\right|^q\int_0^1\left[\frac{\left|f'(a)\right|}{\left|f'(b)\right|}\right]^{qt}dt\right)^{1/q}$$
$$\le \frac{-1}{2}\left(\left|f(a)\right|-\left|f(b)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}$$
$$\times\left(b^p\int_0^1\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt-b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt\right.$$
$$-b^p\int_0^1\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt+\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt$$
$$\left.+b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt-b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt\right)^{1/p}$$
$$\times\left(\frac{\left|f'(b)\right|^q}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^q}\left[\left|\frac{f'(a)}{f'(b)}\right|^q-1\right]\right)^{1/q}$$
$$\le \frac{-1}{2}\left(\left|f(a)\right|-\left|f(b)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}$$
$$\times\left(b^p\left(\frac{a}{b}\right)^p\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^p\right]^{i-1}}{(p\alpha+1)_i}\right.$$
$$-b^p\sum_{i=0}^{\infty}\frac{\left[\ln(a/b)^p\right]^{i-1}}{(p\alpha+1)_i}-2b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}t^{p\alpha}dt$$
$$\left.+2b^p\int_0^{1/2}\left(\frac{a}{b}\right)^{pt}(1-t)^{p\alpha}dt\right)^{1/p}$$
$$\times\left(\frac{\left|f'(b)\right|^q}{\ln\left(\left|f'(a)\right|/\left|f'(b)\right|\right)^q}\left[\left|\frac{f'(a)}{f'(b)}\right|^q-1\right]\right)^{1/q}$$
$$\le \frac{-1}{2}\left(\left|f(a)\right|-\left|f(b)\right|\right)$$
$$+\frac{\ln b-\ln a}{2}$$
$$\times\left(b^p\left(\frac{a}{b}\right)^p\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left[\ln(a/b)^p\right]^{i-1}}{(p\alpha+1)_i}\right.$$

$$
\begin{aligned}
&- b^p \sum_{i=1}^{\infty} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&- \left(\frac{1}{2}\right)^{p\alpha} b^p \left(\frac{a}{b}\right)^{p/2} \sum_{i=1}^{\infty} \frac{\left[-(1/2)\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&+ 2b^p \sum_{i=0}^{\infty} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \left[1 - \left(\frac{a}{b}\right)^{p/2} \left(1-\frac{1}{2}\right)^{p\alpha+i}\right]\Bigg)^{1/p} \\
&\times \left(\frac{\left|f'(b)\right|^q}{\ln \left(\left|f'(a)\right| / \left|f'(b)\right|\right)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q} \\
\leq & \frac{-1}{2} \left(\left|f(a)\right| - \left|f(b)\right|\right) \\
&+ \frac{\ln b - \ln a}{2} \\
&\times \Bigg(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} - b^p \sum_{i=1}^{\infty} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&- \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{\left[-(1/2)\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&+ 2b^p \sum_{i=0}^{\infty} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&- 2b^p \left(\frac{a}{b}\right)^{p/2} \sum_{i=0}^{\infty} \left(1-\frac{1}{2}\right)^{p\alpha+i} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i}\Bigg)^{1/p} \\
&\times \left(\frac{\left|f'(b)\right|^q}{\ln \left(\left|f'(a)\right| / \left|f'(b)\right|\right)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q} \\
\leq & \frac{-1}{2} \left(\left|f(a)\right| - \left|f(b)\right|\right) \\
&+ \frac{\ln b - \ln a}{2} \\
&\times \Bigg(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} - b^p \sum_{i=1}^{\infty} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&- \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{\left[-(1/2)\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&+ 2b^p \sum_{i=1}^{\infty} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} + 2b^p \left[\ln \left(\frac{a}{b}\right)^p\right]^{-1} \\
&- 2b^p \left(\frac{a}{b}\right)^{p/2} \sum_{i=0}^{\infty} \left(1-\frac{1}{2}\right)^{p\alpha+i} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i}\Bigg)^{1/p} \\
&\times \left(\frac{\left|f'(b)\right|^q}{\ln \left(\left|f'(a)\right| / \left|f'(b)\right|\right)^q} \left[\left|f'(a)/f'(b)\right|^q - 1\right]\right)^{1/q} \\
\leq & \frac{1}{2} \left(\left|f(b)\right| - \left|f(a)\right|\right) \\
&+ \frac{\ln b - \ln a}{2} \\
&\times \Bigg(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} + b^p \sum_{i=1}^{\infty} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&- \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{\left[-(1/2)\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&- \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{\left[(1/2)\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&- \left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} \left[\ln \left(\frac{a}{b}\right)^p\right]^{-1} \\
&+ 2b^p \left[\ln \left(\frac{a}{b}\right)^p\right]^{-1}\Bigg)^{1/p} \\
&\times \left(\frac{\left|f'(b)\right|^q}{\ln \left(\left|f'(a)\right| / \left|f'(b)\right|\right)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q} \\
\leq & \frac{1}{2} \left(\left|f(b)\right| - \left|f(a)\right|\right) \\
&+ \frac{\ln b - \ln a}{2} \\
&\times \Bigg(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\left[(1/2)\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} \\
&+ b^p \sum_{i=1}^{\infty} \frac{\left[\ln (a/b)^p\right]^{i-1}}{(p\alpha+1)_i} - \frac{(1/2)^{p\alpha-1}\left(ab^{p/2}\right)}{p\alpha+1} \\
&- \left[\ln \left(\frac{a}{b}\right)^p\right]^{-1} \left[\left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} + 2b^p\right]\Bigg)^{1/p} \\
&\times \left(\frac{\left|f'(b)\right|^q}{\ln \left(\left|f'(a)\right| / \left|f'(b)\right|\right)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q}.
\end{aligned} \tag{27}
$$

