Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 896591, 20 pages
http://dx.doi.org/10.1155/2014/896591

Research Article

Stochastic Separated Continuous Conic Programming: Strong Duality and a Solution Method

Xiaoqing Wang

Lingnan (University) College, Sun Yat-sen University, Guangzhou, Guangdong 510275, China

Correspondence should be addressed to Xiaoqing Wang; xqwang2@gmail.com

Received 3 November 2013; Accepted 29 November 2013; Published 9 January 2014

Academic Editor: Dongdong Ge

We study a new class of optimization problems called *stochastic separated continuous conic programming* (SSCCP). SSCCP is an extension to the optimization model called *separated continuous conic programming* (SCCP) which has applications in robust optimization and sign-constrained linear-quadratic control. Based on the relationship among SSCCP, its dual, and their discretization counterparts, we develop a strong duality theory for the SSCCP. We also suggest a polynomial-time approximation algorithm that solves the SSCCP to any predefined accuracy.

1. Introduction

Stochastic programming is one of the branches of optimization which enjoys a fast development in recent years. It tries to find optimal decisions in problems involving uncertain data, so it is also called "optimization under uncertainty" [1]. Since the problems in reality often involve uncertain data, stochastic programming has a lot of applications.

Many deterministic optimization models have their stochastic counterpart; for example, the stochastic counterpart of linear programming is stochastic linear programming. In this paper, we consider the stochastic counterpart of a kind of optimization model called *separated continuous conic programming* (SCCP) which has the following form:

$$\begin{aligned}
\text{(SCCP)}\quad \max\quad & \int_0^T \left[(\gamma + (T-t)c)' u(t) + d' x(t)\right] dt \\
\text{s.t.}\quad & \alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \in \mathcal{K}_1, \\
& b - Hu(t) \in \mathcal{K}_2, \\
& u(t) \in \mathcal{K}_3, \quad x(t) \in \mathcal{K}_4, \quad t \in [0,T].
\end{aligned} \tag{1}$$

Here the control and state variables (both are decision variables), $u(t)$ and $x(t)$, are vectors of bounded measurable functions of time $t \in [0,T]$. $\mathcal{K}_i$, $i = 1,2,3,4$, are closed convex cones in the Euclidean space with appropriate dimensions, $\gamma, c, d, \alpha, a, b$ are vectors, G, F, H are matrices, and the superscript $'$ denotes the transpose operation.

SCCP was first studied by Wang et al. [2]. They developed a strong duality theory for SCCP under some mild and verifiable conditions and suggested an approximation algorithm to solve SCCP with predefined precision. SCCP has a variety of applications in robust optimization and sign-constrained linear-quadratic control. However, many applications of SCCP are stochastic in nature in the sense that the values of some parameters in the resulted SCCP models may change over time with some probability distribution. To incorporate this kind of randomness into the SCCP model, we introduce the following stochastic counterpart of SCCP which we call *stochastic separated continuous conic programming* (SSCCP) problem:

$$\begin{aligned}
\max\quad & \int_0^{T_1} \left[(\gamma + (T-t)c)' u(t) + d' x(t)\right] dt \\
& + \mathbf{E}_\xi \left(\int_{T_1}^{T_2} \left[(\gamma(\xi) + (T-t)c(\xi))' u(t) + d(\xi)' x(t)\right] dt \right) \\
\text{s.t.}\quad & \alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \in \mathcal{K}_1, \quad t \in [0, T_1],
\end{aligned}$$

$$
\begin{aligned}
& b - Hu(t) \in \mathcal{K}_2, \quad t \in [0, T_1], \\
& u(t) \in \mathcal{K}_3, \quad x(t) \in \mathcal{K}_4, \quad t \in [0, T_1], \\
& \alpha + T_1 a + (t - T_1) a(\xi) - \int_0^{T_1} Gu(s)\, ds \\
& \quad - \int_{T_1}^{t} Gu(s)\, ds - Fx(t) \in \mathcal{K}(\xi)_1, \\
& \qquad\qquad t \in [T_1, T_2], \\
& b(\xi) - Hu(t) \in \mathcal{K}(\xi)_2, \quad t \in [T_1, T_2], \\
& u(t) \in \mathcal{K}(\xi)_3, \quad x(t) \in \mathcal{K}(\xi)_4, \quad t \in [T_1, T_2],
\end{aligned} \tag{2}
$$

where ξ is a random variable.

SSCCP is formulated with the similar idea as that of the stochastic linear programming [1, 3]. There are two stages in this problem; the values of some parameters in the second stage depend on the value of a random variable ξ.

Our goal in this paper is developing the strong duality for SSCCP and suggesting a solution method to solve it approximately with predefined precision. Here is a summary of our main results. Through discretization, we connect SSCCP and its dual to two ordinary conic programs, and we show that strong duality holds for SSCCP and its dual under some mild (and verifiable) conditions on these two ordinary conic programs. Furthermore, the optimal values of those two conic programs provide an explicit bound on the duality gap between SSCCP and its dual, based on which we suggest a polynomial-time approximation algorithm that solves SSCCP to any predefined accuracy. According to our knowledge, we are the first to raise the SSCCP model and there have been no other results on SSCCP besides those in this paper.

The paper is organized as follows. In Section 2, we present an overview on the related literature. We also give a concrete example to show the application of SSCCP. In Section 3, we construct a dual for SSCCP. We also discretize SSCCP and its dual into two ordinary conic programs, and bring out their relations. In Section 4, we discuss the strong feasibility for SSCCP, its dual, and their discretizations. We then establish the strong duality result for SSCCP and its dual in Section 5. This leads to a polynomial-time approximation algorithm with an explicit error bound, detailed in Section 6. In Section 7, we summarize what we get for SSCCP and point out some future research directions.

For simpler presentation, in the remainder of this paper, we will concentrate on the following problem, which is the corresponding SSCCP when ξ is a discrete variable and only takes two different values with probability θ and $1-\theta$, that is, there are only two scenarios in the second stage of SSCCP:

$$
\begin{aligned}
\max \quad & \int_0^{T_1} \left[(\gamma + (T - t) c)' u(t) + d' x(t) \right] dt \\
& + \int_{T_1}^{T} \left[(\breve{\gamma}_1 + (T - t) \breve{c}_1)' v(t) + \breve{d}_1' y(t) \right] dt \\
& + \int_{T_1}^{T} \left[(\breve{\gamma}_2 + (T - t) \breve{c}_2)' w(t) + \breve{d}_2' z(t) \right] dt \\
\text{s.t.} \quad & \alpha + ta - \int_0^{t} Gu(s)\, ds - Fx(t) \in \mathcal{K}_1, \quad t \in [0, T_1], \\
& b - Hu(t) \in \mathcal{K}_2, \quad t \in [0, T_1], \\
& u(t) \in \mathcal{K}_3, \quad x(t) \in \mathcal{K}_4, \quad t \in [0, T_1], \\
& \alpha_1 + ta_1 - \int_0^{T_1} Gu(s)\, ds \\
& \quad - \int_{T_1}^{t} Gv(s)\, ds - Fy(t) \in \mathcal{K}_{11}, \quad t \in (T_1, T], \\
& b_1 - Hv(t) \in \mathcal{K}_{21}, \quad t \in (T_1, T], \\
& v(t) \in \mathcal{K}_{31}, \quad y(t) \in \mathcal{K}_{41}, \quad t \in (T_1, T], \\
& \alpha_2 + ta_2 - \int_0^{T_1} Gu(s)\, ds \\
& \quad - \int_{T_1}^{t} Gw(s)\, ds - Fz(t) \in \mathcal{K}_{12}, \\
& \qquad\qquad t \in (T_1, T], \\
& b_2 - Hw(t) \in \mathcal{K}_{22}, \quad t \in (T_1, T], \\
& w(t) \in \mathcal{K}_{32}, \quad z(t) \in \mathcal{K}_{42}, \quad t \in (T_1, T],
\end{aligned} \tag{3}
$$

where the first-stage control and state variables are $u(t)$ and $x(t)$, $t \in [0, T_1]$, and the second-stage control and state variables are $v(t), w(t)$, $y(t)$, and $z(t)$, $t \in (T_1, T]$. Also $\alpha_1 = \alpha + T_1 a - T_1 a_1$, $\alpha_2 = \alpha + T_1 a - T_1 a_2$, $\breve{\gamma}_1 = \theta\gamma_1$, $\breve{c}_1 = \theta c_1$, $\breve{d}_1 = \theta d_1$, $\breve{\gamma}_2 = (1-\theta)\gamma_2$, $\breve{c}_2 = (1-\theta)c_2$, $\breve{d}_2 = (1-\theta)d_2$.

Note that although (3) is a deterministic optimization problem, it is *not* an SCCP. To see why this is the case, one can try to formulate (3) into the form of SCCP and it then becomes clear that (3) cannot fit into the SCCP form.

In the rest of this paper, we will use some results on conic programming without explanations. Interested readers can consult the books on conic programming (e.g., [4]) for the related results.

2. Literature Review

Bellman [5, 6] first introduced the so-called *continuous linear programming* (CLP), which has the following form:

$$
\begin{aligned}
\text{(CLP)} \quad \max \quad & \int_0^{T} c(t)' x(t)\, dt \\
\text{s.t.} \quad & Bx(t) - \int_0^{t} Kx(s)\, ds \le b(t) \\
& x(t) \ge 0, \quad t \in [0, T].
\end{aligned} \tag{4}
$$

Here $x(t)$ is a decision variable. The model has wide-ranging applications (e.g., the bottleneck problem [5]). But CLP is very difficult to solve in its general form. Later, Anderson [7] introduced *separated continuous linear programming* (SCLP) (see (5)), a special case of CLP, to model the job-shop scheduling problems:

$$\begin{aligned}
\text{(SCLP)}\quad & \max \quad \int_0^T \left[(\gamma + (T-t)c)' u(t) + d' x(t) \right] dt \\
& \text{s.t.} \quad \int_0^t G u(s)\, ds + F x(t) \le \alpha + t a, \\
& \qquad Hu(t) \le b, \\
& \qquad u(t) \ge 0, \quad x(t) \ge 0, \quad t \in [0,T].
\end{aligned} \tag{5}$$

The word "separated" refers to the fact that there are two kinds of constraints in SCLP: the constraints involving integration and the instantaneous constraints [7].

Anderson et al. [8] studied the properties of the extreme solutions of the SCLP, based on which Anderson and Philpott [9] developed a simplex type of algorithm for a network-based SCLP. Refer to Anderson and Philpott [10] and Anderson and Nash [11] for their other results on SCLP. Pullan [12–18] continues studying SCLP in a series of papers. He systematically developed a duality theory and solution algorithms for the SCLP.

There are other researches focused on other forms of SCLP, including Luo and Bertsimas [19], Shapiro [20], Fleischer and Sethuraman [21], Weiss [22], and Nasrabadi et al. [23].

One of the extensions of SCLP is SCCP introduced by Wang et al. [2] in which the constraints involve the convex cone in their right hand side. When all the convex cones are nonnegative orthants, SCCP reduces to SCLP. In [2], based on the relationship among SCCP, its dual, and their discretization counterparts, they develop a strong duality theory for the SCCP. They also suggest a polynomial-time approximation algorithm that solves the SCCP to any predefined accuracy.

Wang [24, 25] extends SCCP to *generalized separated continuous conic programming* (GSCCP) by allowing the parameters in (1) to be piece-wise constants and extends the results of [2] for SCCP to GSCCP. In this paper, we extend SCCP to SSCCP by allowing the changes of values of some parameters in SCCP in the second stage. We also extend the results of [2] for SCCP to SSCCP.

2.1. A Motived Example for SSCCP. We consider a problem which appears in [2]; for completeness, we reproduce the problem description and the formulation below.

A network processes a continuous flow of jobs at two machines. The jobs visit machines 1 and 2 in the order $1 \to 2 \to 1$, that is, a total of three processing steps; see Figure 1. Corresponding to each processing step, there is a buffer holding the fluid. At $t = 0$, the initial levels of fluid at the three steps are 50, 20, and 120 units. The input rates of fluid from outside to the three buffers are 0.01, 0.01, and 0.01. To process each unit of job ("fluid"), the time requirements at the three steps are 0.4, 0.8, and 0.2 time units.

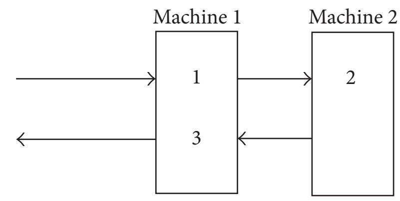


Figure 1: The fluid network in the example in Section 2.1.

The problem is to find the processing rates at the three steps, $u_i(t)$, $i = 1,2,3$, which determine the fluid levels in the three buffers, $x_i(t)$, $i = 1,2,3$, during a given time interval $[0,T]$ such that the fluid levels in the three buffers are maintained as close as possible to a prespecified constant level $d = (30 \;\; 10 \;\; 80)'$.

The problem can be formulated as follows:

$$\begin{aligned}
& \min \quad \int_0^T \left[(x(t) - d)' (x(t) - d) \right] dt \\
& \text{s.t.} \quad \int_0^t G u(s)\, ds + x(t) = \alpha + t a, \\
& \qquad b - Hu(t) \ge 0, \\
& \qquad u(t) \ge 0, \quad x(t) \ge 0, \quad t \in [0,T],
\end{aligned} \tag{6}$$

where

$$\begin{gathered}
G = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} 0.4 & 0 & 0.2 \\ 0 & 0.8 & 0 \end{pmatrix}, \\
\alpha = \begin{pmatrix} 50 \\ 20 \\ 120 \end{pmatrix}, \qquad a = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{gathered} \tag{7}$$

We can further express the above problem in the form of SCCP. Please refer to [2] for the details.

In reality, the values of a and b could be changed during $[0,T]$ for example, when the machine 1 experiences partial breakdown within $[T_1, T]$; where $0 \le T_1 \le T$, the corresponding value of capacity vector for machine 1, b_1, will change during $[T_1, T]$. This makes the formulation of the problem an SSCCP. We omit the details here.

3. The Dual and Discretizations

3.1. The Dual. The dual of SSCCP that we will focus on is the following problem:

$$\begin{aligned}
& (\text{SSCCP}^*) \\
& \min \quad \int_0^{T-T_1} \left[(\alpha_1 + (T-t) a_1)' h(t) + b_1' l(t) \right] dt \\
& \qquad + \int_0^{T-T_1} \left[(\alpha_2 + (T-t) a_2)' p(t) + b_2' q(t) \right] dt \\
& \qquad + \int_{T-T_1}^{T} \left[(\alpha + (T-t) a)' f(t) + b' g(t) \right] dt
\end{aligned}$$

$$\begin{aligned}
\text{s.t.}\quad & \int_0^t G' h(s)\,ds + H' l(t) - (\check{\gamma}_1 + t\check{c}_1) \in \mathcal{K}_{31}^*, \\
& \qquad t \in [0, T - T_1], \\
& F' h(t) - \check{d}_1 \in \mathcal{K}_{41}^*, \quad t \in [0, T - T_1], \\
& h(t) \in \mathcal{K}_{11}^*, \quad l(t) \in \mathcal{K}_{21}^*, \quad t \in [0, T - T_1], \\
& \int_0^t G' p(s)\,ds + H' q(t) - (\check{\gamma}_2 + t\check{c}_2) \in \mathcal{K}_{32}^*, \\
& \qquad t \in [0, T - T_1], \\
& F' p(t) - \check{d}_2 \in \mathcal{K}_{42}^*, \quad t \in [0, T - T_1], \\
& p(t) \in \mathcal{K}_{12}^*, \quad q(t) \in \mathcal{K}_{22}^*, \quad t \in [0, T - T_1], \\
& \int_0^{T-T_1} G' (h(s) + p(s))\,ds \\
& \quad + \int_{T-T_1}^t G' f(s)\,ds + H' g(t) - (\gamma + tc) \in \mathcal{K}_3^*, \\
& \qquad t \in (T - T_1, T], \\
& F' f(t) - d \in \mathcal{K}_4^*, \quad t \in (T - T_1, T], \\
& f(t) \in \mathcal{K}_1^*, \quad g(t) \in \mathcal{K}_2^*, \quad t \in (T - T_1, T],
\end{aligned} \tag{8}$$

where the decision variables $h(t)$, $l(t)$, $p(t)$, $q(t)$, $f(t)$, and $g(t)$ are bounded measurable functions. $\mathcal{K}_i^*$, $\mathcal{K}_{ij}^*$ are the dual cones of $\mathcal{K}_i$ and $\mathcal{K}_{ij}$, $i = 1, 2, 3, 4$, $j = 1, 2$, respectively.

The derivation of the above dual problem is similar to the derivation of the dual problem for LP (see, e.g., [26]) and we omit the details here. Because SSCCP involves time, to achieve some degree of symmetry in the dual (to facilitate the later analysis), we choose to write the dual in the reversed time; that is, t in the dual is $T - t$ in the primal.

The following weak duality is readily shown from the derivation of (SSCCP*).

Proposition 1. *The weak duality holds between SSCCP and SSCCP*; that is, if $u(t), x(t)$, $t \in [0, T_1]$, $v(t)$, $y(t)$, $w(t)$, $z(t)$, $t \in (T_1, T]$ is a feasible solution for SSCCP and $h(t)$, $l(t)$, $p(t)$, $q(t)$, $t \in [0, T - T_1)$, $f(t)$, $g(t)$, $t \in [T - T_1]$ is a feasible solution for SSCCP*, then*

$$\begin{aligned}
& \int_0^{T_1} \left[(\gamma + (T - t)c)' u(t) + d' x(t) \right] dt \\
& \quad + \int_{T_1}^T \left[(\check{\gamma}_1 + (T - t)\check{c}_1)' v(t) + \check{d}_1' y(t) \right] dt \\
& \quad + \int_{T_1}^T \left[(\check{\gamma}_2 + (T - t)\check{c}_2)' w(t) + \check{d}_2' z(t) \right] dt \\
& \le \int_0^{T-T_1} \left[(\alpha_1 + (T - t)a_1)' h(t) + b_1' l(t) \right] dt \\
& \quad + \int_0^{T-T_1} \left[(\alpha_2 + (T - t)a_2)' p(t) + b_2' q(t) \right] dt \\
& \quad + \int_{T-T_1}^T \left[(\alpha + (T - t)a)' f(t) + b' g(t) \right] dt.
\end{aligned} \tag{9}$$

Next we will introduce the discretizations for SSCCP and SSCCP*, respectively, and discuss the relationships among SSCCP, SSCCP*, and their discretizations. But first, we need the following notation and conventions which mostly follow what is used in [2].

