Internat. J. Math. & Math. Sci.
Vol. 8 No. 1 (1985) 109-112

ON THE SPECTRUM OF WEAKLY ALMOST PERIODIC SOLUTIONS OF CERTAIN ABSTRACT DIFFERENTIAL EQUATIONS

ARIBINDI SATYANARAYAN RAO
Department of Mathematics
Sir George Williams Campus
Concordia University
Montreal, Quebec, Canada

and

L.S. DUBE
Department of Mathematics
Vanier College
821 Ste-Mis Croix Blvd.
St.-Laurent
Quebec, H4L 3x9, Canada

(Received October 15, 1984)

ABSTRACT. In a sequentially weakly complete Banach space, if the dual operator of a linear operator A satisfies certain conditions, then the spectrum of any weakly almost periodic solution of the differential equation $u' = Au + f$ is identical with the spectrum of f except at the origin, where f is a weakly almost periodic function.

KEY WORDS AND PHRASES. *Strongly (weakly) almost periodic function, sequentially weakly complete Banach space, densely defined linear operator, dual operator, Hilbert space, nonnegative self-adjoint operator.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 34C25, 34G05; 43A60

1. INTRODUCTION.

Suppose X is a Banach space and X^* is the dual space of X. Let J be the interval $-\infty < t < \infty$. A continuous function $f : J \to X$ is said to be strongly almost periodic if, given $\varepsilon > 0$, there is a positive real number $\ell = \ell(\varepsilon)$ such that any interval of the real line of length ℓ contains at least one point τ for which

$$\sup_{t \in J} \|f(t+\tau)-f(t)\| \leq \varepsilon. \tag{1.1}$$

We say that a function $f : J \to X$ is weakly almost periodic if the scalar-valued function $\langle x^*, f(t)\rangle = x^*f(t)$ is almost periodic for each $x^* \in X^*$.

It is known that, if X is sequentially weakly complete, $f : J \to X$ is weakly almost periodic, and λ is a real number, then the weak limit

$$m(e^{-i\lambda t}f(t)) = \underset{T\to\infty}{w\text{-}\lim} \frac{1}{T}\int_0^T e^{-i\lambda t}f(t)dt \tag{1.2}$$

exists in X and is different from the null element Θ of X for at most a countable set $\{\lambda_n\}_{n=1}^{\infty}$, called the spectrum of $f(t)$ (see Theorem 6, p. 43, Amerio-Prouse [1]). We denote by $\sigma(f(t))$ the spectrum of $f(t)$.

2. RESULTS

Our first result is as follows (see Theorem 9, p. 79, Amerio-Prouse [1] for the spectrum of an S^1-almost periodic function).

THEOREM 1. Suppose X is a sequentially weakly complete Banach space, A is a densely defined linear operator with domain $D(A)$ and range $R(A)$ in X, and the dual operator A^* is densely defined in X^*, with $R(i\lambda - A^*)$ being dense in X^* for all real $\lambda \neq 0$. Further, suppose $f : J \to X$ is a weakly almost periodic (or an S^1-almost periodic continuous) function. If a differentiable function $u: J \to D(A)$ is a weakly almost periodic solution of the differential equation

$$u'(t) = Au(t) + f(t) \tag{1.3}$$

on J, with u' being weakly continuous on J, then $\sigma(u(t)) \setminus \{0\} = \sigma(f(t)) \setminus \{0\}$.

PROOF OF THEOREM 1. First we note that u is bounded on J, since u is weakly almost periodic. Hence, for $x^* \in X^*$, we have

$$\frac{1}{T}\int_0^T e^{-i\lambda t}x^*u'(t)dt = x^* \frac{1}{T} \{ [e^{-i\lambda t}u(t)]_0^T + \frac{i\lambda}{T}\int_0^T e^{-i\lambda t}u(t)dt\} \tag{2.1}$$

$$\to i\lambda x^*m(e^{-i\lambda t}u(t)) \text{ as } T \to \infty .$$

So, for $x^* \in D(A^*)$, it follows from (1.3) that

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-\lambda t}x^*Au(t)dt &= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda t}(A^*x^*)u(t)dt \\ &= \lim_{T\to\infty} (A^*x^*) \left[\frac{1}{T}\int_0^T e^{-i\lambda t}u(t)dt \right] \\ &= (A^*x^*)m(e^{-i\lambda t}u(t)) \\ &= i\lambda x^*m(e^{-i\lambda t}u(t)) - x^*m(e^{-i\lambda t}f(t)).\end{aligned} \tag{2.2}$$

Consequently, we have

$$x^*m(e^{-i\lambda t}f(t)) = (i\lambda x^* - A^*x^*)m(e^{-i\lambda t}u(t)). \tag{2.3}$$

Now suppose that $\lambda \in \sigma(f(t)) \setminus \{0\}$. Then, since $D(A^*)$ is dense in X^*, there exists $x_1^* \in D(A^*)$ such that

$$0 \neq x_1^*m(e^{-\lambda t}f(t)) = (-\lambda x_1^* - A^*x_1^*)m(e^{-\lambda t}u(t)). \tag{2.4}$$

Therefore $m(e^{-i\lambda t}u(t)) \neq 0$ and so $\lambda \in \sigma(u(t)) \setminus \{0\}$.