The proof is done. □

4. Applications to Special Means

Consider the following special means (see [23]) for arbitrary real numbers $x, y, x \neq y$ as follows:

(M_1) $A(x, y) = (x + y)/2$, $x, y \in \mathbb{R}$;

(M_2) $H(x, y) = 2/(1/x + 1/y)$, $x, y \in \mathbb{R} \setminus \{0\}$;

(M_3) $G(x, y) = \sqrt{xy}$;

(M_4) $L(x, y) = (y - x)/(\ln|y| - \ln|x|)$, $|x| \neq |y|$, $xy \neq 0$;

(M_5) $L_n(x,y) = [(y^{n+1}-x^{n+1})/(n+1)(y-x)]^{1/n}, n \in \mathbb{Z} \setminus \{-1,0\}, x, y \in \mathbb{R}, x \neq y$.

We give some applications to special means of real numbers.

Proposition 18. *Let $a, b \in \mathbb{R}^+ \setminus \{0\}$, $0 \le a < b$, $x \in [0,b]$. Then,*

$$|A(a,b) - L(a,b)| \le \frac{\ln b - \ln a}{2}\frac{63a-b}{192};$$

$$|A(a,b) - L(a,b)| \le \frac{\ln b - \ln a}{2}\left(\frac{1}{2}\right)^{1/q} \times \left(\frac{a^p-2b^p}{p+2} + \frac{b^p}{p+1} - \frac{2b^p+a^p}{2^{p+1}(p+1)} - \frac{b^p-a^p}{2^{p+2}(p+2)} + \frac{a^p}{(p+1)(p+2)} - \frac{a^p}{2^{p+1}(p+1)(p+2)}\right)^{1/p};$$

$$|L(a,b) - A(a,b)| \le \frac{\ln b - \ln a}{2}\frac{63a-b}{192};$$

$$|L(a,b) - A(a,b)| \le \frac{\ln b - \ln a}{2}\left(\frac{1}{2}\right)^{1/q} \times \left(\frac{a^p-2b^p}{p+2} + \frac{b^p}{p+1} - \frac{2b^p+a^p}{2^{p+1}(p+1)} - \frac{b^p-a^p}{2^{p+2}(p+2)} + \frac{a^p}{(p+1)(p+2)} - \frac{a^p}{2^{p+1}(p+1)(p+2)}\right)^{1/p};$$

$$|A(a,b) - L(a,b)| \le b\frac{\ln b - \ln a}{2} \times \left[\frac{a/b-1}{\ln(a/b)} + 2\frac{(a/b)^{1/2} - a/b + (a/b)^2 - 1}{(\ln(a/b))^2}\right]. \tag{28}$$