Notation and Conventions

(i) When we say $(u(t), x(t), v(t), y(t), w(t), z(t))$ is a feasible solution to SSCCP, we mean $(u(t), x(t), t \in [0, T_1], v(t), y(t), w(t), z(t), t \in (T_1, T])$ is a feasible solution to SSCCP.

(ii) By default, all vectors are column vectors. One exception is when we denote the solutions to SSCCP and its dual (or their variations) as (u, x, v, y, w, z) and (h, l, p, q, f, g), we mean $(u', x', v', y', w', z')'$ and $(h', l', p', q', f', g')'$.

(iii) $\pi = \{t_0, \ldots, t_{m_1}, \ldots, t_{m_1+m_2}\}$ denotes a partition of $[0, T_1, T]$ into $m_1 + m_2$ segments:

$$0 = t_0 < t_1 < \cdots < t_{m_1} = T_1 < t_{m_1+1} < \cdots < t_{m_1+m_2} = T, \tag{10}$$

where m_1 and m_2 are positive integer numbers.

(iv) Given a partition $\pi = \{t_0, \ldots, t_m\}$ and a vector $\hat{r} := (\hat{r}(t_0), \hat{r}(t_1), \ldots, \hat{r}(t_m))$, where $\hat{r}(\cdot)$ is a right continuous function, the following (continuous) function

$$r(t) = \left(\frac{t_i - t}{t_i - t_{i-1}} \right) \hat{r}(t_{i-1}) + \left(\frac{t - t_{i-1}}{t_i - t_{i-1}} \right) \hat{r}(t_i), \quad \text{for } t \in [t_{i-1}, t_i], \quad i = 1, \ldots, m, \tag{11}$$

is called a piecewise linear extension of $\hat{r}$, whereas the following (right-continuous) function

$$r(t) = \begin{cases} \hat{r}(t_{i-1}), & t \in [t_{i-1}, t_i), \quad \text{for } i = 1, \ldots, m, \\ \hat{r}(t_{m-1}), & t = T, \end{cases} \tag{12}$$

is called a piecewise constant extension of $\hat{r}$.

(v) When $(u(t), x(t), v(t), y(t), w(t), z(t))$ is a feasible solution to SSCCP, with $u(t)$, $v(t)$, $w(t)$ being piecewise constant and $x(t)$, $y(t)$, $z(t)$ piecewise linear, we assume $u(t)$, $v(t)$, $w(t)$ is right continuous, and $x(t)$, $y(t)$, $z(t)$ is continuous, with $y(T_1+) = x(T_1)$, $z(T_1+) = x(T_1)$, and the pieces of both u and x correspond to a common partition for $[0, T_1]$, and the pieces of both v, w and y, z correspond to a common partition for $(T_1, T]$.

When $(h(t), l(t), p(t), q(t), f(t), g(t))$ is a feasible solution to SSCCP*, with $h(t), p(t), f(t)$ being piecewise constant and $l(t), q(t), g(t)$ piecewise linear, we assume $h(t)$, $p(t)$, and $f(t)$ are right continuous and $l(t)$, $q(t)$ and $g(t)$ are continuous, with $g(T-T_1+) = l(T-T_1) + q(T-T_1)$, and the pieces of h, l, p, and q correspond to a common partition for $[0, T-T_1]$, and the pieces of both f and g correspond to a common partition for $(T-T_1, T]$.

(vi) For $i = 1, 2, 3, 4$, denote $\mathscr{K}_{i,m} := \underbrace{\mathscr{K}_i \times \cdots \times \mathscr{K}_i}_{m}$, and similarly denote $\mathscr{K}^*_{i,m} := \underbrace{\mathscr{K}^*_i \times \cdots \times \mathscr{K}^*_i}_{m}$.

3.2. The Discretizations. We start with introducing the following discretization of SSCCP based on the partition π of $[0, T_1, T]$, where $0 = t_0 < t_1 < \cdots < t_{m_1} = T_1 < t_{m_1+1} < \cdots < t_{m_1+m_2} = T$:

$$\begin{aligned}
&(\mathrm{SCP}_1(\pi))\\
&\max \quad \sum_{i=1}^{m_1}\left(\left(\gamma + \left(T - \frac{t_i + t_{i-1}}{2}\right)c\right)'\widehat{u}_i + d'\left[\widehat{x}_i + \widehat{x}_{i-1}\right]\frac{t_i - t_{i-1}}{2}\right)\\
&\qquad + \sum_{i=1}^{m_2}\left(\left(\breve{\gamma}_1 + \left(T - \frac{t_{m_1+i} + t_{m_1+i-1}}{2}\right)\breve{c}_1\right)'\widehat{v}_i + \breve{d}_1'\left[\widehat{y}_i + \widehat{y}_{i-1}\right]\frac{t_{m_1+i} - t_{m_1+i-1}}{2}\right)\\
&\qquad + \sum_{i=1}^{m_2}\left(\left(\breve{\gamma}_2 + \left(T - \frac{t_{m_1+i} + t_{m_1+i-1}}{2}\right)\breve{c}_2\right)'\widehat{w}_i + \breve{d}_2'\left[\widehat{z}_i + \widehat{z}_{i-1}\right]\frac{t_{m_1+i} - t_{m_1+i-1}}{2}\right)\\
&\text{s.t.} \quad \alpha + t_i a - \left[G\widehat{u}_1 + \cdots + G\widehat{u}_i + F\widehat{x}_i\right] \in \mathscr{K}_1, \quad i = 1, 2, \ldots, m_1;\\
&\qquad \left(t_i - t_{i-1}\right)b - H\widehat{u}_i \in \mathscr{K}_2, \quad i = 1, \ldots, m_1;\\
&\qquad \widehat{u}_i \in \mathscr{K}_3, \quad \widehat{x}_i \in \mathscr{K}_4, \quad i = 1, \ldots, m_1;\\
&\qquad \alpha_1 + t_{m_1+i}a_1 - \left[G\widehat{u}_1 + \cdots + G\widehat{u}_{m_1}\right] - \left[G\widehat{v}_1 + \cdots + G\widehat{v}_i\right] - F\widehat{y}_i \in \mathscr{K}_{11}, \quad i = 1, 2, \ldots, m_2;\\
&\qquad \left(t_{m_1+i} - t_{m_1+i-1}\right)b_1 - H\widehat{v}_i \in \mathscr{K}_{21}, \quad i = 1, \ldots, m_2;\\
&\qquad \widehat{v}_i \in \mathscr{K}_{31}, \quad \widehat{y}_i \in \mathscr{K}_{41}, \quad i = 1, \ldots, m_2;\\
&\qquad \alpha_2 + t_{m_1+i}a_2 - \left[G\widehat{u}_1 + \cdots + G\widehat{u}_{m_1}\right] - \left[G\widehat{w}_1 + \cdots + G\widehat{w}_i\right] - F\widehat{z}_i \in \mathscr{K}_{12}, \quad i = 1, 2, \ldots, m_2;\\
&\qquad \left(t_{m_1+i} - t_{m_1+i-1}\right)b_2 - H\widehat{w}_i \in \mathscr{K}_{22}, \quad i = 1, \ldots, m_2;\\
&\qquad \widehat{w}_i \in \mathscr{K}_{32}, \quad \widehat{z}_i \in \mathscr{K}_{42}, \quad i = 1, \ldots, m_2.
\end{aligned} \tag{13}$$

Note that here we require that $\widehat{y}_0 = \widehat{z}_0 = \widehat{x}_{m_1}$ and

$$\alpha - F\widehat{x}_0 \in \mathscr{K}_1, \qquad \widehat{x}_0 \in \mathscr{K}_4. \tag{14}$$

Clearly, $(\mathrm{SCP}_1(\pi))$ is a conic program.

Lemma 2. *From a feasible solution for $\mathrm{SCP}_1(\pi)$, one can get a feasible solution for SSCCP with the same objective values, if $\mathscr{K}_1 \subseteq \mathscr{K}_{11}$, $\mathscr{K}_1 \subseteq \mathscr{K}_{12}$, $\mathscr{K}_4 \subseteq \mathscr{K}_{41}$, $\mathscr{K}_4 \subseteq \mathscr{K}_{42}$.*

Proof. Suppose $(\widehat{u}, \widehat{x}, \widehat{v}, \widehat{y}, \widehat{w}, \widehat{z})$ is a feasible solution for $\mathrm{SCP}_1(\pi)$. Let

$$\begin{aligned}
u(t) &= \begin{cases} \dfrac{\widehat{u}_i}{t_i - t_{i-1}}, & t \in [t_{i-1}, t_i), \quad i = 1, \ldots, m_1,\\[2mm] \dfrac{\widehat{u}_{m_1}}{t_{m_1} - t_{m_1-1}}, & t = t_{m_1}, \end{cases}\\
x(0) &= \widehat{x}_0,\\
x(t) &= \frac{t_i - t}{t_i - t_{i-1}}\widehat{x}_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}}\widehat{x}_i, \quad i = 1, \ldots, m_1,\\
v(t) &= \frac{\widehat{v}_i}{t_{m_1+i} - t_{m_1+i-1}}, \quad t \in \left(t_{m_1+i-1}, t_{m_1+i}\right], \quad i = 1, \ldots, m_2,\\
y(T_1+) &= \widehat{x}_{m_1},\\
y(t) &= \frac{t_{m_1+i} - t}{t_{m_1+i} - t_{m_1+i-1}}\widehat{y}_{i-1} + \frac{t - t_{m_1+i-1}}{t_{m_1+i} - t_{m_1+i-1}}\widehat{y}_i, \quad i = 1, \ldots, m_2,\\
w(t) &= \frac{\widehat{w}_i}{t_{m_1+i} - t_{m_1+i-1}}, \quad t \in \left(t_{m_1+i-1}, t_{m_1+i}\right], \quad i = 1, \ldots, m_2,\\
z(T_1+) &= \widehat{x}_{m_1},\\
z(t) &= \frac{t_{m_1+i} - t}{t_{m_1+i} - t_{m_1+i-1}}\widehat{z}(t_{i-1}) + \frac{t - t_{m_1+i-1}}{t_{m_1+i} - t_{m_1+i-1}}\widehat{z}_i, \quad i = 1, \ldots, m_2;
\end{aligned} \tag{15}$$

then we have $u(t) \in \mathscr{K}_3$, $x(t) \in \mathscr{K}_4$, $t \in [0, T_1]$, $v(t) \in \mathscr{K}_{31}$, $t \in (T_1, T]$, $w(t) \in \mathscr{K}_{32}$, and $t \in (T_1, T]$.

Because $\hat{x}_{m_1} \in \mathcal{K}_4$ and $\hat{y}_0 = \hat{z}_0 = \hat{x}_{m_1}$, $\hat{y}_0 \in \mathcal{K}_4$ and $\hat{z}_0 \in \mathcal{K}_4$. When $\mathcal{K}_4 \subseteq \mathcal{K}_{41}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{42}$, we have $y(t) \in \mathcal{K}_{41}$, and $z(t) \in \mathcal{K}_{42}$, and $t \in (T_1, T]$.

For $t = 0$,

$$\begin{aligned} &\alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \\ &\quad = \alpha - F\hat{x}_0 \in \mathcal{K}_1. \end{aligned} \tag{16}$$

For $t \in (t_0, t_1)$,

$$\begin{aligned} &\alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \\ &\quad = \alpha + ta - G\frac{\hat{u}_1}{t_1}t - F\left(\frac{t_1 - t}{t_1}\hat{x}_0 + \frac{t}{t_1}\hat{x}_1\right) \\ &\quad = \frac{t}{t_1}\left(\alpha + t_1 a - G\hat{u}_1 - F\hat{x}_1\right) + \frac{t_1 - t}{t_1}\left(\alpha - F\hat{x}_0\right) \\ &\quad \in \mathcal{K}_1. \end{aligned} \tag{17}$$

For $t \in [t_{i-1}, t_i)$, $i = 2, \ldots, m_1$,

$$\begin{aligned} &\alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \\ &\quad = \alpha + ta - \left(\int_0^{t_1} Gu(s)\,ds + \cdots + \int_{t_{i-1}}^t Gu(s)\,ds\right) - Fx(t) \\ &\quad = \alpha + ta - \left(G\frac{\hat{u}_1}{t_1 - t_0}(t_1 - t_0) + \cdots + G\frac{\hat{u}_i}{t_i - t_{i-1}}(t - t_{i-1})\right) \\ &\qquad - F\left(\frac{t_i - t}{t_i - t_{i-1}}\hat{x}_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}}\hat{x}_i\right) \\ &\quad = \frac{t - t_{i-1}}{t_i - t_{i-1}}\left(\alpha + t_i a - \left(G\hat{u}_1 + \cdots + G\hat{u}_{i-1} + G\hat{u}_i\right) - F\hat{x}_i\right) \\ &\qquad + \frac{t_i - t}{t_i - t_{i-1}}\left(\alpha + t_{i-1}a - \left(G\hat{u}_1 + \cdots + G\hat{u}_{i-1}\right) - F\hat{x}_{i-1}\right) \\ &\quad \in \mathcal{K}_1. \end{aligned} \tag{18}$$

For $t = t_{m_1}$,

$$\begin{aligned} &\alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \\ &\quad = \alpha + t_{m_1}a - \left(G\hat{u}_1 + \cdots + G\hat{u}_{m_1-1} + G\hat{u}_{m_1}\right) - F\hat{x}_{m_1} \\ &\quad \in \mathcal{K}_1. \end{aligned} \tag{19}$$

For $t \in [t_{i-1}, t_i)$, $i = 1, \ldots, m_1$,

$$\begin{aligned} b - Hu(t) &= b - H\frac{\hat{u}_i}{t_i - t_{i-1}} \\ &= \frac{1}{t_i - t_{i-1}}\left(\left(t_i - t_{i-1}\right)b - H\hat{u}_i\right) \\ &\in \mathcal{K}_2. \end{aligned} \tag{20}$$

For $t = t_{m_1}$,

$$\begin{aligned} b - Hu(t) &= b - H\frac{\hat{u}_{m_1}}{t_{m_1} - t_{m_1-1}} \\ &= \frac{1}{t_{m_1} - t_{m_1-1}}\left(\left(t_{m_1} - t_{m_1-1}\right)b - H\hat{u}_{m_1}\right) \\ &\in \mathcal{K}_2. \end{aligned} \tag{21}$$

For $t \in (t_{m_1}, t_{m_1+1}]$,

$$\begin{aligned} &\alpha_1 + ta_1 - \int_0^{T_1} Gu(s)\,ds - \int_{T_1}^t Gv(s)\,ds - Fy(t) \\ &\quad = \alpha_1 + ta_1 - \left(G\hat{u}_1 + \cdots + G\hat{u}_{m_1-1} + G\hat{u}_{m_1}\right) \\ &\qquad - G\frac{\hat{v}_1}{t_{m_1+1} - t_{m_1}}\left(t - t_{m_1}\right) \\ &\qquad - F\left(\frac{t_{m_1+1} - t}{t_{m_1+1} - t_{m_1}}\hat{y}_0 + \frac{t - t_{m_1}}{t_{m_1+1} - t_{m_1}}\hat{y}_1\right) \\ &\quad = \frac{t - t_{m_1}}{t_{m_1+1} - t_{m_1}}\left(\alpha_1 + t_{m_1+1}a_1 - \left(G\hat{u}_1 + \cdots + G\hat{u}_{m_1-1}\right.\right. \\ &\qquad\qquad \left.\left. + G\hat{u}_{m_1}\right) - G\hat{v}_1 - F\hat{y}_1\right) \\ &\qquad + \frac{t_{m_1+1} - t}{t_{m_1+1} - t_{m_1}}\left(\alpha + T_1 a - \left(G\hat{u}_1 + \cdots + G\hat{u}_{m_1-1}\right.\right. \\ &\qquad\qquad \left.\left. + G\hat{u}_{m_1}\right) - F\hat{x}_{m_1}\right) \\ &\quad \in \mathcal{K}_{11}, \quad \text{when } \mathcal{K}_1 \subseteq \mathcal{K}_{11}. \end{aligned} \tag{22}$$

For $t \in (t_{m_1+i-1}, t_{m_1+i}]$, $i = 2, \ldots, m_2$,

$$\begin{aligned} &\alpha_1 + ta_1 - \int_0^{T_1} Gu(s)\,ds - \int_{T_1}^t Gv(s)\,ds - Fy(t) \\ &\quad = \alpha_1 + ta_1 - \left(G\hat{u}_1 + \cdots + G\hat{u}_{m_1-1} + G\hat{u}_{m_1}\right) \\ &\qquad - \left(G\frac{\hat{v}_1}{t_{m_1+1} - t_{m_1}}\left(t_{m_1+1} - t_{m_1}\right) + \cdots\right. \\ &\qquad\qquad \left. + G\frac{\hat{v}_i}{t_{m_1+i} - t_{m_1+i-1}}\left(t - t_{m_1+i-1}\right)\right) - Fy(t) \\ &\quad = \alpha_1 + ta_1 - \left(G\hat{u}_1 + \cdots + G\hat{u}_{m_1-1} + G\hat{u}_{m_1}\right) \\ &\qquad - \left(G\hat{v}_1 + \cdots + G\hat{v}_{i-1} + G\frac{\hat{v}_i}{t_{m_1+i} - t_{m_1+i-1}}\left(t - t_{m_1+i-1}\right)\right) \\ &\qquad - F\left(\frac{t_{m_1+i} - t}{t_{m_1+i} - t_{m_1+i-1}}\hat{y}_{i-1} + \frac{t - t_{m_1+i-1}}{t_{m_1+i} - t_{m_1+i-1}}\hat{y}_i\right) \\ &\quad = \frac{t - t_{m_1+i-1}}{t_{m_1+i} - t_{m_1+i-1}} \end{aligned}$$

$$\begin{aligned}
&\times\left(\alpha_1 + t_{m_1+i}a_1 - \left(G\widehat{u}_1 + \cdots + G\widehat{u}_{m_1-1} + G\widehat{u}_{m_1}\right)\right.\\
&\qquad\left.- \left(G\widehat{v}_1 + \cdots + G\widehat{v}_{i-1} + G\widehat{v}_i\right) - F\widehat{y}_i\right)\\
&+ \frac{t_{m_1+i} - t}{t_{m_1+i} - t_{m_1+i-1}}\\
&\times\left(\alpha_1 + t_{m_1+i-1}a_1 - \left(G\widehat{u}_1 + \cdots + G\widehat{u}_{m_1-1} + G\widehat{u}_{m_1}\right)\right.\\
&\qquad\left.- \left(G\widehat{v}_1 + \cdots + G\widehat{v}_{i-1}\right) - F\widehat{y}_{i-1}\right)\\
&\in \mathscr{K}_{11}.
\end{aligned} \tag{23}$$

For $t \in (t_{m_1+i-1}, t_{m_1+i}]$, $i = 1, \ldots, m_2$

$$\begin{aligned}
b_1 - Hv(t) &= b_1 - H\frac{\widehat{v}_i}{t_{m_1+i} - t_{m_1+i-1}}\\
&= \frac{1}{t_{m_1+i} - t_{m_1+i-1}}\left(\left(t_{m_1+i} - t_{m_1+i-1}\right)b_1 - H\widehat{v}_i\right)\\
&\in \mathscr{K}_{21}.
\end{aligned} \tag{24}$$

Similarly, if $\mathscr{K}_1 \subseteq \mathscr{K}_{12}$, then

$$\begin{gathered}
\alpha_2 + ta_2 - \int_0^{T_1} Gu(s)\,ds - \int_{T_1}^{t} Gw(s)\,ds - Fz(t) \in \mathscr{K}_{12},\\
t \in (T_1, T],\\
b_2 - Hw(t) \in \mathscr{K}_{22}, \quad t \in (T_1, T].
\end{gathered} \tag{25}$$

So $(u(t), x(t), v(t), y(t), w(t), z(t))$ is a feasible solution for SSCCP when $\mathscr{K}_1 \subseteq \mathscr{K}_{11}$, $\mathscr{K}_1 \subseteq \mathscr{K}_{12}$, $\mathscr{K}_4 \subseteq \mathscr{K}_{41}$, and $\mathscr{K}_4 \subseteq \mathscr{K}_{42}$.