Thus we have

$$\sigma(f(t)) \setminus \{0\} \subseteq \sigma(u(t)) \setminus \{0\}. \tag{2.5}$$

Now assume that $\lambda \in \sigma(u(t) \setminus \{0\}$. Then, since $R(i\lambda - A^*)$ is dense in X^*, there exists $x_2^* \in D(A^*)$ such that

$$0 \neq (-i\lambda x_2^* - A^* x_2^*)m(e^{-i\lambda t}u(t)) = x_2^* m(e^{-i\lambda t}f(t)). \tag{2.6}$$

Therefore $m(e^{-i\lambda t}f(t)) \neq 0$ and so $\lambda \in \sigma(f(t)) \setminus \{0\}$.

Consequently, we have

$$\sigma(u(t)) \setminus \{0\} \subseteq \sigma(f(t)) \setminus \{0\}. \tag{2.7}$$

It follows from (2.5) and (2.7) that $\sigma(u(t)) \setminus \{0\} = \sigma(f(t)) \setminus \{0\}$, which completes the proof of the theorem.

REMARK 1. The conclusion of Theorem 1 remains valid if $D(A^*)$ is total and $R(i\lambda - A^*)$ is total for all real $\lambda \neq 0$, instead of dense in X^*.
We indicate the proof of the following result.

THEOREM 2. In a sequentially weakly complete Banach space X, suppose A is a densely defined linear operator, the dual operator A^* is densely defined in X^*, with $R(\lambda^2 + A^*)$ being dense in X^* for all real $\lambda \neq 0$, and $f : J \to X$ is a weakly almost periodic (or an S^1-almost periodic continuous) function. If a twice differentiable function $u : J \to D(A)$ is a weakly almost periodic solution of the differential equation

$$u''(t) = Au(t) + f(t) \tag{3.1}$$

on J, with u'' being weakly continuous and u' bounded on J, then

$$\sigma(u(t)) \setminus \{0\} = \sigma(f(t)) \setminus \{0\}.$$

PROOF. For $x^* \in D(A^*)$, we have

$$\frac{1}{T}\int_0^T e^{-i\lambda t}x^*u''(t)dt = x^* \{\frac{1}{T}[e^{-i\lambda t}u'(t)]_0^T + \frac{i\lambda}{T}\int_0^T e^{-i\lambda t}u'(t)dt\}$$

$$= x^* \{\frac{1}{T}[e^{-i\lambda t}u'(t)]_0^T + \frac{i\lambda}{T}[e^{-i\lambda t}u(t)]_0^T - \frac{\lambda^2}{T}\int_0^T e^{-i\lambda t}u(t)dt\}$$

$$\to -\lambda^2 x^* m(e^{-i\lambda t}u(t)) \text{ as } T \to \infty. \tag{3.2}$$

Hence it follows from (3.1) that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda t}x^*Au(t)dt = (A^*x^*)m(e^{-i\lambda t}u(t))$$

$$= -\lambda^2 x^* m(e^{-i\lambda t}u(t)) - x^* m(e^{-i\lambda t}f(t)). \tag{3.3}$$

Thus we have

$$-x^* m(e^{-i\lambda t}f(t)) = (\lambda^2 x^* + A^*x^*)m(e^{-i\lambda t}u(t)). \tag{3.4}$$

Now the rest of the proof parallels that of Theorem 1.

REMARK 2. The conclusion of Theorem 2 also remains valid if $D(A^*)$ is total and $R(\lambda^2 + A^*)$ is total for all real $\lambda \neq 0$, instead of dense in X^*.

REMARK 3. If X is a Hilbert space and A is a nonnegative self-adjoint operator, then the hypotheses on A in Theorem 2 are verified (see Corollary 2, p. 208, Yosida [2]) and so Theorem 2 is a generalization of a result of Zaidman [3].

NOTE. As a consequence of our Theorem 1, we have the following result:

THEOREM 3. In a Hilbert space H, suppose A is a self-adjoint operator and $f : J \to H$ is a weakly almost periodic (or an S^1-almost periodic continuous) function. If a differentiable function $u : J \to D(A)$ is a weakly almost periodic solution of the differential equation

$$u'(t) = Au(t) + f(t)$$

on J, with u' being weakly continuous on J, then

$$\sigma(u(t)) \setminus \{0\} = \sigma(f(t)) \setminus \{0\}.$$

PROOF. By Example 4, p. 210, Yosida [2], $R(i\lambda - A) = H$ for all real $\lambda \neq 0$.

ACKNOWLEDGEMENT. This work was supported by the National Research Council of Canada Grant Nos. 4056 and A-9085.

REFERENCES

1. AMERIO, L. and PROUSE, G. Almost Periodic Functions and Functional Equations, Van Nostrand Reinhold Company (1971).

2. YOSIDA, K. Functional Analysis, Springer-Verlag New York Inc. (1971).

3. ZAIDMAN, S. Spectrum of Almost Periodic Solutions for some abstract Differential Equations, Math. Anal. Appl., 28 (1969), pp. 336-338