By using Lemmas 7, 8, and 9, we have

$$|A(a,b) - L(a,b)| \le b\frac{\ln b - \ln a}{2} \times \left(I[p+1] - H\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) - 2J\left(p+1, \left(\frac{a}{b}\right)^p\right) + 2bR\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right)\right)^{1/p};$$

$$|L(a,b) - A(a,b)| \le b\frac{\ln b - \ln a}{2} \times \left[\frac{a/b-1}{\ln(a/b)} + 2\frac{(a/b)^{1/2} - a/b + (a/b)^2 - 1}{(\ln(a/b))^2}\right];$$

$$|L(a,b) - A(a,b)| \le b\frac{\ln b - \ln a}{2} \times \left(I[p+1] - H\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) - 2J\left(p+1, \left(\frac{a}{b}\right)^p\right) + 2bR\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right)\right)^{1/p}. \tag{29}$$

Proof. Applying Theorems 10, 11, 14, 15, 12, 13, 16, and 17, for $f(x) = x$ and $\alpha = 1$, one can obtain the results immediately. □

Proposition 19. *Let $a, b \in \mathbb{R}^+ \setminus \{0\}$, $0 \le a < b$, $x \in [0,b]$, $n \ge 2$. Then,*

$$\left|A(a^n, b^n) - L_{n-1}^{n-1}(a,b)L(a,b)\right| \le \frac{\ln b - \ln a}{2}\frac{22na^n + 12nba^{n-1} + 9nab^{n-1} - 12nb^n}{96};$$

$$\left|A(a^n, b^n) - L_{n-1}^{n-1}(a,b)L(a,b)\right| \le \frac{\ln b - \ln a}{2}\left(\frac{n^q a^{q(n-1)} + n^q b^{q(n-1)}}{2}\right)^{1/q} \times \left(\frac{a^p-2b^p}{p+2} + \frac{b^p}{p+1} - \frac{1}{2^{p+1}}\frac{b^p}{p+1} + \frac{1}{2^{p+2}}\frac{b^p-a^p}{p+2} - \frac{1}{2^{p+3}}\frac{1}{p+3} - \frac{a^p}{p(p+1)} + \frac{2a^p}{(p+1)(p+2)}\right)^{1/p};$$

$$\left|L_{n-1}^{n-1}(a,b)L(a,b) - A^n(x,y)\right| \le \frac{nb^{n-1} - na^{n-1}}{2} + \frac{\ln b - \ln a}{2}\frac{22na^n + 12nba^{n-1} + 9nab^{n-1} - 12nb^n}{96};$$

$$\left|L_{n-1}^{n-1}(a,b)L(a,b)-A^n(x,y)\right|$$
$$\leq \frac{nb^{n-1}-na^{n-1}}{2}$$
$$+\frac{\ln b-\ln a}{2}\left(\frac{n^q a^{q(n-1)}+n^q b^{q(n-1)}}{2}\right)^{1/q}$$
$$\times\left(\frac{a^p-2b^p}{p+2}+\frac{b^p}{p+1}-\frac{1}{2^{p+1}}\frac{b^p}{p+1}\right.$$
$$+\frac{1}{2^{p+2}}\frac{b^p-a^p}{p+2}-\frac{1}{2^{p+3}}\frac{1}{p+3}$$
$$\left.-\frac{a^p}{p(p+1)}+\frac{2a^p}{(p+1)(p+2)}\right)^{1/p};$$
$$\left|A(a^n,b^n)-L_{n-1}^{n-1}(a,b)L(a,b)\right|$$
$$\leq nb^n\frac{\ln b-\ln a}{2}$$
$$\times\left(\frac{a^n/b^n-1}{\ln(a^n/b^n)}+\frac{4(a^n/b^n)-2(a^n/b^n)-2}{(\ln(a^n/b^n))^2}\right). \tag{30}$$