It is easy to see that the objective value of $(u(t), x(t), v(t), y(t), w(t), z(t))$ is the same as that of $(\widehat{u}, \widehat{x}, \widehat{v}, \widehat{y}, \widehat{w}, \widehat{z})$. We omit the details here. □

We now introduce the following discretization of SSCCP* based on the partition π' of $[0, T - T_1, T]$, $\pi' = \{t_0, t_1, \ldots, t_{m_2}, \ldots, t_{m_2+m_1}\}$, where $t_0 = 0$, $t_{m_2} = T - T_1$, $t_{m_2+m_1} = T$.

$$\left(\mathrm{SCP}_2\left(\pi'\right)\right)$$

$$\begin{aligned}
\min\quad &\sum_{i=1}^{m_2}\left(\left(\alpha_1 + \left(T - \frac{t_i + t_{i-1}}{2}\right)a_1\right)'\widehat{h}_i\right.\\
&\qquad\left. + b_1'\left[\widehat{l}_i + \widehat{l}_{i-1}\right]\frac{t_i - t_{i-1}}{2}\right)\\
&+ \sum_{i=1}^{m_2}\left(\left(\alpha_2 + \left(T - \frac{t_i + t_{i-1}}{2}\right)a_2\right)'\widehat{p}_i\right.\\
&\qquad\left. + b_2'\left[\widehat{q}_i + \widehat{q}_{i-1}\right]\frac{t_i - t_{i-1}}{2}\right)\\
&+ \sum_{i=1}^{m_1}\left(\left(\alpha + \left(T - \frac{t_{m_2+i} + t_{m_2+i-1}}{2}\right)a\right)'\widehat{f}_i\right.\\
&\qquad\left. + b'\left[\widehat{g}_i + \widehat{g}_{i-1}\right]\frac{t_{m_2+i} - t_{m_2+i-1}}{2}\right)\\
\text{s.t.}\quad &G'\left(\widehat{h}_1 + \widehat{h}_2 + \cdots + \widehat{h}_i\right) + H'\widehat{l}_i\\
&\quad - \left(\check{\gamma}_1 + t_i\check{c}_1\right) \in \mathscr{K}_{31}^*,\\
&\qquad i = 1, \ldots, m_2;\\
&F'\widehat{h}_i - \left(t_i - t_{i-1}\right)\check{d}_1 \in \mathscr{K}_{41}^*, \quad i = 1, \ldots, m_2;\\
&\widehat{h}_i \in \mathscr{K}_{11}^*, \quad \widehat{l}_i \in \mathscr{K}_{21}^*, \quad i = 1, \ldots, m_2;\\
&G'\left(\widehat{p}_1 + \widehat{p}_2 + \cdots + \widehat{p}_i\right) + H'\widehat{q}_i\\
&\quad - \left(\check{\gamma}_2 + t_i\check{c}_2\right) \in \mathscr{K}_{32}^*,\\
&\qquad i = 1, \ldots, m_2;\\
&F'\widehat{p}_i - \left(t_i - t_{i-1}\right)\check{d}_2 \in \mathscr{K}_{42}^*, \quad i = 1, \ldots, m_2;\\
&\widehat{p}_i \in \mathscr{K}_{12}^*, \quad \widehat{q}_i \in \mathscr{K}_{22}^*, \quad i = 1, \ldots, m_2;\\
&G'\left(\sum_{j=1}^{m_2}\widehat{h}_j + \sum_{j=1}^{m_2}\widehat{p}_j\right) + G'\left(\widehat{f}_1 + \widehat{f}_2 + \cdots + \widehat{f}_i\right)\\
&\quad + H'\widehat{g}_i - \left(\gamma + t_{m_2+i}c\right) \in \mathscr{K}_3^*,\\
&\qquad i = 1, \ldots, m_1;\\
&F'\widehat{f}_i - \left(t_{m_2+i} - t_{m_2+i-1}\right)d \in \mathscr{K}_4^*,\\
&\qquad i = 1, \ldots, m_1;\\
&\widehat{f}_i \in \mathscr{K}_1^*, \quad \widehat{g}_i \in \mathscr{K}_2^*, \quad i = 1, \ldots, m_1.
\end{aligned} \tag{26}$$

Note here we require that $\widehat{g}_0 = \widehat{l}_{m_2} + \widehat{q}_{m_2}$ and

$$\begin{aligned}
H'\widehat{l}_0 - \check{\gamma}_1 \in \mathscr{K}_{31}^*, &\qquad \widehat{l}_0 \in \mathscr{K}_{21}^*,\\
H'\widehat{q}_0 - \check{\gamma}_2 \in \mathscr{K}_{32}^*, &\qquad \widehat{q}_0 \in \mathscr{K}_{22}^*.
\end{aligned} \tag{27}$$

Clearly $\mathrm{SCP}_2(\pi')$ is also a conic program.

We now show the following.

Lemma 3. *For any two convex cones $\mathscr{K}_1, \mathscr{K}_2$,*

$$\mathscr{K}_1 \subseteq \mathscr{K}_2 \Longleftrightarrow \mathscr{K}_2^* \subseteq \mathscr{K}_1^*. \tag{28}$$

Proof. $\Rightarrow$: Because $\mathscr{K}_1 \subseteq \mathscr{K}_2$, for any $x_1 \in \mathscr{K}_1$, $x_1 \in \mathscr{K}_2$, so for any $y_2 \in \mathscr{K}_2^*$, $x_1'y_2 \ge 0$. So $y_2 \in \mathscr{K}_1^*$. So $\mathscr{K}_2^* \subseteq \mathscr{K}_1^*$.

$\Leftarrow$: Because $\mathscr{K}_2^* \subseteq \mathscr{K}_1^*$, for any $y_2 \in \mathscr{K}_2^*$, $y_2 \in \mathscr{K}_1^*$, So for any $x_1 \in \mathscr{K}_1$, $x_1'y_2 \ge 0$. So $x_1 \in \mathscr{K}_2$. So $\mathscr{K}_1 \subseteq \mathscr{K}_2$. □

Lemma 4. *From a feasible solution for $SCP_2(\pi')$, one can get a feasible solution for SSCCP* with the same objective values,* *if* $\mathcal{K}_2 \subseteq \mathcal{K}_{21}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{22}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{31}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{32}$, $\breve{\gamma}_1 + \breve{\gamma}_2 = \gamma$, $\breve{c}_1 + \breve{c}_2 = c$.

Proof. Suppose $(\widehat{h}, \widehat{l}, \widehat{p}, \widehat{q}, \widehat{f}, \widehat{g})$ is a feasible solution for $\mathrm{SCP}_2(\pi')$. Let

$$h(t) = \begin{cases} \dfrac{\widehat{h}_i}{t_i - t_{i-1}}, & t \in [t_{i-1}, t_i), \quad i = 1, \ldots, m_2, \\ \dfrac{\widehat{h}_{m_2}}{t_{m_2} - t_{m_2-1}}, & t = t_{m_2}, \end{cases}$$
$$l(0) = \widehat{l}_0,$$
$$l(t) = \frac{t_i - t}{t_i - t_{i-1}} \widehat{l}_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \widehat{l}_i,$$
$$t \in [t_{i-1}, t_i], \quad i = 1, \ldots, m_2,$$
$$p(t) = \begin{cases} \dfrac{\widehat{p}_i}{t_i - t_{i-1}}, & t \in [t_{i-1}, t_i), \quad i = 1, \ldots, m_2, \\ \dfrac{\widehat{p}_{m_2}}{t_{m_2} - t_{m_2-1}}, & t = t_{m_2}, \end{cases}$$
$$q(0) = \widehat{q}_0, \tag{29}$$
$$q(t) = \frac{t_i - t}{t_i - t_{i-1}} \widehat{q}_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} \widehat{q}_i,$$
$$t \in [t_{i-1}, t_i], \quad i = 1, \ldots, m_2,$$
$$f(t) = \frac{\widehat{f}_i}{t_{m_2+i} - t_{m_2+i-1}}, \quad t \in (t_{m_2+i-1}, t_{m_2+i}],$$
$$i = 1, \ldots, m_1,$$
$$g(T - T_1 +) = \widehat{g}_0,$$
$$g(t) = \frac{t_{m_2+i} - t}{t_{m_2+i} - t_{m_2+i-1}} \widehat{g}_{i-1} + \frac{t - t_{m_2+i-1}}{t_{m_2+i} - t_{m_2+i-1}} \widehat{g}_i,$$
$$t \in (t_{m_2+i-1}, t_{m_2+i}], \quad i = 1, \ldots, m_1;$$

then $h(t) \in \mathcal{K}_{11}^*$, $l(t) \in \mathcal{K}_{21}^*$, $p(t) \in \mathcal{K}_{12}^*$, $q(t) \in \mathcal{K}_{22}^*$, $t \in [0, T - T_1]$, $f(t) \in \mathcal{K}_1^*$, and $t \in (T - T_1, T]$. When $\mathcal{K}_2 \subseteq \mathcal{K}_{21}$, and $\mathcal{K}_2 \subseteq \mathcal{K}_{22}$, from Lemma 3, we have $\mathcal{K}_{21}^* \subseteq \mathcal{K}_2^*$, and $\mathcal{K}_{22}^* \subseteq \mathcal{K}_2^*$, then; because $\widehat{g}_0 = \widehat{l}_{m_2} + \widehat{q}_{m_2}$, $\widehat{l}_{m_2} \in \mathcal{K}_{21}^*$ and $\widehat{q}_{m_2} \in \mathcal{K}_{22}^*$, we have $\widehat{g}_0 \in \mathcal{K}_2^*$, so $g(t) \in \mathcal{K}_2^*$ and $t \in (T - T_1, T]$.

The remaining proof is similar to that in proving Lemma 2 and we omit the details here. □

4. Strong Feasibility

We say that (u, x, v, y, w, z) is a *strongly feasible* solution to SSCCP, if for the closed and convex cones $\mathcal{K}_i$, $\mathcal{K}_{ij}$, $i = 1, 2, 3, 4$, $j = 1, 2$, with nonempty interiors, the following holds:

$$\begin{gathered} \alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \in \operatorname{int} \mathcal{K}_1, \quad t \in (0, T_1], \\ \alpha - Fx(0) \in \mathcal{K}_1, \\ b - Hu(t) \in \operatorname{int} \mathcal{K}_2, \quad t \in [0, T_1], \\ u(t) \in \operatorname{int} \mathcal{K}_3, \quad x(t) \in \operatorname{int} \mathcal{K}_4, \\ t \in (0, T_1], \quad x(0) \in \mathcal{K}_4, \\ \alpha_1 + ta_1 - \int_0^{T_1} Gu(s)\,ds \\ - \int_{T_1}^t Gv(s)\,ds - Fy(t) \in \operatorname{int} \mathcal{K}_{11}, \quad t \in (T_1, T], \\ b_1 - Hv(t) \in \operatorname{int} \mathcal{K}_{21}, \quad t \in (T_1, T], \\ v(t) \in \operatorname{int} \mathcal{K}_{31}, \quad y(t) \in \operatorname{int} \mathcal{K}_{41}, \quad t \in (T_1, T], \\ \alpha_2 + ta_2 - \int_0^{T_1} Gu(s)\,ds \\ - \int_{T_1}^t Gw(s)\,ds - Fz(t) \in \operatorname{int} \mathcal{K}_{12}, \quad t \in (T_1, T], \\ b_2 - Hw(t) \in \operatorname{int} \mathcal{K}_{22}, \quad t \in (T_1, T], \\ w(t) \in \operatorname{int} \mathcal{K}_{32}, \quad z(t) \in \operatorname{int} \mathcal{K}_{42}, \quad t \in (T_1, T]. \end{gathered} \tag{30}$$

We say that SSCCP is *strongly feasible* if there exists a strongly feasible solution. The similar notions apply to the dual problem SSCCP*.

Next we will show that the strong feasibility of SSCCP and SSCCP* can be determined by the strong feasibility of the following two conic programs:

$$\begin{aligned} (\mathrm{CP}_1) \quad \max \quad & T_1 c' \widehat{u} - \left(T_1 d - (T - T_1)\left(\breve{d}_1 + \breve{d}_2\right)\right)' \widehat{x} \\ & + (T - T_1)\, \breve{c}_1' \widehat{v} + (T - T_1)\, \breve{d}_1' \widehat{y} \\ & + (T - T_1)\, \breve{c}_2' \widehat{w} + (T - T_1)\, \breve{d}_2' \widehat{z} \\ \text{s.t.} \quad & \alpha + T_1 a - G\widehat{u} - F\widehat{x} \in \mathcal{K}_1, \\ & T_1 b - H\widehat{u} \in \mathcal{K}_2, \\ & \widehat{u} \in \mathcal{K}_3, \quad \widehat{x} \in \mathcal{K}_4, \\ & \alpha_1 + Ta_1 - G\widehat{u} - G\widehat{v} - F\widehat{y} \in \mathcal{K}_{11}, \\ & (T - T_1)\, b_1 - H\widehat{v} \in \mathcal{K}_{21}, \\ & \widehat{v} \in \mathcal{K}_{31}, \quad \widehat{y} \in \mathcal{K}_{41}, \\ & \alpha_2 + Ta_2 - G\widehat{u} - G\widehat{w} - F\widehat{z} \in \mathcal{K}_{12}, \\ & (T - T_1)\, b_2 - H\widehat{w} \in \mathcal{K}_{22}, \\ & \widehat{w} \in \mathcal{K}_{32}, \quad \widehat{z} \in \mathcal{K}_{42}, \end{aligned} \tag{31}$$

$$\begin{aligned}
(\mathrm{CP}_2)\quad \min\quad & (T-T_1)\,a_1'\widehat{h} + ((T-T_1)\,b_1 - T_1 b)'\widehat{l} \\
& + (T-T_1)\,a_2'\widehat{p} + ((T-T_1)\,b_2 - T_1 b)'\widehat{q} \\
& + T_1 a'\widehat{f} + T_1 b'\widehat{g} \\
\text{s.t.}\quad & G'\widehat{h} + H'\widehat{l} - (\check{\gamma}_1 + (T-T_1)\,\check{c}_1) \in \mathcal{K}_{31}^*, \\
& F'\widehat{h} - (T-T_1)\,\check{d}_1 \in \mathcal{K}_{41}^*, \\
& \widehat{h} \in \mathcal{K}_{11}^*, \quad \widehat{l} \in \mathcal{K}_{21}^*, \\
& G'\widehat{p} + H'\widehat{q} - (\check{\gamma}_2 + (T-T_1)\,\check{c}_2) \in \mathcal{K}_{32}^*, \\
& F'\widehat{p} - (T-T_1)\,\check{d}_2 \in \mathcal{K}_{42}^*, \\
& \widehat{p} \in \mathcal{K}_{12}^*, \quad \widehat{q} \in \mathcal{K}_{22}^*, \\
& G'\left(\widehat{h} + \widehat{p}\right) + G'\widehat{f} + H'\widehat{g} - (\gamma + Tc) \in \mathcal{K}_3^*, \\
& F'\widehat{f} - T_1 d \in \mathcal{K}_4^*, \\
& \widehat{f} \in \mathcal{K}_1^*, \quad \widehat{g} \in \mathcal{K}_2^*.
\end{aligned} \tag{32}$$

Note that the constraints of CP_1 and CP_2 above are the same as the constraints of $\mathrm{SCP}_1(\pi)$ and $\mathrm{SCP}_2(\pi')$, respectively, when $m_1 = m_2 = 1$. The objectives of CP_1 and CP_2, however, are different from those of $\mathrm{SCP}_1(\pi)$ and $\mathrm{SCP}_2(\pi')$. The choice of these objectives is to facilitate the explicit derivation of a bound on the duality gap; see the proof of Theorem 11.

Lemma 5. *If the conic programs CP_1 are strongly feasible and $\mathcal{K}_1 \subseteq \mathcal{K}_{11}$, $\mathcal{K}_1 \subseteq \mathcal{K}_{12}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{41}$, and $\mathcal{K}_4 \subseteq \mathcal{K}_{42}$, then SSCCP is strongly feasible, and so is $SCP_1(\pi)$.*

Proof. (1) We first show that when CP_1 is strongly feasible, SSCCP is strongly feasible.

Suppose $(\widehat{u}, \widehat{x}, \widehat{v}, \widehat{y}, \widehat{w}, \widehat{z})$ is a strongly feasible solution to CP_1. We have

$$\begin{aligned}
& \alpha + T_1 a - G\widehat{u} - F\widehat{x} \in \operatorname{int} \mathcal{K}_1, \\
& T_1 b - H\widehat{u} \in \operatorname{int} \mathcal{K}_2, \\
& \widehat{u} \in \operatorname{int} \mathcal{K}_3, \quad \widehat{x} \in \operatorname{int} \mathcal{K}_4, \\
& \alpha_1 + Ta_1 - G\widehat{u} - G\widehat{v} - F\widehat{y} \in \operatorname{int} \mathcal{K}_{11}, \\
& (T-T_1)\,b_1 - H\widehat{v} \in \operatorname{int} \mathcal{K}_{21}, \\
& \widehat{v} \in \operatorname{int} \mathcal{K}_{31}, \quad \widehat{y} \in \operatorname{int} \mathcal{K}_{41}, \\
& \alpha_2 + Ta_2 - G\widehat{u} - G\widehat{w} - F\widehat{z} \in \operatorname{int} \mathcal{K}_{12}, \\
& (T-T_1)\,b_2 - H\widehat{w} \in \operatorname{int} \mathcal{K}_{22}, \\
& \widehat{w} \in \operatorname{int} \mathcal{K}_{32}, \quad \widehat{z} \in \operatorname{int} \mathcal{K}_{42},
\end{aligned} \tag{33}$$

and $\widehat{x}_0$ is such a constant that

$$\alpha - F\widehat{x}_0 \in \mathcal{K}_1, \quad \widehat{x}_0 \in \mathcal{K}_4. \tag{34}$$

Let

$$\begin{aligned}
u(t) &= \frac{\widehat{u}}{T_1}, \quad t \in [0, T_1], \\
x(0) &= \widehat{x}_0, \quad x(T_1) = \widehat{x}, \\
x(t) &= \frac{T_1 - t}{T_1}\widehat{x}_0 + \frac{t}{T_1}\widehat{x}, \quad t \in (0, T_1), \\
v(t) &= \frac{\widehat{v}}{T - T_1}, \quad t \in (T_1, T], \\
y(T) = \widehat{y}, \quad y(t) &= \frac{T - t}{T - T_1}\widehat{x} + \frac{t - T_1}{T - T_1}\widehat{y}, \quad t \in (T_1, T), \\
w(t) &= \frac{\widehat{w}}{T - T_1}, \quad t \in (T_1, T], \\
z(T) = \widehat{z}, \quad z(t) &= \frac{T - t}{T - T_1}\widehat{x} + \frac{t - T_1}{T - T_1}\widehat{z}, \\
& \qquad t \in (T_1, T).
\end{aligned} \tag{35}$$

We have $u(t) \in \operatorname{int} \mathcal{K}_3$, $t \in [0, T_1]$, $x(t) \in \operatorname{int} \mathcal{K}_4$, $t \in (0, T_1]$, $x(0) \in \mathcal{K}_4$, $v(t) \in \operatorname{int} \mathcal{K}_{31}$, $w(t) \in \operatorname{int} \mathcal{K}_{32}$, and $t \in (T_1, T]$.

Because $\mathcal{K}_4 \subseteq \mathcal{K}_{41}$ and $\mathcal{K}_4 \subseteq \mathcal{K}_{42}$, then $y(t) \in \operatorname{int} \mathcal{K}_{41}$, $t \in (T_1, T]$, $z(t) \in \operatorname{int} \mathcal{K}_{42}$ and $t \in (T_1, T]$.

For $t = 0$,

$$\alpha - Fx(0) = \alpha - F\widehat{x}_0 \in \mathcal{K}_1. \tag{36}$$

For $t \in (0, T_1]$,

$$\begin{aligned}
& \alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \\
& \quad = \alpha + ta - G\frac{\widehat{u}}{T_1}t - F\left(\frac{T_1 - t}{T_1}\widehat{x}_0 + \frac{t}{T_1}\widehat{x}\right) \\
& \quad = \frac{t}{T_1}\left(\alpha + T_1 a - G\widehat{u} - F\widehat{x}\right) + \frac{T_1 - t}{T_1}\left(\alpha - F\widehat{x}_0\right) \\
& \quad \in \operatorname{int} \mathcal{K}_1.
\end{aligned} \tag{37}$$

For $t \in [0, T_1]$,

$$b - Hu(t) = b - H\frac{\widehat{u}}{T_1} = \frac{1}{T_1}\left(T_1 b - H\widehat{u}\right) \in \operatorname{int} \mathcal{K}_2. \tag{38}$$

For $t \in (T_1, T]$,

$$\begin{aligned}
& \alpha_1 + ta_1 - \int_0^{T_1} Gu(s)\,ds - \int_{T_1}^t Gv(s)\,ds - Fy(t) \\
& \quad = \alpha_1 + ta_1 - G\frac{\widehat{u}}{T_1}T_1 - G\frac{\widehat{v}}{T - T_1}(t - T_1) \\
& \qquad - F\left(\frac{T - t}{T - T_1}\widehat{x} + \frac{t - T_1}{T - T_1}\widehat{y}\right) \\
& \quad = \frac{t - T_1}{T - T_1}\left(\alpha_1 + Ta_1 - G\widehat{u} - G\widehat{v} - F\widehat{y}\right) \\
& \qquad + \frac{T - t}{T - T_1}\left(\alpha + T_1 a - G\widehat{u} - F\widehat{x}\right) \\
& \quad \in \operatorname{int} \mathcal{K}_{11}.
\end{aligned} \tag{39}$$

Note that when $\mathcal{K}_1 \subseteq \mathcal{K}_{11}$, $\alpha + T_1 a - G\widehat{u} - F\widehat{x} \in \operatorname{int} \mathcal{K}_{11}$.

For $t \in (T_1, T]$,

$$b_1 - Hv(t) = b_1 - H\frac{\hat{v}}{T-T_1} = \frac{1}{T-T_1}\left((T-T_1)b_1 - H\hat{v}\right) \in \operatorname{int}\mathscr{K}_{21}. \tag{40}$$

Similarly, we can get (by noting that $\mathscr{K}_1 \subseteq \mathscr{K}_{12}$)

$$\begin{gathered}\alpha_2 + ta_2 - \int_0^{T_1} Gu(s)\,ds - \int_{T_1}^{t} Gw(s)\,ds - Fz(t) \in \operatorname{int}\mathscr{K}_{12}, \\ t \in (T_1, T], \\ b_2 - Hw(t) \in \operatorname{int}\mathscr{K}_{22}, \quad t \in (T_1, T].\end{gathered} \tag{41}$$

We can see that $(u(t), x(t), v(t), y(t), w(t), z(t))$ is a (two-piece) strongly feasible solution for SSCCP. So SSCCP is strongly feasible.

(2) Now we will show that from this strongly feasible solution for SSCCP, we can get a strongly feasible solution for $\mathrm{SCP}_1(\pi)$.

Let

$$\begin{gathered}\hat{u}_i = u(t)(t_i - t_{i-1}), \quad \hat{x}_i = x(t_i), \quad i = 1, \ldots, m_1; \\ \hat{v}_i = v(t)\left(t_{m_1+i} - t_{m_1+i-1}\right), \quad \hat{y}_i = y\left(t_{m_1+i}\right), \quad i = 1, \ldots, m_2; \\ \hat{w}_i = w(t)\left(t_{m_1+i} - t_{m_1+i-1}\right), \quad \hat{z}_i = z\left(t_{m_1+i}\right), \quad i = 1, \ldots, m_2.\end{gathered} \tag{42}$$

Then $\hat{u}_i \in \operatorname{int}\mathscr{K}_3$, $\hat{x}_i \in \operatorname{int}\mathscr{K}_4$, $i = 1, \ldots, m_1$, $\hat{v}_i \in \operatorname{int}\mathscr{K}_{31}$, $\hat{y}_i \in \operatorname{int}\mathscr{K}_{41}$, $\hat{w}_i \in \operatorname{int}\mathscr{K}_{32}$, $\hat{z}_i \in \operatorname{int}\mathscr{K}_{42}$, $i = 1, \ldots, m_2$.

For $i = 1, \ldots, m_1$,

$$\begin{aligned}\alpha + t_i a &- \left[G\hat{u}_1 + \cdots + G\hat{u}_i + F\hat{x}_i\right] \\ &= \alpha + t_i a - \left[Gu(t)(t_1 - t_0) + \cdots + Gu(t)(t_i - t_{i-1}) + Fx(t_i)\right] \\ &= \alpha + t_i a - \left[\int_0^{t_1} Gu(s)\,ds + \cdots + \int_{t_{i-1}}^{t_i} Gu(s)\,ds + Fx(t_i)\right] \\ &= \alpha + t_i a - \int_0^{t_i} Gu(s)\,ds - Fx(t_i) \\ &\in \operatorname{int}\mathscr{K}_1, \\ (t_i - t_{i-1})\,b &- H\hat{u}_i \\ &= (t_i - t_{i-1})\,b - Hu(t)(t_i - t_{i-1}) \\ &= (t_i - t_{i-1})(b - Hu(t)) \\ &\in \operatorname{int}\mathscr{K}_2.\end{aligned} \tag{43}$$

For $i = 1, \ldots, m_2$,

$$\begin{aligned}\alpha_1 + t_{m_1+i}a_1 &- \left[G\hat{u}_1 + \cdots + G\hat{u}_{m_1}\right] - \left[G\hat{v}_1 + \cdots + G\hat{v}_i\right] - F\hat{y}_i \\ &= \alpha_1 + t_{m_1+i}a_1 - \int_0^{T_1} Gu(s)\,ds - \int_{T_1}^{t_{m_1+i}} Gv(s)\,ds - Fy\left(t_{m_1+i}\right) \\ &\in \operatorname{int}\mathscr{K}_{11}, \\ \left(t_{m_1+i} - t_{m_1+i-1}\right) b_1 &- H\hat{v}_i \\ &= \left(t_{m_1+i} - t_{m_1+i-1}\right)(b_1 - Hv(t)) \\ &\in \operatorname{int}\mathscr{K}_{21}.\end{aligned} \tag{44}$$

Similarly, we can get.

For $i = 1, \ldots, m_2$,

$$\begin{gathered}\alpha_2 + t_{m_1+i}a_2 - \left[G\hat{u}_1 + \cdots + G\hat{u}_{m_1}\right] - \left[G\hat{w}_1 + \cdots + G\hat{w}_i\right] - F\hat{z}_i \in \operatorname{int}\mathscr{K}_{12}, \\ \left(t_{m_1+i} - t_{m_1+i-1}\right) b_2 - H\hat{w}_i \in \operatorname{int}\mathscr{K}_{22}.\end{gathered} \tag{45}$$

So $(\hat{u}, \hat{x}, \hat{v}, \hat{y}, \hat{w}, \hat{z})$ is a strongly feasible solution for $\mathrm{SCP}_1(\pi)$, and $\mathrm{SCP}_1(\pi)$ is strongly feasible. □

Lemma 6. *If the conic programs CP_2 are strongly feasible, $\mathscr{K}_2 \subseteq \mathscr{K}_{21}$, $\mathscr{K}_2 \subseteq \mathscr{K}_{22}$, $\mathscr{K}_3 \subseteq \mathscr{K}_{31}$, $\mathscr{K}_3 \subseteq \mathscr{K}_{32}$, $\check{\gamma}_1 + \check{\gamma}_2 = \gamma$, $\check{c}_1 + \check{c}_2 = c$, then SSCCP* is strongly feasible and so is $\mathrm{SCP}_2(\pi')$.*

Proof. The proof is similar to that of Lemma 5; the details are omitted here. □

We shall focus on one partition for $[0, T_1, T]$, denoted by $\pi(\epsilon_1, \epsilon_2)$ and one partition for $[0, T - T_1, T]$, denoted by $\pi'(\epsilon_1, \epsilon_2)$. $\pi(\epsilon_1, \epsilon_2)$ divides the interval $[0, T_1]$ into m_1 equal segments, each of length ϵ_1, and divides the interval $[T_1, T]$ into m_2 equal segments, each of length ϵ_2. $\pi'(\epsilon_1, \epsilon_2)$ divides the interval $[0, T - T_1]$ into m_2 equal segments, each of length ϵ_2, and divides the interval $[T - T_1, T]$ into m_1 equal segments, each of length ϵ_1.

If we reverse the inner order of $\hat{u}, \hat{x}, \hat{v}, \hat{y}, \hat{w}$ and $\hat{z}$ in $\mathrm{SCP}_1(\pi(\epsilon_1, \epsilon_2)$, that is, for example, change $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_{m_1})$ to $(\hat{u}_{m_1}, \ldots, \hat{u}_1)$, we get the following problem:

$$(\mathrm{SCP}_1(\pi(\epsilon_1, \epsilon_2)))$$

$$\begin{aligned}\max \quad & h_{l1}'\hat{u} + d_{l1}'\hat{x} + h_{l_2}'\hat{v} + d_{l_2}'\hat{y} \\ & + h_{l_3}'\hat{w} + d_{l_3}'\hat{z} + \frac{\epsilon_1}{2}d'\hat{x}_0\end{aligned}$$

$$
\begin{aligned}
\text{s.t.}\quad & g_{l1}-\widehat{G}_{m_1}\widehat{u}-\widehat{F}_{m_1}\widehat{x}\in\mathscr{K}_{1,m_1},\\
& f_{l1}-\widehat{H}_{m_1}\widehat{u}\in\mathscr{K}_{2,m_1},\\
& g_{l2}-\widehat{G}_{m_2,m_1}\widehat{u}-\widehat{G}_{m_2}\widehat{v}-\widehat{F}_{m_2}\widehat{y}\in\mathscr{K}_{11,m_2},\\
& f_{l2}-\widehat{H}_{m_2}\widehat{v}\in\mathscr{K}_{21,m_2},\\
& g_{l3}-\widehat{G}_{m_2,m_1}\widehat{u}-\widehat{G}_{m_2}\widehat{w}-\widehat{F}_{m_2}\widehat{z}\in\mathscr{K}_{12,m_2},\\
& f_{l3}-\widehat{H}_{m_2}\widehat{w}\in\mathscr{K}_{22,m_2},\\
& \widehat{u}\in\mathscr{K}_{3,m_1},\quad \widehat{x}\in\mathscr{K}_{4,m_1},\\
& \widehat{v}\in\mathscr{K}_{31,m_2},\quad \widehat{y}\in\mathscr{K}_{41,m_2},\\
& \widehat{w}\in\mathscr{K}_{32,m_2},\quad \widehat{z}\in\mathscr{K}_{42,m_2},
\end{aligned}
\tag{46}
$$

where

$$
\widehat{G}_{m_1}=m_1\left\{\begin{pmatrix} G & G & \cdots & G\\ & G & \cdots & G\\ & & \ddots & \vdots\\ & & & G\end{pmatrix}\right.,
$$

$$
\widehat{G}_{m_2}=m_2\left\{\begin{pmatrix} G & G & \cdots & G\\ & G & \cdots & G\\ & & \ddots & \vdots\\ & & & G\end{pmatrix}\right.,
$$

$$
\widehat{G}_{m_2,m_1}=\underbrace{m_2\left\{\begin{pmatrix} G & G & \cdots & G\\ G & G & \cdots & G\\ \vdots & \vdots & & \vdots\\ G & G & \cdots & G\end{pmatrix}\right.}_{m_1},
$$

$$
\widehat{F}_{m_1}=m_1\left\{\begin{pmatrix} F & & & \\ & F & & \\ & & \ddots & \\ & & & F\end{pmatrix}\right.,
$$

$$
\widehat{F}_{m_2}=m_2\left\{\begin{pmatrix} F & & & \\ & F & & \\ & & \ddots & \\ & & & F\end{pmatrix}\right.,
$$

$$
\widehat{H}_{m_1}=m_1\left\{\begin{pmatrix} H & & & \\ & H & & \\ & & \ddots & \\ & & & H\end{pmatrix}\right.,
$$

$$
\widehat{H}_{m_2}=m_2\left\{\begin{pmatrix} H & & & \\ & H & & \\ & & \ddots & \\ & & & H\end{pmatrix}\right.,
$$

$$
f_{l1}=m_1\left\{\begin{pmatrix}\epsilon_1 b\\ \epsilon_1 b\\ \vdots\\ \epsilon_1 b\end{pmatrix}\right.,
$$

$$
f_{l2}=m_2\left\{\begin{pmatrix}\epsilon_2 b_1\\ \epsilon_2 b_1\\ \vdots\\ \epsilon_2 b_1\end{pmatrix}\right.,\qquad
f_{l3}=m_2\left\{\begin{pmatrix}\epsilon_2 b_2\\ \epsilon_2 b_2\\ \vdots\\ \epsilon_2 b_2\end{pmatrix}\right.,
$$

$$
g_{l1}=m_1\left\{\begin{pmatrix}\alpha+T_1 a\\ \vdots\\ \alpha+2\epsilon_1 a\\ \alpha+\epsilon_1 a\end{pmatrix}\right.,
$$

$$
g_{l2}=m_2\left\{\begin{pmatrix}\alpha_1+Ta_1\\ \vdots\\ \alpha_1+\left(T_1+2\epsilon_2\right)a_1\\ \alpha_1+\left(T_1+\epsilon_2\right)a_1\end{pmatrix}\right.,
$$

$$
g_{l3}=m_2\left\{\begin{pmatrix}\alpha_2+Ta_2\\ \vdots\\ \alpha_2+\left(T_1+2\epsilon_2\right)a_2\\ \alpha_2+\left(T_1+\epsilon_2\right)a_2\end{pmatrix}\right.,
$$

$$
h_{l1}=m_1\left\{\begin{pmatrix}\gamma+\left(T-T_1+\dfrac{\epsilon_1}{2}\right)c\\ \vdots\\ \gamma+\left(T-\dfrac{3\epsilon_1}{2}\right)c\\ \gamma+\left(T-\dfrac{\epsilon_1}{2}\right)c\end{pmatrix}\right.,
$$

$$
h_{l2}=m_2\left\{\begin{pmatrix}\breve{\gamma}_1+\dfrac{\epsilon_2}{2}\breve{c}_1\\ \vdots\\ \breve{\gamma}_1+\left(T-T_1-\dfrac{3\epsilon_2}{2}\right)\breve{c}_1\\ \breve{\gamma}_1+\left(T-T_1-\dfrac{\epsilon_2}{2}\right)\breve{c}_1\end{pmatrix}\right.,
$$

$$
h_{l3}=m_2\left\{\begin{pmatrix}\breve{\gamma}_2+\dfrac{\epsilon_2}{2}\breve{c}_2\\ \vdots\\ \breve{\gamma}_2+\left(T-T_1-\dfrac{3\epsilon_2}{2}\right)\breve{c}_2\\ \breve{\gamma}_2+\left(T-T_1-\dfrac{\epsilon_2}{2}\right)\breve{c}_2\end{pmatrix}\right.,
$$

$$
d_{l1}=m_1\left\{\begin{pmatrix}\dfrac{\epsilon_1}{2}d+\dfrac{\epsilon_2}{2}\breve{d}_1+\dfrac{\epsilon_2}{2}\breve{d}_2\\ \text{same}\left\{\begin{pmatrix}\epsilon_1 d\\ \vdots\\ \epsilon_1 d\end{pmatrix}\right.\end{pmatrix}\right.,
$$

$$d_{l2} = m_2 \left\{ \begin{pmatrix} \frac{\epsilon_2}{2}\breve{d}_1 \\ \text{same} \left\{ \begin{pmatrix} \epsilon_2 \breve{d}_1 \\ \vdots \\ \epsilon_2 \breve{d}_1 \end{pmatrix} \right. \end{pmatrix} \right. ,$$

$$d_{l3} = m_2 \left\{ \begin{pmatrix} \frac{\epsilon_2}{2}\breve{d}_2 \\ \text{same} \left\{ \begin{pmatrix} \epsilon_2 \breve{d}_2 \\ \vdots \\ \epsilon_2 \breve{d}_2 \end{pmatrix} \right. \end{pmatrix} \right. . \tag{47}$$