By using Lemmas 7, 8, and 9, we have

$$\left|A(a^n,b^n)-L_{n-1}^{n-1}(a,b)L(a,b)\right|$$
$$\leq b\frac{\ln b-\ln a}{2}\left(I[p+1]-H\left(p+1,\left(\frac{a}{b}\right)^p,\frac{1}{2}\right)\right.$$
$$-2J\left(p+1,\left(\frac{a}{b}\right)^p\right)$$
$$\left.+2bR\left(p+1,\left(\frac{a}{b}\right)^p,\frac{1}{2}\right)\right)^{1/p}$$
$$\times\left(\frac{\left|nb^{n-1}\right|^q}{\ln(a^{n-1}/b^{n-1})^q}\left[\left(\frac{a^{n-1}}{b^{n-1}}\right)^q-1\right]\right)^{1/q};$$
$$\left|L_{n-1}^{n-1}(a,b)L(a,b)-A^n(x,y)\right|$$
$$\leq \frac{nb^{n-1}-na^{n-1}}{2}$$
$$+nb^n\frac{\ln b-\ln a}{2}$$
$$\times\left(\frac{a^n/b^n-1}{\ln(a^n/b^n)}+\frac{4(a^n/b^n)-2(a^n/b^n)-2}{(\ln(a^n/b^n))^2}\right);$$
$$\left|L_{n-1}^{n-1}(a,b)L(a,b)-A^n(x,y)\right|$$
$$\leq \frac{nb^{n-1}-na^{n-1}}{2}$$
$$+b\frac{\ln b-\ln a}{2}\left(I[p+1]-H\left(p+1,\left(\frac{a}{b}\right)^p,\frac{1}{2}\right)\right.$$
$$-2J\left(p+1,\left(\frac{a}{b}\right)^p\right)$$
$$\left.+2bR\left(p+1,\left(\frac{a}{b}\right)^p,\frac{1}{2}\right)\right)^{1/p}$$
$$\times\left(\frac{\left|nb^{n-1}\right|^q}{\ln(a^{n-1}/b^{n-1})^q}\left[\left(\frac{a^{n-1}}{b^{n-1}}\right)^q-1\right]\right)^{1/q}. \tag{31}$$

Proof. Applying Theorems 10, 11, 14, 15, 12, 13, 16, and 17 for $f(x)=x^n$ and $\alpha=1$, one can obtain the results immediately. □

Proposition 20. *Let $a,b\in\mathbb{R}^+\setminus\{0\}$, $0\leq a<b$, $x\in[0,b]$, $n\geq 2$. Then,*

$$\left|H^{-1}(a,b)-G^{-2}(a,b)L(a,b)\right|$$
$$\leq\frac{\ln b-\ln a}{2}$$
$$\times\left(450(1/a)+60\left(a/b^2\right)-30\left(b/a^2\right)\right.$$
$$\left.-32(1/a)+42(1/a)\right)(1920)^{-1};$$
$$\left|H^{-1}(a,b)-G^{-2}(a,b)L(a,b)\right|$$
$$\leq\frac{\ln b-\ln a}{2}\left(\frac{a^{-2q}+b^{-2q}}{2}\right)^{1/q}$$
$$\times\left(\frac{a^p-2b^p}{p+2}+\frac{b^p}{p+1}-\frac{1}{2^{p+2}}\frac{2a^p+3b^p}{p+1}\right.$$
$$+\frac{1}{2^{p+2}}\frac{2b^p-a^p}{p+2}-\frac{1}{p(p+1)}$$
$$\left.-\frac{2a^p+b^p}{2^{p+2}(p+1)(p+2)}\right)^{1/p};$$
$$\left|G^{-2}(a,b)L(a,b)-A^{-1}(a,b)\right|$$
$$\leq\frac{b^2-a^2}{2a^2b^2}+\frac{\ln b-\ln a}{2}$$
$$\times\left(450(1/a)+60\left(a/b^2\right)-30\left(b/a^2\right)\right.$$
$$\left.-32(1/a)+42(1/a)\right)(1920)^{-1};$$

$$\begin{aligned}&\left|G^{-2}(a,b)L(a,b)-A^{-1}(a,b)\right|\\&\quad\le\frac{b^2-a^2}{2a^2b^2}\\&\qquad+\frac{\ln b-\ln a}{2}\left(\frac{a^{-2q}+b^{-2q}}{2}\right)^{1/q}\\&\qquad\times\left(\frac{a^p-2b^p}{p+2}+\frac{b^p}{p+1}-\frac{1}{2^{p+2}}\frac{2a^p+3b^p}{p+1}\right.\\&\qquad\quad+\frac{1}{2^{p+2}}\frac{2b^p-a^p}{p+2}-\frac{1}{p(p+1)}\\&\qquad\quad\left.-\frac{2a^p+b^p}{2^{p+2}(p+1)(p+2)}\right)^{1/p};\\&\left|H^{-1}(a,b)-G^{-2}(a,b)L(a,b)\right|\\&\quad\le\frac{\ln b-\ln a}{2b}\left(\frac{b/a-1}{\ln(b/a)}+\frac{4(b/a)^{1/2}-2(b/a)-2}{(\ln(b/a))^2}\right).\end{aligned}\tag{32}$$

By using Lemmas 7, 8, and 9, we have

$$\begin{aligned}&\left|H^{-1}(a,b)-G^{-2}(a,b)L(a,b)\right|\\&\quad\le b\frac{\ln b-\ln a}{2}\left(I[p+1]-H\left(p+1,\left(\frac{a}{b}\right)^p,\frac{1}{2}\right)\right.\\&\qquad\quad-2J\left(p+1,\left(\frac{a}{b}\right)^p\right)\\&\qquad\quad\left.+2bR\left(p+1,\left(\frac{a}{b}\right)^p,\frac{1}{2}\right)\right)^{1/p}\\&\qquad\times\left(\frac{\left|1/b^2\right|^q}{\ln\left|b^2/a^2\right|^q}\left[\left|\frac{b^2}{a^2}\right|^q-1\right]\right)^{1/q};\\&\left|G^{-2}(a,b)L(a,b)-A^{-1}(a,b)\right|\\&\quad\le\frac{b^2-a^2}{2a^2b^2}\\&\qquad+\frac{\ln b-\ln a}{2b}\left(\frac{b/a-1}{\ln(b/a)}+\frac{4(b/a)^{1/2}-2(b/a)-2}{(\ln(b/a))^2}\right);\\&\left|G^{-2}(a,b)L(a,b)-A^{-1}(a,b)\right|\\&\quad\le\frac{b^2-a^2}{2a^2b^2}\\&\qquad+b\frac{\ln b-\ln a}{2}\left(I[p+1]-H\left(p+1,\left(\frac{a}{b}\right)^p,\frac{1}{2}\right)\right.\\&\qquad\quad-2J\left(p+1,\left(\frac{a}{b}\right)^p\right)\\&\qquad\quad\left.+2bR\left(p+1,\left(\frac{a}{b}\right)^p,\frac{1}{2}\right)\right)^{1/p}\\&\qquad\times\left(\frac{\left|1/b^2\right|^q}{\ln\left|b^2/a^2\right|^q}\left[\left|b^2/a^2\right|^q-1\right]\right)^{1/q}.\end{aligned}\tag{33}$$

Proof. Applying Theorems 10, 11, 14, 15, 12, 13, 16, and 17 for $f(x)=1/x$ and $\alpha=1$, one can obtain the results immediately. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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