Note that in g_{l1}, the difference between two adjacent items is $\epsilon_1 a$; in g_{l2}, the difference between two adjacent items is $\epsilon_2 a_1$; in g_{l3}, the difference between two adjacent items is $\epsilon_2 a_2$. In h_{l1}, the difference between two adjacent items is $\epsilon_1 c$; in h_{l2}, the difference between two adjacent items is $\epsilon_2 \check{c}_1$; in h_{l3}, the difference between two adjacent items is $\epsilon_2 \check{c}_2$. All the items in f_{l1} are the same, and this is also true for f_{l2} and f_{l3}.

The dual of $\mathrm{SCP}_1(\pi, \epsilon_1, \epsilon_2)$ *without* $(\epsilon_1/2) d' \hat{x}_0$ in the objective of $\mathrm{SCP}_1(\pi(\epsilon_1, \epsilon_2))$ is the following problem:

$$\begin{aligned}
&\left(\mathrm{SCP}_1^* \left(\pi\left(\epsilon_1, \epsilon_2\right)\right)\right): \\
&\min \quad g_{l1}' \hat{f} + f_{l1}' \hat{g} + g_{l2}' \hat{h} \\
&\qquad + f_{l2}' \hat{l} + g_{l3}' \hat{p} + f_{l3}' \hat{q} \\
&\text{s.t.} \quad \widehat{G}_{m_1}' \hat{f} + \widehat{H}_{m_1}' \hat{g} + \widehat{G}_{m_1,m_2}' \left(\hat{h} + \hat{p}\right) \\
&\qquad - h_{l1} \in \mathcal{K}_{3,m_1}^*, \\
&\qquad \widehat{F}_{m_1}' \hat{f} - d_{l1} \in \mathcal{K}_{4,m_1}^*, \\
&\qquad \hat{f} \in \mathcal{K}_{1,m_1}^*, \quad \hat{g} \in \mathcal{K}_{2,m_1}^*, \\
&\qquad \widehat{G}_{m_2}' \hat{h} + \widehat{H}_{m_2}' \hat{l} - h_{l2} \in \mathcal{K}_{31,m_2}^*, \\
&\qquad \widehat{F}_{m_2}' \hat{h} - d_{l2} \in \mathcal{K}_{41,m_2}^*, \\
&\qquad \hat{h} \in \mathcal{K}_{11,m_2}^*, \quad \hat{l} \in \mathcal{K}_{21,m_2}^*, \\
&\qquad \widehat{G}_{m_2}' \hat{p} + \widehat{H}_{m_2}' \hat{q} - h_{l3} \in \mathcal{K}_{32,m_2}^*, \\
&\qquad \widehat{F}_{m_2}' \hat{p} - d_{l3} \in \mathcal{K}_{42,m_2}^*, \\
&\qquad \hat{p} \in \mathcal{K}_{12,m_2}^*, \quad \hat{q} \in \mathcal{K}_{22,m_2}^*.
\end{aligned} \tag{48}$$

Similarly, we have

$$\begin{aligned}
&\left(\mathrm{SCP}_2 \left(\pi' \left(\epsilon_1, \epsilon_2\right)\right)\right) \\
&\min \quad g_{u2}' \hat{h} + f_{u2}' \hat{l} + \frac{\epsilon_2}{2} b_1' \hat{l}_0 + g_{u3}' \hat{p} \\
&\qquad + f_{u3}' \hat{q} + \frac{\epsilon_2}{2} b_2' \hat{q}_0 + g_{u1}' \hat{f} + f_{u1}' \hat{g} \\
&\text{s.t.} \quad \widehat{G}_{m_2}' \hat{h} + \widehat{H}_{m_2}' \hat{l} - h_{u2} \in \mathcal{K}_{31,m_2}^*, \\
&\qquad \widehat{F}_{m_2}' \hat{h} - d_{u2} \in \mathcal{K}_{41,m_2}^*, \\
&\qquad \hat{f} \in \mathcal{K}_{1,m_1}^*, \quad \hat{g} \in \mathcal{K}_{2,m_1}^*, \\
&\qquad \widehat{G}_{m_2}' \hat{p} + \widehat{H}_{m_2}' \hat{q} - h_{u3} \in \mathcal{K}_{32,m_2}^*, \\
&\qquad \widehat{F}_{m_2}' \hat{p} - d_{u3} \in \mathcal{K}_{42,m_2}^*, \\
&\qquad \hat{h} \in \mathcal{K}_{11,m_2}^*, \quad \hat{l} \in \mathcal{K}_{21,m_2}^*, \\
&\qquad \widehat{G}_{m_1,m_2}' \left(\hat{h} + \hat{p}\right) + \widehat{G}_{m_1}' \hat{f} \\
&\qquad + \widehat{H}_{m_1}' \hat{g} - h_{u1} \in \mathcal{K}_{3,m_1}^*, \\
&\qquad \widehat{F}_{m_1}' \hat{f} - d_{u1} \in \mathcal{K}_{4,m_1}^*, \\
&\qquad \hat{p} \in \mathcal{K}_{12,m_2}^*, \quad \hat{q} \in \mathcal{K}_{22,m_2}^*,
\end{aligned} \tag{49}$$

where

$$\widehat{G}_{m_1}' = m_1 \left\{ \begin{pmatrix} G' & & & \\ G' & G' & & \\ & & \ddots & \\ G' & G' & \cdots & G' \end{pmatrix} \right. ,$$

$$\widehat{G}_{m_2}' = m_2 \left\{ \begin{pmatrix} G' & & & \\ G' & G' & & \\ & & \ddots & \\ G' & G' & \cdots & G' \end{pmatrix} \right. ,$$

$$\widehat{G}_{m_1,m_2}' = m_1 \left\{ \underbrace{\begin{pmatrix} G' & G' & \cdots & G' \\ G' & G' & \cdots & G' \\ \vdots & \vdots & & \vdots \\ G' & G' & \cdots & G' \end{pmatrix}}_{m_2} \right. ,$$

$$f_{u1} = m_1 \left\{ \begin{pmatrix} \text{same} \left\{ \begin{pmatrix} \epsilon_1 b \\ \vdots \\ \epsilon_1 b \end{pmatrix} \right. \\ \frac{\epsilon_1}{2} b \end{pmatrix} \right. ,$$

$$f_{u2} = m_2 \left\{ \begin{pmatrix} \text{same} \left\{ \begin{pmatrix} \epsilon_2 b_1 \\ \vdots \\ \epsilon_2 b_1 \end{pmatrix} \right. \\ \frac{\epsilon_2}{2} b_1 + \frac{\epsilon_1}{2} b \end{pmatrix} \right. ,$$

$$f_{u3} = m_2 \left\{ \begin{pmatrix} \text{same} \left\{ \begin{pmatrix} \epsilon_2 b_2 \\ \vdots \\ \epsilon_2 b_2 \end{pmatrix} \right. \\ \frac{\epsilon_2}{2} b_2 + \frac{\epsilon_1}{2} b \end{pmatrix} \right. ,$$

$$
\begin{gathered}
g_{u1} = m_1 \left\{ \begin{pmatrix} \alpha + \left(T_1 - \frac{\epsilon_1}{2}\right) a \\ \alpha + \left(T_1 - \frac{3\epsilon_1}{2}\right) a \\ \vdots \\ \alpha + \frac{\epsilon_1}{2} a \end{pmatrix} \right., \\
g_{u2} = m_2 \left\{ \begin{pmatrix} \alpha_1 + \left(T - \frac{\epsilon_2}{2}\right) a_1 \\ \alpha_1 + \left(T - \frac{3\epsilon_2}{2}\right) a_1 \\ \vdots \\ \alpha_1 + \left(T_1 + \frac{\epsilon_2}{2}\right) a_1 \end{pmatrix} \right., \\
g_{u3} = m_2 \left\{ \begin{pmatrix} \alpha_2 + \left(T - \frac{\epsilon_2}{2}\right) a_2 \\ \alpha_2 + \left(T - \frac{3\epsilon_2}{2}\right) a_2 \\ \vdots \\ \alpha_2 + \left(T_1 + \frac{\epsilon_2}{2}\right) a_2 \end{pmatrix} \right., \\
h_{u1} = m_1 \left\{ \begin{pmatrix} \gamma + \left(T - T_1 + \epsilon_1\right) c \\ \gamma + \left(T - T_1 + 2\epsilon_1\right) c \\ \vdots \\ \gamma + Tc \end{pmatrix} \right., \\
h_{u2} = m_2 \left\{ \begin{pmatrix} \breve{\gamma}_1 + \epsilon_2 \breve{c}_1 \\ \breve{\gamma}_1 + 2\epsilon_2 \breve{c}_1 \\ \vdots \\ \breve{\gamma}_1 + \left(T - T_1\right) \breve{c}_1 \end{pmatrix} \right., \\
h_{u3} = m_2 \left\{ \begin{pmatrix} \breve{\gamma}_2 + \epsilon_2 \breve{c}_2 \\ \breve{\gamma}_2 + 2\epsilon_2 \breve{c}_2 \\ \vdots \\ \breve{\gamma}_2 + \left(T - T_1\right) \breve{c}_2 \end{pmatrix} \right., \\
d_{u1} = m_1 \left\{ \begin{pmatrix} \epsilon_1 d \\ \epsilon_1 d \\ \vdots \\ \epsilon_1 d \end{pmatrix} \right., \qquad d_{u2} = m_2 \left\{ \begin{pmatrix} \epsilon_2 \breve{d}_1 \\ \epsilon_2 \breve{d}_1 \\ \vdots \\ \epsilon_2 \breve{d}_1 \end{pmatrix} \right., \\
d_{u3} = m_2 \left\{ \begin{pmatrix} \epsilon_2 \breve{d}_2 \\ \epsilon_2 \breve{d}_2 \\ \vdots \\ \epsilon_2 \breve{d}_2 \end{pmatrix} \right..
\end{gathered}
\tag{50}
$$

Note that in g_{u1}, the difference between two adjacent items is $\epsilon_1 a$; in g_{u2}, the difference between two adjacent items is $\epsilon_2 a_1$; in g_{u3}, the difference between two adjacent items is $\epsilon_2 a_2$. In h_{u1}, the difference between two adjacent items is $\epsilon_1 c$; in h_{u2}, the difference between two adjacent items is $\epsilon_2 \breve{c}_1$; in h_{u3}, the difference between two adjacent items is $\epsilon_2 \breve{c}_2$. All the items in d_{u1} are the same, and this is also true for d_{u2} and d_{u3}.

Now we write down the relationships between the input parameters in $\mathrm{SCP}_1(\pi(\epsilon_1, \epsilon_2))$ and $\mathrm{SCP}_2(\pi'(\epsilon_1, \epsilon_2))$, and these following relationships will be used in proving Theorem 11 in Section 5:

$$
\begin{gathered}
g_{u1} - g_{l1} = m_1, \qquad \text{same} \left\{ \begin{pmatrix} -\frac{\epsilon_1}{2} a \\ \vdots \\ -\frac{\epsilon_1}{2} a \end{pmatrix} \right., \\
g_{u2} - g_{l2} = m_2, \qquad \text{same} \left\{ \begin{pmatrix} -\frac{\epsilon_2}{2} a_1 \\ \vdots \\ -\frac{\epsilon_2}{2} a_1 \end{pmatrix} \right., \\
g_{u3} - g_{l3} = m_2, \qquad \text{same} \left\{ \begin{pmatrix} -\frac{\epsilon_2}{2} a_2 \\ \vdots \\ -\frac{\epsilon_2}{2} a_2 \end{pmatrix} \right., \\
f_{u1} - f_{l1} = m_1, \qquad \left\{ \begin{pmatrix} \text{same} \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right. \\ -\frac{\epsilon_1}{2} b \end{pmatrix} \right., \\
f_{u2} - f_{l2} = m_2, \qquad \left\{ \begin{pmatrix} \text{same} \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right. \\ -\frac{\epsilon_2}{2} b_1 + \frac{\epsilon_1}{2} b \end{pmatrix} \right., \\
f_{u3} - f_{l3} = m_2, \qquad \left\{ \begin{pmatrix} \text{same} \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right. \\ -\frac{\epsilon_2}{2} b_2 + \frac{\epsilon_1}{2} b \end{pmatrix} \right., \\
h_{u1} - h_{l1} = m_1, \qquad \text{same} \left\{ \begin{pmatrix} \frac{\epsilon_1}{2} c \\ \vdots \\ \frac{\epsilon_1}{2} c \end{pmatrix} \right., \\
h_{u2} - h_{l2} = m_2, \qquad \text{same} \left\{ \begin{pmatrix} \frac{\epsilon_2}{2} \breve{c}_1 \\ \vdots \\ \frac{\epsilon_2}{2} \breve{c}_1 \end{pmatrix} \right., \\
h_{u3} - h_{l3} = m_2, \qquad \text{same} \left\{ \begin{pmatrix} \frac{\epsilon_2}{2} \breve{c}_2 \\ \vdots \\ \frac{\epsilon_2}{2} \breve{c}_2 \end{pmatrix} \right.,
\end{gathered}
$$

$$d_{u1}-d_{l1}=m_1\left\{\begin{pmatrix}\frac{\epsilon_1}{2}d-\frac{\epsilon_2}{2}\check{d}_1-\frac{\epsilon_2}{2}\check{d}_2\\ \text{same}\left\{\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\right.\end{pmatrix}\right.,$$

$$d_{u2}-d_{l2}=m_2\left\{\begin{pmatrix}\frac{\epsilon_2}{2}\check{d}_1\\ \text{same}\left\{\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\right.\end{pmatrix}\right.,$$

$$d_{u3}-d_{l3}=m_2\left\{\begin{pmatrix}\frac{\epsilon_2}{2}\check{d}_2\\ \text{same}\left\{\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\right.\end{pmatrix}\right.. \tag{51}$$

5. Strong Duality

In this section, we will prove that under some mild and verifiable conditions, strong duality holds between SSCCP and its dual.

Let π_1 denote a partition of $[0,T-T_1,T]$, $\pi_1=\{t_0,t_1,\ldots,t_{m_2},t_{m_2+1},\ldots,t_{m_2+m_1+2}\}$, with $t_0=0$, $t_{m_2+1}=T-T_1$, $t_{m_2+m_1+2}=T$, and $t_1-t_0=t_{m_2+1}-t_{m_2}=(\epsilon_2/2)$, $t_{m_2+2}-t_{m_2+1}=t_{m_2+m_1+2}-t_{m_2+m_1+1}=(\epsilon_1/2)$, while $t_i-t_{i-1}=\epsilon_2$, $i=2,\ldots,m_2$; $t_j-t_{j-1}=\epsilon_1$, $j=m_2+3,\ldots,m_2+m_1+1$.

Lemma 7. *If the conic programs $SCP_2(\pi_1)$ are strongly feasible and $\mathscr{K}_1\subseteq\mathscr{K}_{11}$, $\mathscr{K}_1\subseteq\mathscr{K}_{12}$, $\mathscr{K}_4\subseteq\mathscr{K}_{41}$, $\mathscr{K}_4\subseteq\mathscr{K}_{42}$, then $SCP_1^*(\pi(\epsilon_1,\epsilon_2))$ is strongly feasible.*

Proof. The constraints of $(\mathrm{SCP}_2(\pi_1))$ are:

$$\begin{gathered}
\begin{pmatrix}\widehat{G}'_{m_2} & \\ G'\ldots G' & G'\end{pmatrix}\widehat{h}+\begin{pmatrix}\widehat{H}'_{m_2} & \\ & H'\end{pmatrix}\widehat{l}-\begin{pmatrix}h_{l2}\\ \check{\gamma}_1+(T-T_1)\check{c}_1\end{pmatrix}\in\mathscr{K}^*_{31,m_2+1},\\
\begin{pmatrix}\widehat{F}'_{m_2} & \\ & F'\end{pmatrix}\widehat{h}-\begin{pmatrix}d_{l2}\\ \frac{\epsilon_2}{2}\check{d}_1\end{pmatrix}\in\mathscr{K}^*_{41,m_2+1},\\
\widehat{h}\in\mathscr{K}^*_{11,m_2+1},\quad \widehat{l}\in\mathscr{K}^*_{21,m_2+1},\\
\begin{pmatrix}\widehat{G}'_{m_2} & \\ G'\ldots G' & G'\end{pmatrix}\widehat{p}+\begin{pmatrix}\widehat{H}'_{m_2} & \\ & H'\end{pmatrix}\widehat{q}-\begin{pmatrix}h_{l3}\\ \check{\gamma}_2+(T-T_1)\check{c}_2\end{pmatrix}\in\mathscr{K}^*_{32,m_2+1},\\
\begin{pmatrix}\widehat{F}'_{m_2} & \\ & F'\end{pmatrix}\widehat{p}-\begin{pmatrix}d_{l3}\\ \frac{\epsilon_2}{2}\check{d}_2\end{pmatrix}\in\mathscr{K}^*_{42,m_2+1},\\
\widehat{p}\in\mathscr{K}^*_{12,m_2+1},\quad \widehat{q}\in\mathscr{K}^*_{22,m_2+1},\\
\begin{pmatrix} & G'\\ \widehat{G}'_{m_1,m_2} & \vdots\\ & G'\\ G'\cdots G' & G'\end{pmatrix}(\widehat{h}+\widehat{p})+\begin{pmatrix}\widehat{G}'_{m_1} & \\ G'\cdots G' & G'\end{pmatrix}\widehat{f}+\begin{pmatrix}\widehat{H}'_{m_1} & \\ & H'\end{pmatrix}\widehat{g}-\begin{pmatrix}h_{l1}\\ \gamma+Tc\end{pmatrix}\in\mathscr{K}^*_{3,m_1+1},\\
\begin{pmatrix}\widehat{F}'_{m_1} & \\ & F'\end{pmatrix}\widehat{f}-\begin{pmatrix}\widehat{d}_{l1}-\begin{pmatrix}\frac{\epsilon_2}{2}\check{d}_1+\frac{\epsilon_2}{2}\check{d}_2\\ 0\\ \vdots\\ 0\end{pmatrix}\\ \frac{\epsilon_1}{2}d\end{pmatrix}\in\mathscr{K}^*_{4,m_1+1},\\
\widehat{f}\in\mathscr{K}^*_{1,m_1+1},\quad \widehat{g}\in\mathscr{K}^*_{2,m_1+1}.
\end{gathered}\tag{52}$$

When $\mathscr{K}_1\subseteq\mathscr{K}_{11}$, $\mathscr{K}_1\subseteq\mathscr{K}_{12}$ and $\mathscr{K}_4\subseteq\mathscr{K}_{41}$, $\mathscr{K}_4\subseteq\mathscr{K}_{42}$, we have $\mathscr{K}^*_{11}\subseteq\mathscr{K}^*_1$, $\mathscr{K}^*_{12}\subseteq\mathscr{K}^*_1$, $\mathscr{K}^*_{41}\subseteq\mathscr{K}^*_4$, $\mathscr{K}^*_{42}\subseteq\mathscr{K}^*_4$. Comparing the above with the constraints of $\mathrm{SCP}_1^*(\pi(\epsilon_1,\epsilon_2))$, we observe that if

$$\begin{gathered}\left(\left(\widehat{h}_1,\ldots,\widehat{h}_{m_2+1}\right),\left(\widehat{l}_1,\ldots,\widehat{l}_{m_2+1}\right),\left(\widehat{p}_1,\ldots,\widehat{p}_{m_2+1}\right),\right.\\ \left.\left(\widehat{q}_1,\ldots,\widehat{q}_{m_2+1}\right),\left(\widehat{f}_1,\ldots,\widehat{f}_{m_1+1}\right),\left(\widehat{g}_1,\ldots,\widehat{g}_{m_1+1}\right)\right)\end{gathered}\tag{53}$$

is a strongly feasible solution of $\mathrm{SCP}_2(\pi_1)$, then

$$\begin{gathered}\left(\left(\widehat{h}_1,\ldots,\widehat{h}_{m_2}\right),\left(\widehat{l}_1,\ldots,\widehat{l}_{m_2}\right),\left(\widehat{p}_1,\ldots,\widehat{p}_{m_2}\right),\left(\widehat{q}_1,\ldots,\widehat{q}_{m_2}\right),\right.\\ \left.\left(\widehat{f}_1+\widehat{h}_{m_2+1}+\widehat{p}_{m_2+1},\widehat{f}_2,\ldots,\widehat{f}_{m_1}\right),\left(\widehat{g}_1,\ldots,\widehat{g}_{m_1}\right)\right)\end{gathered}\tag{54}$$

is a strongly feasible solution to $\mathrm{SCP}_1^*(\pi(\epsilon_1,\epsilon_2))$. □

From Lemma 5, we know if the conic programs CP_1 are strongly feasible and $\mathscr{K}_1\subseteq\mathscr{K}_{11}$, $\mathscr{K}_1\subseteq\mathscr{K}_{12}$, $\mathscr{K}_4\subseteq\mathscr{K}_{41}$ and $\mathscr{K}_4\subseteq\mathscr{K}_{42}$, then $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ is strongly feasible.

From Lemma 6, we know if the conic programs CP_2 are strongly feasible and $\mathscr{K}_2\subseteq\mathscr{K}_{21}$, $\mathscr{K}_2\subseteq\mathscr{K}_{22}$, $\mathscr{K}_3\subseteq\mathscr{K}_{31}$, $\mathscr{K}_3\subseteq\mathscr{K}_{32}$, $\check{\gamma}_1+\check{\gamma}_2=\gamma$ and $\check{c}_1+\check{c}_2=c$, $\mathrm{SCP}_2(\pi')$ is strongly feasible. Now, from Lemma 7, if additionally $\mathscr{K}_1\subseteq\mathscr{K}_{11}$, $\mathscr{K}_1\subseteq\mathscr{K}_{12}$ and $\mathscr{K}_4\subseteq\mathscr{K}_{41}$, $\mathscr{K}_4\subseteq\mathscr{K}_{42}$, then $\mathrm{SCP}_1^*(\pi(\epsilon_1,\epsilon_2))$ is strongly feasible.

So under the condition that the conic programs (CP_1) and (CP_2) are strongly feasible, $\mathscr{K}_1\subseteq\mathscr{K}_{11}$, $\mathscr{K}_1\subseteq\mathscr{K}_{12}$, $\mathscr{K}_2\subseteq\mathscr{K}_{21}$, $\mathscr{K}_2\subseteq\mathscr{K}_{22}$, $\mathscr{K}_3\subseteq\mathscr{K}_{31}$, $\mathscr{K}_3\subseteq\mathscr{K}_{32}$, $\mathscr{K}_4\subseteq\mathscr{K}_{41}$, $\mathscr{K}_4\subseteq\mathscr{K}_{42}$, $\check{\gamma}_1+\check{\gamma}_2=\gamma$, $\check{c}_1+\check{c}_2=c$, $(\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2)))$ is solvable.

The dual of $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$ *without* the constant terms $(\epsilon_2/2)b_1'\widehat{l}_0 + (\epsilon_2/2)b_2'\widehat{q}_0$ in the objective is the following:

$$\begin{aligned}
&\left(\mathrm{SCP}_2^*\left(\pi'\left(\epsilon_1,\epsilon_2\right)\right)\right)\\
&\quad \max \quad h_{u1}'\widehat{u} + d_{u1}'\widehat{x} + h_{u2}'\widehat{v}\\
&\qquad\qquad + d_{u2}'\widehat{y} + h_{u3}'\widehat{w} + d_{u3}'\widehat{z}\\
&\quad \text{s.t.} \quad g_{u1} - \widehat{G}_{m_1}\widehat{u} - \widehat{F}_{m_1}\widehat{x} \in \mathcal{K}_{1,m_1},\\
&\qquad\quad f_{u1} - \widehat{H}_{m_1}\widehat{u} \in \mathcal{K}_{2,m_1},\\
&\qquad\quad g_{u2} - \widehat{G}_{m_2,m_1}\widehat{u} - \widehat{G}_{m_2}\widehat{v} - \widehat{F}_{m_2}\widehat{y} \in \mathcal{K}_{11,m_2},\\
&\qquad\quad f_{u2} - \widehat{H}_{m_2}\widehat{v} \in \mathcal{K}_{21,m_2},\\
&\qquad\quad g_{u3} - \widehat{G}_{m_2,m_1}\widehat{u} - \widehat{G}_{m_2}\widehat{w} - \widehat{F}_{m_2}\widehat{z} \in \mathcal{K}_{12,m_2},\\
&\qquad\quad f_{u3} - \widehat{H}_{m_2}\widehat{w} \in \mathcal{K}_{22,m_2},\\
&\qquad\quad \widehat{u} \in \mathcal{K}_{3,m_1}, \quad \widehat{x} \in \mathcal{K}_{4,m_1},\\
&\qquad\quad \widehat{v} \in \mathcal{K}_{31,m_2}, \quad \widehat{y} \in \mathcal{K}_{41,m_2},\\
&\qquad\quad \widehat{w} \in \mathcal{K}_{32,m_2}, \quad \widehat{z} \in \mathcal{K}_{42,m_2}.
\end{aligned} \tag{55}$$

We use π_2 to denote the partition $\{t_0, t_1, t_2, \ldots, t_{m_1+1}, \ldots, t_{m_1+m_2+2}\}$, with $t_0 = 0$, $t_{m_1+1} = T_1$, $t_{m_1+m_2+2} = T$, and $t_1 - t_0 = t_{m_1+1} - t_{m_1} = \epsilon_1/2$, $t_{m_1+2} - t_{m_1+1} = t_{m_1+m_2+2} - t_{m_1+m_2+1} = \epsilon_2/2$, while $t_i - t_{i-1} = \epsilon_1$, $i = 2, \ldots, m_1$; $t_j - t_{j-1} = \epsilon_2$, $j = m_1+3, \ldots, m_1+m_2+1$.

Lemma 8. *If the conic programs $\mathrm{SCP}_1(\pi_2)$ are strongly feasible and $\mathcal{K}_2 \subseteq \mathcal{K}_{21}, \mathcal{K}_2 \subseteq \mathcal{K}_{22}, \mathcal{K}_3 \subseteq \mathcal{K}_{31}$ and $\mathcal{K}_3 \subseteq \mathcal{K}_{32}$, then $\mathrm{SCP}_2^*(\pi'(\epsilon_1,\epsilon_2))$ is strongly feasible.*

Proof. If we reverse the inner order of $\widehat{u}, \widehat{x}, \widehat{v}, \widehat{y}, \widehat{w}, \widehat{z}$, the constraints of $\mathrm{SCP}_1(\pi_2)$ are:

$$\begin{gathered}
\begin{pmatrix} \alpha + T_1 a \\ g_{u1} \end{pmatrix} - \begin{pmatrix} G & G\cdots G \\ & \widehat{G}_{m_1} \end{pmatrix}\widehat{u} - \begin{pmatrix} F & \\ & \widehat{F}_{m_1} \end{pmatrix}\widehat{x} \in \mathcal{K}_{1,m_1+1},\\
\begin{pmatrix} \dfrac{\epsilon_1}{2}b \\ f_{u1} \end{pmatrix} - \begin{pmatrix} H & \\ & \widehat{H}_{m_1} \end{pmatrix}\widehat{u} \in \mathcal{K}_{2,m_1+1},\\
\begin{pmatrix} \alpha_1 + Ta_1 \\ g_{u2} \end{pmatrix} - \begin{pmatrix} G & G\cdots G \\ G & \\ \vdots & \widehat{G}_{m_2,m_1} \\ G & \end{pmatrix}\widehat{u} - \begin{pmatrix} G & G\cdots G \\ & \widehat{G}_{m_2} \end{pmatrix}\widehat{v}\\
- \begin{pmatrix} F & \\ & \widehat{F}_{m_2} \end{pmatrix}\widehat{y} \in \mathcal{K}_{11,m_2+1},\\
\begin{pmatrix} \dfrac{\epsilon_2}{2}b_1 \\ f_{u2} - \begin{pmatrix} 0 \\ \vdots \\ \dfrac{\epsilon_1}{2}b \end{pmatrix} \end{pmatrix} - \begin{pmatrix} H & \\ & \widehat{H}_{m_2} \end{pmatrix}\widehat{v} \in \mathcal{K}_{21,m_2+1},\\
\begin{pmatrix} \alpha_2 + Ta_2 \\ g_{u3} \end{pmatrix} - \begin{pmatrix} G & G\cdots G \\ G & \\ \vdots & \widehat{G}_{m_2,m_1} \\ G & \end{pmatrix}\widehat{u} - \begin{pmatrix} G & G\cdots G \\ & \widehat{G}_{m_2} \end{pmatrix}\widehat{w}\\
- \begin{pmatrix} F & \\ & \widehat{F}_{m_2} \end{pmatrix}\widehat{z} \in \mathcal{K}_{12,m_2+1},\\
\begin{pmatrix} \dfrac{\epsilon_2}{2}b_2 \\ f_{u3} - \begin{pmatrix} 0 \\ \vdots \\ \dfrac{\epsilon_1}{2}b \end{pmatrix} \end{pmatrix} - \begin{pmatrix} H & \\ & \widehat{H}_{m_2} \end{pmatrix}\widehat{w} \in \mathcal{K}_{22,m_2+1},\\
\widehat{u} \in \mathcal{K}_{3,m_1+1}, \quad \widehat{x} \in \mathcal{K}_{4,m_1+1}, \quad \widehat{v} \in \mathcal{K}_{31,m_2+1},\\
\widehat{y} \in \mathcal{K}_{41,m_2+1}, \quad \widehat{w} \in \mathcal{K}_{32,m_2+1}, \quad \widehat{z} \in \mathcal{K}_{42,m_2+1}.
\end{gathered} \tag{56}$$

When $\mathcal{K}_2 \subseteq \mathcal{K}_{21}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{22}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{31}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{32}$, comparing the above with the constraints of $\mathrm{SCP}_2^*(\pi'(\epsilon_1,\epsilon_2))$, we observe that if

$$\begin{aligned}
&\left(\left(\widehat{u}_1, \ldots, \widehat{u}_{m_1+1}\right), \left(\widehat{x}_1, \ldots, \widehat{x}_{m_1+1}\right), \left(\widehat{v}_1, \ldots, \widehat{v}_{m_2+1}\right),\right.\\
&\left.\left(\widehat{y}_1, \ldots, \widehat{y}_{m_2+1}\right), \left(\widehat{w}_1, \ldots, \widehat{w}_{m_2+1}\right), \left(\widehat{z}_1, \ldots, \widehat{z}_{m_2+1}\right)\right)
\end{aligned} \tag{57}$$

is a strongly feasible solution of $\mathrm{SCP}_1(\pi_2)$, then

$$\begin{aligned}
&\left(\left(\widehat{u}_1, \ldots, \widehat{u}_{m_1}\right), \left(\widehat{x}_1, \ldots, \widehat{x}_{m_1}\right), \left(\widehat{v}_1 + \widehat{u}_{m_1+1}, \widehat{v}_2, \ldots, \widehat{v}_{m_2}\right),\right.\\
&\left(\widehat{y}_2, \ldots, \widehat{y}_{m_2+1}\right), \left(\widehat{w}_1 + \widehat{u}_{m_1+1}, \widehat{w}_2, \ldots, \widehat{w}_{m_2}\right),\\
&\left.\left(\widehat{z}_2, \ldots, \widehat{z}_{m_2+1}\right)\right)
\end{aligned} \tag{58}$$

is a strongly feasible solution to $\mathrm{SCP}_2^*(\pi'(\epsilon_1,\epsilon_2))$. □

From Lemma 5, we know that if the conic program (CP_1) is strongly feasible, and $\mathcal{K}_1 \subseteq \mathcal{K}_{11}$, $\mathcal{K}_1 \subseteq \mathcal{K}_{12}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{41}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{42}$, then $\mathrm{SCP}_1(\pi_2)$ is strongly feasible. From Lemma 8, if additionally, $\mathcal{K}_2 \subseteq \mathcal{K}_{21}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{22}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{31}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{32}$, we have $\mathrm{SCP}_2^*(\pi'(\epsilon_1,\epsilon_2))$ is strongly feasible.

From Lemma 6, we know that if the conic program (CP_2) is strongly feasible and $\mathcal{K}_2 \subseteq \mathcal{K}_{21}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{22}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{31}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{32}$, $\check{\gamma}_1 + \check{\gamma}_2 = \gamma$ and $\check{c}_1 + \check{c}_2 = c$, then $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$ is strongly feasible.

So under the condition that the conic programs CP_1 and CP_2 are strongly feasible and $\mathcal{K}_1 \subseteq \mathcal{K}_{11}$, $\mathcal{K}_1 \subseteq \mathcal{K}_{12}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{21}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{22}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{31}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{32}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{41}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{42}$, $\check{\gamma}_1 + \check{\gamma}_2 = \gamma$, $\check{c}_1 + \check{c}_2 = c$, $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$ is solvable.

Now we have the following.

Lemma 9. *If the conic programs CP_1 and CP_2 are strongly feasible and $\mathcal{K}_1 \subseteq \mathcal{K}_{11}$, $\mathcal{K}_1 \subseteq \mathcal{K}_{12}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{21}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{22}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{31}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{32}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{41}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{42}$, $\breve{\gamma}_1+\breve{\gamma}_2 = \gamma$, $\breve{c}_1 + \breve{c}_2 = c$, both $SCP_1(\pi(\epsilon_1,\epsilon_2))$ and $SCP_2(\pi'(\epsilon_1,\epsilon_2))$ are solvable, that is, they have optimal solutions and their optimal objective values are finite.*

Proposition 10. *Suppose that $SCP_1(\pi(\epsilon_1,\epsilon_2))$ and $SCP_2(\pi'(\epsilon_1,\epsilon_2))$ are solvable. Then, one has*

$$\begin{aligned} v\left(SCP_1\left(\pi\left(\epsilon_1,\epsilon_2\right)\right)\right) &\le v\left(SSCCP\right) \le v\left(SSCCP^*\right) \\ &\le v\left(SCP_2\left(\pi'\left(\epsilon_1,\epsilon_2\right)\right)\right). \end{aligned} \tag{59}$$

Proof. Because $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ and $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$ are solvable, they have finite optimal objective values $v(\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2)))$ and $v(\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2)))$.

From Lemma 2, we know the optimal solution of $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ can be extended to a feasible solution of SSCCP, hence, the first inequality (since SSCCP is a maximization problem). A similar argument justifies the third inequality. The second inequality follows from the weak duality in Proposition 1. □

Theorem 11. *Suppose CP_1 and CP_2 are strongly feasible, with finite optimal values. $\mathcal{K}_1 \subseteq \mathcal{K}_{11}$, $\mathcal{K}_1 \subseteq \mathcal{K}_{12}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{21}$, $\mathcal{K}_2 \subseteq \mathcal{K}_{22}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{31}$, $\mathcal{K}_3 \subseteq \mathcal{K}_{32}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{41}$, $\mathcal{K}_4 \subseteq \mathcal{K}_{42}$, $\breve{\gamma}_1 + \breve{\gamma}_2 = \gamma$ and $\breve{c}_1 + \breve{c}_2 = c$. If one lets the number of intervals in π, π', m_1 and m_2 be the same and both of them equal to m, then there exists a constant $\Gamma > 0$, which is independent of m, such that*

$$v\left(SCP_2\left(\pi'\left(\epsilon_1,\epsilon_2\right)\right)\right) - v\left(SCP_1\left(\pi\left(\epsilon_1,\epsilon_2\right)\right)\right) \le \frac{\Gamma}{m}. \tag{60}$$

Consequently, one must have $v(SSCCP) = v(SSCCP^*)$; that is, strong duality holds.

Proof. First note that strong duality follows immediately from the inequality in (60) by letting $m \to \infty$, taking into account the inequalities in Proposition 10.

To establish the error bound in (60), consider the following primal-dual pair of conic programs:

$$\begin{aligned} (\mathrm{SCP}) \quad \max \quad & h_{u1}'\hat{u} + d_{u1}'\hat{x} + h_{u2}'\hat{v} + d_{u2}'\hat{y} \\ & + h_{u3}'\hat{w} + d_{u3}'\hat{z} \\ \text{s.t.} \quad & g_{l1} - \widehat{G}_{m_1}\hat{u} - \widehat{F}_{m_1}\hat{x} \in \mathcal{K}_{1,m_1}, \\ & f_{l1} - \widehat{H}_{m_1}\hat{u} \in \mathcal{K}_{2,m_1}, \\ & g_{l2} - \widehat{G}_{m_2,m_1}\hat{u} - \widehat{G}_{m_2}\hat{v} - \widehat{F}_{m_2}\hat{y} \in \mathcal{K}_{11,m_2}, \\ & f_{l2} - \widehat{H}_{m_2}\hat{v} \in \mathcal{K}_{21,m_2}, \\ & g_{l3} - \widehat{G}_{m_2,m_1}\hat{u} - \widehat{G}_{m_2}\hat{w} - \widehat{F}_{m_2}\hat{z} \in \mathcal{K}_{12,m_2}, \\ & f_{l3} - \widehat{H}_{m_2}\hat{w} \in \mathcal{K}_{22,m_2}, \\ & \hat{u} \in \mathcal{K}_{3,m_1}, \quad \hat{x} \in \mathcal{K}_{4,m_1}, \quad \hat{v} \in \mathcal{K}_{31,m_2}, \\ & \hat{y} \in \mathcal{K}_{41,m_2}, \quad \hat{w} \in \mathcal{K}_{32,m_2}, \quad \hat{z} \in \mathcal{K}_{42,m_2}, \end{aligned} \tag{61}$$

$$\begin{aligned} (\mathrm{SCP}^*) \quad \min \quad & g_{l2}'\hat{h} + f_{l2}'\hat{l} + g_{l3}'\hat{p} + f_{l3}'\hat{q} \\ & + g_{l1}'\hat{f} + f_{l1}'\hat{g} \\ \text{s.t.} \quad & \widehat{G}_{m_2}'\hat{h} + \widehat{H}_{m_2}'\hat{l} - h_{u2} \in \mathcal{K}_{31,m_2}^*, \\ & \widehat{F}_{m_2}'\hat{h} - d_{u2} \in \mathcal{K}_{41,m_2}^*, \\ & \widehat{G}_{m_2}'\hat{p} + \widehat{H}_{m_2}'\hat{q} - h_{u3} \in \mathcal{K}_{32,m_2}^*, \\ & \widehat{F}_{m_2}'\hat{p} - d_{u3} \in \mathcal{K}_{42,m_2}^*, \\ & \widehat{G}_{m_1,m_2}'\left(\hat{h} + \hat{p}\right) + \widehat{G}_{m_1}'\hat{f} + \widehat{H}_{m_1}'\hat{g} \\ & - h_{u1} \in \mathcal{K}_{3,m_1}^*, \\ & \widehat{F}_{m_1}'\hat{f} - d_{u1} \in \mathcal{K}_{4,m_1}^*, \\ & \hat{f} \in \mathcal{K}_{1,m_1}^*, \quad \hat{g} \in \mathcal{K}_{2,m_1}^*, \quad \hat{h} \in \mathcal{K}_{11,m_2}^*, \\ & \hat{l} \in \mathcal{K}_{21,m_2}^*, \quad \hat{p} \in \mathcal{K}_{12,m_2}^*, \quad \hat{q} \in \mathcal{K}_{22,m_2}^*. \end{aligned} \tag{62}$$

Note that the problem in (62) has the same constraints as $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$ but the objective function of $\mathrm{SCP}_1^*(\pi(\epsilon_1,\epsilon_2))$, whereas the problem in (8) has the constraints of $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ but the objective function of $\mathrm{SCP}_2^*(\pi'(\epsilon_1,\epsilon_2))$. Hence, both primal and dual are strongly feasible, since $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ and $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$ are. Consequently they both have optimal solutions and their respective optimal objective values coincide.

We denote the optimal solutions for SCP as $(\hat{u}^*, \hat{x}^*, \hat{v}^*, \hat{y}^*, \hat{w}^*, \hat{z}^*)$ and the optimal solution for SCP^* as $(\hat{h}^*, \hat{l}^*, \hat{p}^*, \hat{q}^*, \hat{f}^*, \hat{g}^*)$. Note that these are feasible solutions to $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ and $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$, respectively. Hence, we have

$$\begin{aligned} v\left(\mathrm{SCP}_1\left(\pi\left(\epsilon_1,\epsilon_2\right)\right)\right) &\ge h_{l1}'\hat{u}^* + d_{l1}'\hat{x}^* + h_{l_2}'\hat{v}^* \\ &\quad + d_{l_2}'\hat{y}^* + h_{l_3}'\hat{w}^* + d_{l_3}'\hat{z}^* + \frac{\epsilon_1}{2}d'\hat{x}_0, \\ v\left(\mathrm{SCP}_2\left(\pi'\left(\epsilon_1,\epsilon_2\right)\right)\right) &\le g_{u2}'\hat{h}^* + f_{u2}'\hat{l}^* + \frac{\epsilon_2}{2}b_1'\hat{l}_0 + g_{u3}'\hat{p}^* \\ &\quad + f_{u3}'\hat{q}^* + \frac{\epsilon_2}{2}b_2'\hat{q}_0 + g_{u1}'\hat{f}^* + f_{u1}'\hat{g}^*. \end{aligned} \tag{63}$$

Hence,

$$\begin{aligned} & v\left(\mathrm{SCP}_2\left(\pi'\left(\epsilon_1,\epsilon_2\right)\right)\right) - v\left(\mathrm{SCP}_1\left(\pi\left(\epsilon_1,\epsilon_2\right)\right)\right) \\ &\quad \le g_{u2}'\hat{h}^* + f_{u2}'\hat{l}^* + \frac{\epsilon_2}{2}b_1'\hat{l}_0 + g_{u3}'\hat{p}^* + f_{u3}'\hat{q}^* \\ &\quad + \frac{\epsilon_2}{2}b_2'\hat{q}_0 + g_{u1}'\hat{f}^* + f_{u1}'\hat{g}^* \end{aligned}$$

$$
\begin{aligned}
&-\Big(h_{l1}'\widehat{u}^* + d_{l1}'\widehat{x}^* + h_{l2}'\widehat{v}^* + d_{l2}'\widehat{y}^* + h_{l3}'\widehat{w}^* \\
&\qquad + d_{l3}'\widehat{z}^* + \frac{\epsilon_1}{2}d'\widehat{x}_0\Big) \\
&= \left[\widehat{g}_{l2} - \begin{pmatrix} \frac{\epsilon_2}{2}a_1 \\ \frac{\epsilon_2}{2}a_1 \\ \vdots \\ \frac{\epsilon_2}{2}a_1 \end{pmatrix}\right]'\widehat{h}^* \\
&\quad + \left[\widehat{f}_{l2} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\epsilon_1}{2}b - \frac{\epsilon_2}{2}b_1 \end{pmatrix}\right]'\widehat{l}^* \\
&\quad + \left[\widehat{g}_{l3} - \begin{pmatrix} \frac{\epsilon_2}{2}a_2 \\ \frac{\epsilon_2}{2}a_2 \\ \vdots \\ \frac{\epsilon_2}{2}a_2 \end{pmatrix}\right]'\widehat{p}^* \\
&\quad + \left[\widehat{f}_{l3} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\epsilon_1}{2}b - \frac{\epsilon_2}{2}b_2 \end{pmatrix}\right]'\widehat{q}^* \\
&\quad + \left[\widehat{g}_{l1} - \begin{pmatrix} \frac{\epsilon_1}{2}a \\ \frac{\epsilon_1}{2}a \\ \vdots \\ \frac{\epsilon_1}{2}a \end{pmatrix}\right]'\widehat{f}^* \\
&\quad + \left[\widehat{f}_{l1} - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\epsilon_1}{2}b \end{pmatrix}\right]'\widehat{g}^* + \frac{\epsilon_2}{2}\left(b_1'\widehat{l}_0 + b_2'\widehat{q}_0\right) \\
&\quad - \left\{\left[\widehat{h}_{u1} - \begin{pmatrix} \frac{\epsilon_1}{2}c \\ \frac{\epsilon_1}{2}c \\ \vdots \\ \frac{\epsilon_1}{2}c \end{pmatrix}\right]'\widehat{u}^*\right. \\
&\quad + \left[\widehat{d}_{u1} - \begin{pmatrix} -\frac{\epsilon_2}{2}(\breve{d}_1 + \breve{d}_2) + \frac{\epsilon_1}{2}d \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]'\widehat{x}^* \\
&\quad + \left[\widehat{h}_{u2} - \begin{pmatrix} \frac{\epsilon_2}{2}\breve{c}_1 \\ \frac{\epsilon_2}{2}\breve{c}_1 \\ \vdots \\ \frac{\epsilon_2}{2}\breve{c}_1 \end{pmatrix}\right]'\widehat{v}^* \\
&\quad + \left[\widehat{d}_{u2} - \begin{pmatrix} \frac{\epsilon_2}{2}\breve{d}_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]'\widehat{y}^* \\
&\quad + \left[\widehat{h}_{u3} - \begin{pmatrix} \frac{\epsilon_2}{2}\breve{c}_2 \\ \frac{\epsilon_2}{2}\breve{c}_2 \\ \vdots \\ \frac{\epsilon_2}{2}\breve{c}_2 \end{pmatrix}\right]'\widehat{w}^* \\
&\quad \left. + \left[\widehat{d}_{u3} - \begin{pmatrix} \frac{\epsilon_2}{2}\breve{d}_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]'\widehat{z}^* + \frac{\epsilon_1}{2}d'x_0\right\} \\
&= -\frac{\epsilon_2}{2}a_1'\left(\sum_{j=1}^{m_2}\widehat{h}_j^*\right) + \left(\frac{\epsilon_1}{2}b - \frac{\epsilon_2}{2}b_1\right)'\widehat{l}_{m_2}^* \\
&\quad - \frac{\epsilon_2}{2}a_2'\left(\sum_{j=1}^{m_2}\widehat{p}_j^*\right) + \left(\frac{\epsilon_1}{2}b - \frac{\epsilon_2}{2}b_2\right)'\widehat{q}_{m_2}^* \\
&\quad - \frac{\epsilon_1}{2}a'\left(\sum_{j=1}^{m_1}\widehat{f}_j^*\right) - \frac{\epsilon_1}{2}b'\widehat{g}_{m_1}^* + \frac{\epsilon_2}{2}\left(b_1'\widehat{b} + b_2'\widehat{q}_0\right) \\
&\quad - \left[-\frac{\epsilon_1}{2}c'\left(\sum_{j=1}^{m_1}\widehat{u}_j^*\right)\right. \\
&\qquad + \left(-\frac{\epsilon_2}{2}\left(\breve{d}_1 + \breve{d}_2\right) + \frac{\epsilon_1}{2}d\right)'\widehat{x}_{m_1}^* \\
&\qquad - \frac{\epsilon_2}{2}\breve{c}_1'\left(\sum_{j=1}^{m_2}\widehat{v}_j^*\right) - \frac{\epsilon_2}{2}\breve{d}_1'\widehat{y}_{m_2}^* \\
&\qquad \left. - \frac{\epsilon_2}{2}\breve{c}_2'\left(\sum_{j=1}^{m_2}\widehat{w}_j^*\right) - \frac{\epsilon_2}{2}\breve{d}_2'\widehat{z}_{m_2}^* + \frac{\epsilon_1}{2}d'\widehat{x}_0\right].
\end{aligned}
\tag{64}
$$

If $m_1 = m_2 = m$, then $\epsilon_1 = T_1/m$, $\epsilon_2 = (T - T_1)/m$. Hence, we have

$$\begin{aligned}
&\nu\left(\mathrm{SCP}_2\left(\pi'\left(\epsilon_1,\epsilon_2\right)\right)\right) - \nu\left(\mathrm{SCP}_1\left(\pi\left(\epsilon_1,\epsilon_2\right)\right)\right) \\
&\quad\le \frac{1}{2m}\Bigg[-\left(T-T_1\right)a_1'\left(\sum_{j=1}^{m}\widehat{h}_j^*\right)+\left(T_1 b-\left(T-T_1\right)b_1\right)'\widehat{l}_m^* \\
&\qquad - \left(T-T_1\right)a_2'\left(\sum_{j=1}^{m}\widehat{p}_j^*\right)+\left(T_1 b-\left(T-T_1\right)b_2\right)'\widehat{q}_m^* \\
&\qquad - T_1 a'\left(\sum_{j=1}^{m}\widehat{f}_j^*\right)-T_1 b'\widehat{g}_m^*+\left(T-T_1\right)\left(b_1'\widehat{l}_0+b_2'\widehat{q}_0\right) \\
&\qquad + T_1 c'\left(\sum_{j=1}^{m}\widehat{u}_j^*\right)-\left(-\left(T-T_1\right)\left(\check{d}_1+\check{d}_2\right)+T_1 d\right)'\widehat{x}_m^* \\
&\qquad + \left(T-T_1\right)\check{c}_1'\left(\sum_{j=1}^{m}\widehat{v}_j^*\right)+\left(T-T_1\right)\check{d}_1'\widehat{y}_m^* \\
&\qquad + \left(T-T_1\right)\check{c}_2'\left(\sum_{j=1}^{m}\widehat{w}_j^*\right)+\left(T-T_1\right)\check{d}_2'\widehat{z}_m^*-T_1 d'\widehat{x}_0\Bigg].
\end{aligned} \tag{65}$$

From the primal feasibility of $(\widehat{u}^*, \widehat{x}^*, , \widehat{v}^*, \widehat{y}^*, \widehat{w}^*, \widehat{z}^*)$, we have

$$\begin{aligned}
&\alpha + T_1 a - G\left(\sum_{i=1}^{m}\widehat{u}_i^*\right) - F\widehat{x}_m^* \in \mathscr{K}_1, \\
&T_1 b - H\left(\sum_{i=1}^{m}\widehat{u}_i^*\right) \in \mathscr{K}_2, \\
&\alpha_1 + T a_1 - G\left(\sum_{i=1}^{m}\widehat{u}_i^*\right) - G\left(\sum_{i=1}^{m}\widehat{v}_i^*\right) - F\widehat{y}_m^* \in \mathscr{K}_{11}, \\
&\left(T-T_1\right) b_1 - H\left(\sum_{i=1}^{m}\widehat{v}_i^*\right) \in \mathscr{K}_{21}, \\
&\alpha_2 + T a_2 - G\left(\sum_{i=1}^{m}\widehat{u}_i^*\right) - G\left(\sum_{i=1}^{m}\widehat{w}_i^*\right) - F\widehat{z}_m^* \in \mathscr{K}_{12}, \\
&\left(T-T_1\right) b_2 - H\left(\sum_{i=1}^{m}\widehat{w}_i^*\right) \in \mathscr{K}_{22}, \\
&\sum_{i=1}^{m}\widehat{u}_i^* \in \mathscr{K}_3, \quad \widehat{x}_m^* \in \mathscr{K}_4, \quad \sum_{i=1}^{m}\widehat{v}_i^* \in \mathscr{K}_{31}, \\
&\widehat{y}_m^* \in \mathscr{K}_{41}, \quad \sum_{i=1}^{m}\widehat{w}_i^* \in \mathscr{K}_{32}, \quad \widehat{z}_m^* \in \mathscr{K}_{42}.
\end{aligned} \tag{66}$$

So we know that $(\sum_{i=1}^{m}\widehat{u}_i^*, \widehat{x}_m^*, \sum_{i=1}^{m}\widehat{v}_i^*, \widehat{y}_m^*, \sum_{i=1}^{m}\widehat{w}_i^*, \widehat{z}_m^*)$ is a feasible solution to CP_1. Thus,

$$\begin{aligned}
&T_1 c'\sum_{i=1}^{m}\widehat{u}_i^* - \left(-\left(T-T_1\right)\left(\check{d}_1+\check{d}_2\right)+T_1 d\right)'\widehat{x}_m^* \\
&\qquad + \left(T-T_1\right)\check{c}_1'\sum_{i=1}^{m}\widehat{v}_i^* + \left(T-T_1\right)\check{d}_1'\widehat{y}_m^* \\
&\qquad + \left(T-T_1\right)\check{c}_2'\sum_{i=1}^{m}\widehat{w}_i^* + \left(T-T_1\right)\check{d}_2'\widehat{z}_m^* \\
&\quad \le \nu\left(\mathrm{CP}_1\right).
\end{aligned} \tag{67}$$

Similarly, from the dual feasibility of $(\widehat{h}^*, \widehat{l}^*, \widehat{p}^*, \widehat{q}^*, \widehat{f}^*, \widehat{g}^*)$, we have

$$\begin{aligned}
&G'\left(\sum_{i=1}^{m}\widehat{h}_i^*\right) + H'\widehat{l}_m^* - \left(\check{\gamma}_1 + \left(T-T_1\right)\check{c}_1\right) \in \mathscr{K}_{31}^*, \\
&F'\left(\sum_{i=1}^{m}\widehat{h}_i^*\right) - \left(T-T_1\right)\check{d}_1 \in \mathscr{K}_{41}^*, \\
&G'\left(\sum_{i=1}^{m}\widehat{p}_i^*\right) + H'\widehat{q}_m^* - \left(\check{\gamma}_2 + \left(T-T_1\right)\check{c}_2\right) \in \mathscr{K}_{32}^*, \\
&F'\left(\sum_{i=1}^{m}\widehat{p}_i^*\right) - \left(T-T_1\right)\check{d}_2 \in \mathscr{K}_{42}^*, \\
&G'\left(\sum_{i=1}^{m}\widehat{h}_i^* + \sum_{i=1}^{m}\widehat{p}_i^*\right) + G'\left(\sum_{i=1}^{m}\widehat{f}_i^*\right) \\
&\qquad + H'\widehat{g}_m^* - \left(\gamma + Tc\right) \in \mathscr{K}_3^*, \\
&F'\left(\sum_{i=1}^{m}\widehat{f}_i^*\right) - T_1 d \in \mathscr{K}_4^*, \\
&\sum_{i=1}^{m}\widehat{h}_i^* \in \mathscr{K}_{11}^*, \quad \widehat{l}_m^* \in \mathscr{K}_{21}^*, \quad \sum_{i=1}^{m}\widehat{p}_i^* \in \mathscr{K}_{12}^*, \\
&\widehat{q}_m^* \in \mathscr{K}_{22}^*, \quad \sum_{i=1}^{m}\widehat{f}_i^* \in \mathscr{K}_1^*, \quad \widehat{g}_m^* \in \mathscr{K}_2^*.
\end{aligned} \tag{68}$$

Hence, $(\sum_{i=1}^{m}\widehat{h}_i^*, \widehat{l}_m^*, \sum_{i=1}^{m}\widehat{p}_i^*, \widehat{q}_m^*, \sum_{i=1}^{m}\widehat{f}_i^*, \widehat{g}_m^*)$ is a feasible solution to CP_2, and

$$\begin{aligned}
&\left(T-T_1\right)a_1'\left(\sum_{j=1}^{m}\widehat{h}_j^*\right) - \left(T_1 b - \left(T-T_1\right)b_1\right)'\widehat{l}_m^* \\
&\qquad + \left(T-T_1\right)a_2'\left(\sum_{j=1}^{m}\widehat{p}_j^*\right) - \left(T_1 b - \left(T-T_1\right)b_2\right)'\widehat{q}_m^* \\
&\qquad + T_1 a'\left(\sum_{j=1}^{m}\widehat{f}_j^*\right) + T_1 b'\widehat{g}_m^* \\
&\quad \ge \nu\left(\mathrm{CP}_2\right).
\end{aligned} \tag{69}$$

Putting the above together, we have

$$\begin{aligned} & v\left(\mathrm{SCP}_2\left(\pi'\left(\epsilon_1,\epsilon_2\right)\right)\right) - v\left(\mathrm{SCP}_1\left(\pi\left(\epsilon_1,\epsilon_2\right)\right)\right) \\ & \quad \leq \frac{1}{2m}\left[-v\left(\mathrm{CP}_2\right) + v\left(\mathrm{CP}_1\right) + \left(T - T_1\right)\right. \\ & \qquad \left. \times \left(b_1'\hat{l}_0 + b_2'\hat{q}_0\right) - T_1 d'\hat{x}_0\right]. \end{aligned} \tag{70}$$

Hence, we can let

$$\begin{aligned} \Gamma := \frac{1}{2}&\left[-v\left(\mathrm{CP}_2\right) + v\left(\mathrm{CP}_1\right) + \left(T - T_1\right)\right. \\ & \left. \times \left(b_1'\hat{l}_0 + b_2'\hat{q}_0\right) - T_1 d'\hat{x}_0\right], \end{aligned} \tag{71}$$

and $\Gamma < \infty$, since $v(\mathrm{CP}_1) < \infty$ and $v(\mathrm{CP}_2) < \infty$ as assumed. □

6. The Approximation Algorithm

Proposition 10 and Theorem 11 suggest that we can solve SSCCP and their dual approximately through solving their discretized versions, the ordinary conic program $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ and $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$. The latter is readily solvable by standard algorithms, for example, SeDuMi [27], and the (discrete) solution can then be extended into the piecewise-constant control and piecewise-linear state variables as a feasible solution to SSCCP. Furthermore, the explicit error bound in (60) means that we can achieve any required accuracy by partitions $\pi(\epsilon_1,\epsilon_2)$ and $\pi'(\epsilon_1,\epsilon_2)$ with a sufficiently large number m to construct the discretized conic programs $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ and $\mathrm{SCP}_2(\pi'(\epsilon_1,\epsilon_2))$. Specifically, if δ is the required accuracy, then we can choose

$$m = \left\lceil \frac{\Gamma}{\delta} \right\rceil, \tag{72}$$

where (refer to the end of the proof of Theorem 11)

$$\begin{aligned} \Gamma := \frac{1}{2}&\left[-v\left(CP_2\right) + v\left(CP_1\right) + \left(T - T_1\right)\right. \\ & \left. \times \left(b_1'\hat{l}_0 + b_2'\hat{q}_0\right) - T_1 d'\hat{x}_0\right]. \end{aligned} \tag{73}$$

Then, from (59) and (60), we have

$$\begin{aligned} & v\left(\mathrm{SSCCP}^*\right) - v\left(\mathrm{SSCCP}\right) \leq v\left(\mathrm{SCP}_2\left(\pi'\left(\epsilon_1,\epsilon_2\right)\right)\right) \\ & \quad - v\left(\mathrm{SCP}_1\left(\pi\left(\epsilon_1,\epsilon_2\right)\right)\right) \leq \frac{T\Gamma}{m} \leq \delta. \end{aligned} \tag{74}$$

That is, the duality gap is guaranteed to be no greater than δ.

To select m following (72), we need to first derive Γ. This involves solving the two conic programs CP_1 and CP_2. In addition, we also need to determine $\hat{x}_0$, $\hat{l}_0$, and $\hat{q}_0$. This can be accomplished by solving the following three conic programs:

$$\begin{aligned} \max \quad & d'\hat{x}_0 \\ \text{s.t.} \quad & \alpha - F\hat{x}_0 \in \mathcal{K}_1, \\ & \hat{x}_0 \in \mathcal{K}_4, \\ \min \quad & b'\hat{l}_0 \\ \text{s.t.} \quad & H'\hat{l}_0 - \check{\gamma}_1 \in \mathcal{K}_{31}^*, \\ & \hat{l}_0 \in \mathcal{K}_{21}^*, \\ \min \quad & b'\hat{q}_0 \\ \text{s.t.} \quad & H'\hat{q}_0 - \check{\gamma}_2 \in \mathcal{K}_{32}^*, \\ & \hat{q}_0 \in \mathcal{K}_{22}^*. \end{aligned} \tag{75}$$

Note that the constraints of the above three problems originate from (14) and (27). Clearly, maximizing $d'\hat{x}_0$ and minimizing both $b'\hat{l}_0$ and $b'\hat{q}_0$ improve our estimation of the error bound.

In summary, our algorithm amounts to solving six conic programming problems: CP_1, CP_2, (75), and $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$. Conic programs are known to be polynomially solvable. Hence, ours is a polynomial-time algorithm.

Of course, with m increases, the computational burden in terms of solving the discretized problems increases. However, the discretized problems are all ordinary conic programming problem and they are polynomially solvable. There exist softwares (e.g., SeCuMi [27], CVXOPT [28], etc.) which can solve conic programming problems efficiently. So the increased computational burden does not really pose a problem in this algorithm.

7. Conclusion and Future Work

In this paper, we have developed a duality theory for SSCCP, which is an important extension on SCCP. Specifically, we have shown that the strong duality between SSCCP and its dual is implied by two related ordinary conic programs CP_1 and CP_2 being strongly feasible with finite optimal values. We have also developed a polynomial-time approximation algorithm that solves SSCCP to any desired accuracy with an easily computable error bound, based on the strong duality result.

All these results can be readily generalized for SSCCP with three or more stages and with finite number of scenarios in each stage, without essential difficulty.

From Theorem 11, we know that as $m \rightarrow \infty$, the duality gap tends to 0, and the optimal objective value of the discretized conic program $v(\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2)))$ approaches the optimal objective value of the original SSCCP. In the future, we plan to investigate whether the optimal solution to $\mathrm{SCP}_1(\pi(\epsilon_1,\epsilon_2))$ will also approach the optimal solution to SSCCP.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] J. R. Birge and F. Louveaux, *Introduction to Stochastic Programming*, Springer, New York, NY, USA, 1997.

[2] X. Wang, S. Zhang, and D. D. Yao, "Separated continuous conic programming: strong duality and an approximation algorithm," *SIAM Journal on Control and Optimization*, vol. 48, no. 4, pp. 2118–2138, 2009.

[3] A. Shapiro, D. Dentcheva, and A. Ruszczyński, *Lectures on Stochastic Programming: Modeling and Theory*, vol. 9 of *MOS/SIAM Series on Optimization*, SIAM, Philadelphia, Pa, USA, 2009.

[4] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, MPS/SIAM Series on Optimization, SIAM, Philadelphia, Pa, USA, 2001.

[5] R. Bellman, "Bottleneck problems and dynamic programming," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 39, pp. 947–951, 1953.

[6] R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, NJ, USA, 1957.

[7] E. J. Anderson, *A continuous model for job-shop scheduling [Ph.D. thesis]*, University of Cambridge, Cambridge, UK, 1978.

[8] E. J. Anderson, P. Nash, and A. F. Perold, "Some properties of a class of continuous linear programs," *SIAM Journal on Control and Optimization*, vol. 21, no. 5, pp. 758–765, 1983.

[9] E. J. Anderson and A. B. Philpott, "A continuous-time network simplex algorithm," *Networks*, vol. 19, no. 4, pp. 395–425, 1989.

[10] E. J. Anderson and A. B. Philpott, "On the solutions of a class of continuous linear programs," *SIAM Journal on Control and Optimization*, vol. 32, no. 5, pp. 1289–1296, 1994.

[11] E. J. Anderson and P. Nash, *Linear Programming in Infinite-Dimensional Spaces: Theory and Applications*, John Wiley & Sons, Chichester, UK, 1987.

[12] M. C. Pullan, "An algorithm for a class of continuous linear programs," *SIAM Journal on Control and Optimization*, vol. 31, no. 6, pp. 1558–1577, 1993.

[13] M. C. Pullan, "Forms of optimal solutions for separated continuous linear programs," *SIAM Journal on Control and Optimization*, vol. 33, no. 6, pp. 1952–1977, 1995.

[14] M. C. Pullan, "A duality theory for separated continuous linear programs," *SIAM Journal on Control and Optimization*, vol. 34, no. 3, pp. 931–965, 1996.

[15] M. C. Pullan, "Linear optimal control problems with piecewise analytic solutions," *Journal of Mathematical Analysis and Applications*, vol. 197, no. 1, pp. 207–226, 1996.

[16] M. C. Pullan, "A study of general dynamic network programs with arc time-delays," *SIAM Journal on Optimization*, vol. 7, no. 4, pp. 889–912, 1997.

[17] M. C. Pullan, "Convergence of a general class of algorithms for separated continuous linear programs," *SIAM Journal on Optimization*, vol. 10, no. 3, pp. 722–731, 2000.

[18] M. C. Pullan, "An extended algorithm for separated continuous linear programs," *Mathematical Programming*, vol. 93, no. 3, pp. 415–451, 2002.

[19] X. Luo and D. Bertsimas, "A new algorithm for state-constrained separated continuous linear programs," *SIAM Journal on Control and Optimization*, vol. 37, no. 1, pp. 177–210, 1999.

[20] A. Shapiro, "On duality theory of conic linear problems," in *Semi-Infinite Programming*, M. A. Goberna and M. A. Lopez, Eds., vol. 57, chapter 7, pp. 135–165, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[21] L. Fleischer and J. Sethuraman, "Efficient algorithms for separated continuous linear programs: the multicommodity flow problem with holding costs and extensions," *Mathematics of Operations Research*, vol. 30, no. 4, pp. 916–938, 2005.

[22] G. Weiss, "A simplex based algorithm to solve separated continuous linear programs," *Mathematical Programming*, vol. 115, no. 1, pp. 151–198, 2008.

[23] M. M. Nasrabadi, M. A. Yaghoobi, and M. Mashinchi, "Separated continuous linear programs with fuzzy valued objective function," *Scientia Iranica*, vol. 17, no. 2, pp. 105–118, 2010.

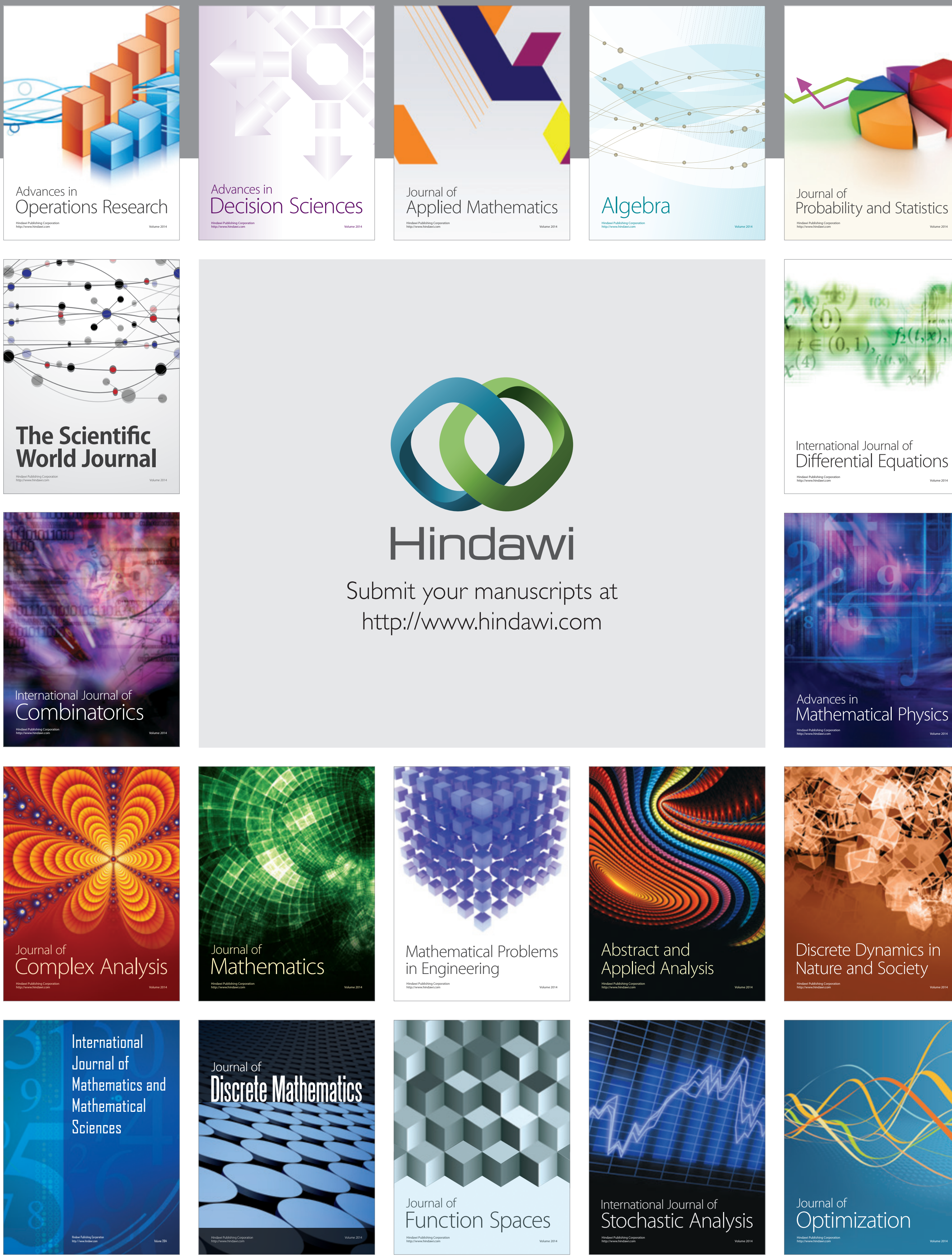
[24] X. Wang, "The duality theory for generalized separated continuous conic programming," in *Proceedings of the 4th International Symposium on Computational Intelligence and Design (ISCID '11)*, pp. 335–339, Hangzhou, China, October 2011.

[25] X. Wang, "An approximation algorithm for solving generalized separated continuous conic programming," *Advanced Materials Research*, vol. 452-453, pp. 1127–1132, 2012.

[26] V. Chvátal, *Linear Programming*, W. H. Freeman and Company, New York, NY, USA, 1983.

[27] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11-12, no. 1–4, pp. 625–653, 1999.

[28] 2013, http://cvxopt.org/index.html .


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization