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Research Article

Motion of Bishop Frenet Offsets of Ruled Surfaces in E^3

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The main goal of this paper is to study the motion of two associated ruled surfaces in Euclidean 3-space E^3. In particular, the motion of Bishop Frenet offsets of ruled surfaces is investigated. Additionally, the characteristic properties for such ruled surfaces are given. Finally, an application is presented and plotted using computer aided geometric design.

1. Introduction

Motion is to add the time element to our curves and surfaces. Motion theory has received a great deal of attention from mathematical physics, biology, dynamical systems, image processing, and computer vision. The problem is interesting since we may set two different subjects on the same theoretical basis. One is a geometrical interpretation of integrable systems. Connections between the differential geometry of curve motions and the integrable systems have been discussed. The analysis is extended to more general types of motion and other integrable systems [1–3]. The other is surface dynamics, the dynamics of shapes in physical and biological systems, as in crystal growth.

A variety of dynamics of shapes in physics, chemistry, and biology are modeled in terms of motion of surfaces and interfaces, and some dynamics of shapes are reduced to motion of curves. These models are specified by velocity fields or acceleration fields which are local or nonlocal functionals of the intrinsic quantities of curves. In physics, it is very interesting to describe motions of patterns such as interfaces, wave fronts, and defects [4]. Applications include kinematics of interfaces in crystal growth [5, 6], deformation of vortex filaments in inviscid fluid, and viscous fingering in a Hele-Shaw cell [7, 8]. The subject of how space curves or surfaces evolve in time is of great interest and has been investigated by many authors [9–22].

Classical differential geometry of the curves may be surrounded by the topics of general helices, involute-evolute curve couples, spherical curves, and Bertrand curves. Such special curves are investigated and used in some real world problems like mechanical design or robotics by the well-known Frenet-Serret equations because we think of curves as the path of a moving particle in the Euclidean space [23]. Thereafter researchers aimed to determine another moving frame for a regular curve. In 1975, Bishop pioneered "Bishop frame" by means of parallel vector fields. This special frame is also called a "parallel" or "alternative" frame of the curves [24].

A practical application of Bishop frame is that it is used in the area of biology and computer graphics. For example, it may be possible to compute information about the shape of sequences of DNA using a curve defined by Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animations [25]. Nowadays a good deal of research has been done on Bishop frames in Euclidean space [26, 27], in Minkowski space [28, 29], and in dual space [30]. Recently, the authors in [31] introduced a new version of the Bishop frame and called it a "type 2 Bishop frame" and this special frame is extended to study many surfaces [32, 33].

Studies related to offset profiles date back to the nineteenth century. Offsets curves play an important role in areas of CAD/CAM, robotics, cam design, and many industrial applications. In particular they are used in mathematical modeling of cutting paths milling machines. The classic work in this area is that of Bertrand [34], who studied curve pairs which have common principal normals. Such curves are referred to as Bertrand curves and can be considered as offsets of one another. The theory of the Mannheim curves has been extended in the three-dimensional Euclidean space by [35, 36].

Recently, there have been a number of studies of offsets ruled surfaces [37, 38], studied Bertrand and Mannheim offsets of ruled surfaces. Pottmann et al. [39] presented classical and circular offsets of rational ruled surfaces. More recently, Soliman et al. [40] studied geometric properties and invariants of Mannheim offsets of timelike ruled surface with timelike ruling.

The aim of this paper is to use the new version of type 2 Bishop frame which is studied in [23, 31, 41] and Frenet frame to construct offsets base curves of two ruled surfaces. Thus, the kinematics of such surfaces in terms of their intrinsic geometric formulas are established. An application of these surfaces and their motions is considered and plotted.

2. Geometry of Motion Curves and Surfaces in E^3

2.1. Motion of Curves. Let $\boldsymbol{\alpha} : I \to E^3$ be an arbitrary curve in E^3. Recall that the curve $\boldsymbol{\alpha}$ is said to be of unit speed if $\langle \boldsymbol{\alpha}', \boldsymbol{\alpha}' \rangle = 1$, where $\langle , \rangle$ is the standard scalar (inner) product of E^3. Denote by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ the moving Frenet frame along the unit speed curve $\boldsymbol{\alpha}$. Then the Frenet formulas are given by [42]

$$\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}. \tag{1}$$

Here, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are the tangent, the principal normal, and the binormal vector fields of the curve $\boldsymbol{\alpha}$, respectively. $\kappa(s)$ and $\tau(s)$ are called curvature and torsion of the curve $\boldsymbol{\alpha}$, respectively.

In the rest of the paper, we suppose everywhere that $\kappa \neq 0$ and $\tau \neq 0$.

Let $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^*(s)$ be a unit speed regular curve in E^3. The type 2 Bishop formulas of $\boldsymbol{\alpha}^*(s)$ are defined by [23, 31, 41]

$$\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{T}^*\\ \mathbf{N}^*\\ \mathbf{B}^*\end{bmatrix} = \begin{bmatrix}0 & 0 & -k_1\\ 0 & 0 & -k_2\\ k_1 & k_2 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}^*\\ \mathbf{N}^*\\ \mathbf{B}^*\end{bmatrix}. \tag{2}$$

Here, $\mathbf{T}^*$, $\mathbf{N}^*$, and $\mathbf{B}^*$ are the tangent, the principal normal, and the binormal vector fields of the curve $\boldsymbol{\alpha}^*$, respectively.

The Bishop frame or parallel transport frame is an alternative to the Frenet frame. Thus, the matrix relation between type 2 Bishop and Frenet-Serret frames can be expressed as

$$\begin{bmatrix}\mathbf{T}^*\\ \mathbf{N}^*\\ \mathbf{B}^*\end{bmatrix} = \begin{bmatrix}\sin\theta(s) & \cos\theta(s) & 0\\ -\cos\theta(s) & \sin\theta(s) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}. \tag{3}$$

Here, the type 2 Bishop curvatures are defined by

$$\begin{aligned} k_1(s) &= -\tau\cos\theta(s),\\ k_2(s) &= -\tau\sin\theta(s). \end{aligned} \tag{4}$$

It can be also deduced as

$$\theta' = \kappa = \frac{f'}{1+f^2}, \quad f = \frac{k_2}{k_1}. \tag{5}$$

The frame $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}^*\}$ is properly oriented, and τ and $\theta(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^*(s)$. We will call the set $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}^*, k_1, k_2\}$ type 2 Bishop invariants of the curve $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^*(s)$.

Using Frenet formulas (1) many geometries [1–3, 9, 15–19] studied connections between integrable evolution and the motion of curves in a 3-dimensional Euclidean space. They considered that $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s,t)$ denote a point on a space curve at the time t. The conventional geometrical model is specified by the velocity fields

$$\frac{\partial\boldsymbol{\alpha}}{\partial t} = v^1\mathbf{T} + v^2\mathbf{N} + v^3\mathbf{B}, \tag{6}$$

where $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are the unit tangent, normal, and binormal vectors along the curve and v^1, v^2, and v^3 are the tangential, normal, and binormal velocities, respectively. Velocity fields are functionals of the intrinsic quantities of curves, for example, curvature, κ, torsion τ, and their s derivatives.

The time evolution equations for Frenet frame $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are given by

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}0 & a_{12} & a_{13}\\ -a_{12} & 0 & a_{23}\\ -a_{13} & -a_{23} & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}, \tag{7}$$

where

$$\begin{aligned} a_{12} &= \left(\frac{\partial v^2}{\partial s} - \tau v^3 + \kappa v^1\right),\\ a_{13} &= \left(\frac{\partial v^3}{\partial s} + \tau v^2\right),\\ a_{23} &= \left(\frac{1}{\kappa}\frac{\partial a_{13}}{\partial s} + \frac{\tau}{\kappa}a_{12}\right). \end{aligned} \tag{8}$$

Using type 2 Bishop frame, Kiziltuğ [41] considered the flow of the curve $\boldsymbol{\alpha}^*$ as the following:

$$\frac{\partial\boldsymbol{\alpha}^*}{\partial t} = v^{*1}\mathbf{T}^* + v^{*2}\mathbf{N}^* + v^{*3}\mathbf{B}^*, \tag{9}$$

and in view of type 2 Bishop formulas (2), Kiziltuğ [41] obtained the time evolution equations for such frame as follows:

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{T}^* \\ \mathbf{N}^* \\ \mathbf{B}^*\end{bmatrix} = \begin{bmatrix}0 & a_{12}^* & a_{13}^* \\ -a_{12}^* & 0 & a_{23}^* \\ -a_{13}^* & -a_{23}^* & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}^* \\ \mathbf{N}^* \\ \mathbf{B}^*\end{bmatrix}, \quad (10)$$

where

$$\begin{aligned} a_{12}^* &= \left(\frac{\partial v^{*2}}{\partial s} + k_2 v^{*3}\right), \\ a_{13}^* &= \left(\frac{\partial v^{*3}}{\partial s} - k_1 v^{*1} - k_2 v^{*2}\right), \\ a_{23}^* &= \left(\frac{1}{k_1}\frac{\partial a_{13}^*}{\partial s} + \frac{k_2}{k_1}a_{12}^*\right). \end{aligned} \quad (11)$$

2.2. Motion of Surfaces. Here, and in the sequel, we assume that the indices $\{i, j, k, l, m\}$ run over the ranges $\{1, 2\}$. The Einstein summation convention will be used; that is, repeated indices, with one upper index and one lower index, denoted summation over its range.

Let our surface, moving in 3-dimensional Euclidean space E^3, be given at time t by the position vector

$$X = x^\gamma e_\gamma, \quad \gamma = 1, 2, 3, \quad (12)$$

where $x^\gamma = x^\gamma(u^i, t)$ are the Cartesian coordinates in some fixed in time Cartesian frame e_γ and u^i are convective curvilinear coordinates. Then, the two tangent vectors and the unit normal vector to the surface are given by

$$\begin{aligned} E_i &= X_{,i}, \quad i = \frac{\partial}{\partial u^i}, \\ E_3 &= \frac{E_1 \times E_2}{\sqrt{g}}, \end{aligned} \quad (13)$$

respectively. Thus, the metric g_{ij} and the coefficients of the second fundamental form h_{ij} are given by

$$\begin{aligned} g_{ij} &= \left\langle E_i, E_j \right\rangle, \\ g &= \operatorname{Det}\left(g_{ij}\right), \\ h_{ij} &= \left\langle E_3, E_{i,j}\right\rangle = \left\langle E_3, X_{,ij}\right\rangle, \end{aligned} \quad (14)$$

where $\langle , \rangle$ is the Euclidean inner product.

Thus, the Gaussian curvature G and the mean curvature H are given by

$$\begin{aligned} G &= \frac{\operatorname{Det}\left(h_{ij}\right)}{\operatorname{Det}\left(g_{ij}\right)}, \\ H &= \frac{1}{2}\operatorname{tr}\left(g^{ij}h_{jk}\right), \end{aligned} \quad (15)$$

respectively, where (g^{kl}) is the associated contravariant metric tensor field of the covariant metric tensor field (g_{kl}); that is, $g^{ik}g_{jk} = \delta^i_j$.

As one moves along the surface (at a fixed time), the tangent and normal vectors change according to the Gauss-Weingarten equations,

$$\frac{\partial}{\partial u^j}\begin{bmatrix}E_1 \\ E_2 \\ \vdots \\ E_3\end{bmatrix} = \begin{bmatrix}\Gamma_{ij}^k & \vdots & h_{ij} \\ \cdots & \vdots & \cdots \\ h_j^k & \vdots & 0\end{bmatrix}\begin{bmatrix}E_1 \\ E_2 \\ \vdots \\ E_3\end{bmatrix}, \quad h_j^k = g^{ki}h_{ij}, \quad (16)$$

where Γ_{ij}^k are called the Christoffel symbols of the 2nd kind, which are given as

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{il}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l}\right). \quad (17)$$

From the compatibility conditions of (16), we get the Gauss-Codazzi equations,

$$\begin{aligned} R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk}, \\ \nabla_i h_{jk} &= \nabla_j h_{ik}, \end{aligned} \quad (18)$$

where R_{ijkl} is the Riemann tensor and ∇_i is the covariant derivative,

$$\begin{aligned} \nabla_i f_j &= f_{j,i} - \Gamma_{ij}^k f_k, \\ \nabla_i f^j &= f^j_{,i} + \Gamma_{ki}^j f^k, \\ \nabla_i \nabla_j f &= f_{,ij} - \Gamma_{ij}^k f_{,k}. \end{aligned} \quad (19)$$

Nakayama et al. [11, 14, 21, 22] introduced the dynamics of the surface, where the velocity of the surface is expressed by

$$\begin{aligned} \frac{\partial X}{\partial t} &= V^\gamma E_\gamma, \\ V^\gamma &= V^\gamma\left(u^{\boldsymbol{\alpha}}, t\right), \\ &\gamma = 1, 2, 3, \end{aligned} \quad (20)$$

where V^i and V^3 are the tangential and the normal velocities, respectively.

Using (13), (16), and (20) we can obtain the time evolution equations for the local frame,

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{bmatrix}E_1 \\ E_2 \\ \vdots \\ E_3\end{bmatrix} \\ &= \begin{bmatrix}-V^3h_i^k + \nabla_i V^k & \vdots & V^3_{,i} + V^j h_{ij} \\ \cdots & \vdots & \cdots \\ -g^{ki}\left(V^3_{,i} + V^j h_{ij}\right) & \vdots & 0\end{bmatrix}\begin{bmatrix}E_1 \\ E_2 \\ \vdots \\ E_3\end{bmatrix}. \end{aligned} \quad (21)$$

Thus, using (13), (16), one can see that the time evolution equations for g_{ij} and h_{ij} are given by

$$\frac{\partial g_{ij}}{\partial t} = -2h_{ij}V^3 + \nabla_j V^i + \nabla_i V^j, \tag{22}$$

$$\frac{\partial h_{ij}}{\partial t} = \nabla_i \nabla_j V^3 - h_{ik}h_j^k V^3 + h_{ik}\nabla_j V^k + h_{jk}\nabla_i V^k + V^k \nabla_j h_{ik}, \tag{23}$$

respectively.

2.3. Bishop Frenet Offsets of Ruled Surfaces. In view of relation (3) and inspired by the concepts of Bertrand and Mannheim offsets of ruled surfaces [35–38], we can reformulate the following definitions of Bishop Frenet offsets for ruled surfaces.

A pair of curves $\boldsymbol{\alpha}^*$ and $\boldsymbol{\alpha}$ are said to be Bishop Frenet curves if there exists a one-to-one correspondence between their points such that both curves have a common binormal vector at their corresponding points ($\mathbf{B}^* = \mathbf{B}$). Such curves will be referred to as "Bishop Frenet offsets."

Thus, we can write the relation between the curves $\boldsymbol{\alpha}^*$ and $\boldsymbol{\alpha}$ as

$$\boldsymbol{\alpha}^* = \boldsymbol{\alpha} + \psi(s)\,\mathbf{B}, \tag{24}$$

where $\psi = \psi(s)$ is distance between corresponding points on the curves $\boldsymbol{\alpha}^*$ and $\boldsymbol{\alpha}$.

If we take the derivative of the above equation and adopt the relation $\mathbf{B}^* = \mathbf{B}$, we can see that $\psi' = 0$. On the other hand, from the distance function between two points, we have

$$d(\boldsymbol{\alpha}^*, \boldsymbol{\alpha}) = \|\boldsymbol{\alpha}^* - \boldsymbol{\alpha}\| = \|\psi(s)\,\mathbf{B}\| = \psi > 0. \tag{25}$$

Thus, we can say that ψ is a nonzero positive constant.

The ruled surface $\mathbf{X}^*$ is said to be Bishop Frenet offset of the ruled surface $\mathbf{X}$ if there exists a one-to-one correspondence between their rulings such that the binormal vector $\mathbf{B}$ of the base curve of $\mathbf{X}$ is the binormal vector $\mathbf{B}^*$ of the base curve of $\mathbf{X}^*$. In this case, $(\mathbf{X}, \mathbf{X}^*)$ is called a pair of Bishop Frenet ruled surfaces.

Thus, we can write the parametric representation of the ruled surfaces $\mathbf{X}$ and $\mathbf{X}^*$ as follows:

$$\mathbf{X}(s, v) = \boldsymbol{\alpha}(s) + v\mathbf{B}(s), \quad \|\mathbf{B}\| = 1, \ v \in R, \tag{26}$$

$$\mathbf{X}^*(s, v) = \boldsymbol{\alpha}^*(s) + v\mathbf{B}^*(s), \quad \|\mathbf{B}^*\| = 1, \ v \in R,$$
$$\text{or } \mathbf{X}^*(s, v) = \boldsymbol{\alpha}(s) + (\psi + v)\,\mathbf{B}(s), \quad \mathbf{B}^* = \mathbf{B}, \tag{27}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ are the base curves of $\mathbf{X}$ and $\mathbf{X}^*$, respectively.

3. Motion of Frenet Ruled Surface

In this section, the fundamental quantities g_{ij}, h_{ij} and their evolution of ruled surfaces (26) and (28) are obtained, respectively. Thus the Gaussian, mean curvatures, and their evolution of such surfaces are given. For this purpose, let a ruled surface generated by the binormal vector $\mathbf{B}$ of the Frenet frame, moving in 3-dimensional Euclidean space E^3, be given at time t by the parametrization [11]:

$$\overline{\mathbf{X}}(s, v, t) = \boldsymbol{\alpha}(s, t) + v\mathbf{B}(s, t), \tag{28}$$

where $\overline{\mathbf{X}}(s, v, 0) = \mathbf{X}(s, v)$, $\boldsymbol{\alpha}(s, 0) = \boldsymbol{\alpha}(s)$, and $\mathbf{B}(s, 0) = \mathbf{B}(s)$.

3.1. Curvatures of $\mathbf{X}$. From (26) and using Serret-Frenet formulas (1) it is easily checked that the coefficients of the first fundamental form g_{ij} of $\mathbf{X}$ are given by

$$g_{11} = \lambda, \quad g_{22} = 1, \quad g_{12} = 0, \quad g = \operatorname{Det}(g_{ij}) = \lambda, \tag{29}$$

where $\lambda = 1 + v^2\tau^2 > 0$.

Using (13), the unit normal vector field to the surface $\mathbf{X}$ is given by

$$E_3 = -\frac{1}{\sqrt{\lambda}}(v\tau\mathbf{T} + \mathbf{N}). \tag{30}$$

This leads to the coefficients of the second fundamental form h_{ij} of $\mathbf{X}$ given by

$$h_{11} = -\frac{1}{\sqrt{\lambda}}\left(\kappa - v\tau' + v^2\kappa\tau^2\right),$$
$$h_{22} = 0,$$
$$h_{12} = \frac{\tau}{\sqrt{\lambda}},$$
$$h = \operatorname{Det}(h_{ij}) = -\frac{\tau^2}{\lambda},$$
$$' = \frac{d}{ds}. \tag{31}$$

Thus, using (15) one can see that the Gaussian and mean curvature functions of $\mathbf{X}$ are given, respectively, by the following.

Lemma 1. *Consider*

$$G = -\frac{\tau^2}{\lambda^2}, \quad H = \frac{1}{2\lambda^{3/2}}\left(v\tau' - \lambda\kappa\right). \tag{32}$$

From (17) one can see that the Christoffel symbols Γ_{ij}^k of $\mathbf{X}$ are given by the following.

Lemma 2. *Consider*

$$\begin{aligned}\Gamma_{11}^{1} &= \frac{1}{\lambda}\left(v^{2}\tau\tau'\right),\\ \Gamma_{12}^{1} = \Gamma_{21}^{1} &= \frac{1}{\lambda}\left(v\tau^{2}\right),\\ \Gamma_{11}^{2} &= -v\tau^{2},\end{aligned} \tag{33}$$

and other components equal zero.

3.2. Curvatures' Evolution of $\overline{\mathbf{X}}$. Actually, here and in the sequel is a remarkable fact that when we calculate $\partial g_{ij}/\partial t$ and $\partial h_{ij}/\partial t$ of $\overline{\mathbf{X}}$, we have to compute the velocities V^{γ} of $\overline{\mathbf{X}}$. Thus, using (6), (7), and (20) with the assumption that velocities of the curve $\boldsymbol{\alpha}$ are $v^{1} = 0$, $v^{2} = \kappa$, and $v^{3} = \tau$, one can see that the tangential velocities V^{i} and the normal velocity V^{3} of $\overline{\mathbf{X}}$ are given by the following.

Corollary 3. *Consider*

$$\begin{aligned}V^{1} &= \frac{1}{\lambda}\left[v^{2}\tau\lambda_{1} - v\lambda_{2} - v\kappa\tau\right],\\ V^{2} &= \tau,\\ V^{3} &= \frac{1}{\sqrt{\lambda}}\left[v\lambda_{1} + v^{2}\tau\lambda_{2} - \kappa\right],\end{aligned} \tag{34}$$

where $\lambda_{1} = (1/\kappa)(2\tau\kappa' + \kappa\tau' + \tau'' - \tau^{3})$ *and* $\lambda_{2} = \kappa\tau + \tau'$.

Using (19), (33), and (34) one can obtain the following:

$$\begin{aligned}\nabla_{1}V^{1} &= \frac{v}{\kappa^{2}\lambda^{2}}\Big\{v\tau\kappa'\lambda_{3}\lambda_{4} - \kappa^{2}\Big[\tau^{3}\left(2v\kappa' - v^{3}\tau'' + 1\right)\\ &\quad + \tau\left(2\kappa' - v\left(\tau'' + 3v\tau'^{2}\right)\right) + \tau'' - v\tau'^{2}\\ &\quad + v^{2}\tau^{2}\left(\tau'' + 2v\tau'^{2}\right) + v^{2}\tau^{5}\Big]\\ &\quad + v\kappa\left[2\tau^{2}\left(\kappa'' - v^{2}\tau'\tau''\right) + 2v^{2}\tau^{4}\kappa''\right] + \tau^{3}\Big(v^{2}\tau^{(3)}\\ &\quad - 2\left(v^{2}\kappa' + 2\right)\tau'\Big) + \tau\left(4\kappa'\tau' + \tau^{(3)}\right) - v^{2}\tau^{5}\tau'\\ &\quad + 2\kappa^{3}\left(2v^{2}\tau^{2} - 1\right)\tau'\Big\},\\ \nabla_{1}V^{2} &= \frac{1}{\kappa\lambda^{2}}\Big\{v\tau\left(v^{2}\tau^{2} - 2\right)\lambda_{4} - \kappa\Big(v^{3}\tau^{3} - 2v^{2}\tau^{2}\\ &\quad - 2v\tau + 1\Big)\tau' + \kappa^{2}\left(4v^{2}\tau^{3} - 2\tau\right)\Big\},\\ \nabla_{2}V^{1} &= \frac{1}{\kappa\lambda}\left\{v^{3}\tau^{3}\lambda_{4} + \kappa\left(v^{3}\tau^{3} + 1\right)\tau' - 2v^{2}\kappa^{2}\tau^{3}\right\},\\ \nabla_{2}V^{2} &= 0,\end{aligned} \tag{35}$$

where $\lambda_{3} = 1 + v\tau^{2}$ *and* $\lambda_{4} = -2\tau\kappa' - \tau'' + \tau^{3}$.

Based on the above results, we have the following.

Corollary 4. *The evolution equations for the metric tensor* g_{ij} *of* $\overline{\mathbf{X}}$ *are given by*

$$\begin{aligned}\frac{\partial g_{11}}{\partial t} &= \frac{2}{\kappa^{2}\lambda^{2}}\Big\{v^{2}\tau\kappa'\lambda\lambda_{4} - v\kappa^{2}\Big[\tau^{3}\left(2v^{2}\kappa' - v^{3}\tau'' + 1\right)\\ &\quad + \tau\left(2\kappa' - v\left(\tau'' + 3v\tau'^{2}\right)\right) + \tau'' + v^{4}\tau^{4}\tau' - v\tau'^{2}\\ &\quad + \tau' + v^{2}\tau^{2}\left(\tau'' + 2v\tau'^{2} + 2\tau'\right) + v^{2}\tau^{5}\Big]\\ &\quad + v^{2}\kappa\Big[2\tau^{2}\left(\kappa'' - v^{2}\tau'\tau''\right) + 2v^{2}\tau^{4}\kappa''\\ &\quad + \tau^{3}\left(v^{2}\tau^{(3)} - 2\left(v^{2}\kappa' + 2\right)\tau'\right)\\ &\quad + \kappa^{3}\left(4v^{3}\tau^{2}\tau' - 2v\tau' + v^{6}\tau^{6} + 3v^{4}\tau^{4} + 3\lambda\tau\right)\Big]\Big\},\\ \frac{\partial g_{12}}{\partial t} &= \frac{\tau}{\kappa\lambda^{2}}\Big\{-v\left(v^{4}\tau^{4} + 2\right)\lambda_{4} + \kappa\Big(v^{5}\tau^{4}\tau'\\ &\quad + v^{2}\tau\left(3\tau' - 4\tau\right) + 2v\tau' - 2v^{4}\tau^{4} - 2\Big) - 2\kappa^{2}\Big(v^{4}\tau^{4}\\ &\quad - \lambda\Big)\Big\},\\ \frac{\partial g_{22}}{\partial t} &= 0.\end{aligned} \tag{36}$$

In view of (19), (31), and (34), one can obtain

$$\begin{aligned}\nabla_{1}\nabla_{1}V^{3} &= \frac{1}{\kappa^{3}\lambda^{5/2}}\Big\{v^{2}\kappa^{4}\Big[-v^{3}\tau'\tau^{5} + v^{4}\left(\tau\tau'' - \tau'^{2}\right)\tau^{4}\\ &\quad + 2v^{2}\left(2\tau\tau'' - 3\tau'^{2}\right)\tau^{2} - 3v\tau'\tau^{3} + 3\left(\tau'^{2} + \tau\tau''\right)\Big]\\ &\quad + \tau^{4}v^{6}\left[\kappa''\tau^{2} + \left(\kappa'\tau' + \tau^{(3)}\right)\tau - \tau'\tau''\right]\\ &\quad - \tau^{2}v^{5}\left[\kappa'\tau^{4} + \tau''\tau^{3} - \tau^{(3)}\tau^{2} + 4\tau'\tau''\tau - 3\tau'^{3}\right]\\ &\quad + \tau v^{4}\Big[\left(\kappa'' - \tau''\right)\tau^{3} + \left(\tau'^{2} + 8\kappa'\tau' + 2\tau^{(3)}\right)\tau^{2}\\ &\quad + 2\tau'\tau''\tau - 4\tau'^{3}\Big]\\ &\quad - v^{3}\left[\left(\tau''\tau^{3} + \left(\tau'^{2} - 2\tau^{(3)}\right)\tau^{2} + 4\tau'\tau''\tau + \tau'^{3}\right)\right]\Big\},\\ \nabla_{1}\nabla_{2}V^{3} = \nabla_{2}\nabla_{1}V^{3} &= \frac{1}{\kappa^{2}\lambda^{5/2}}\Big\{\lambda\kappa'\lambda_{4}\\ &\quad + \kappa^{2}\Big[v^{5}\tau^{5}\left(\tau\kappa' + \tau''\right) + v^{3}\tau^{2}\left(4\tau^{2}\kappa' + 3\tau\tau'' - \tau'^{2}\right)\\ &\quad + v\left(\tau^{2}\left(3\kappa' - \tau'\right) + 2\tau\tau'' + 2\tau'^{2}\right) + \tau'' - v^{4}\tau^{5}\tau'\\ &\quad + v^{2}\tau\left(\tau\tau'' - 2\tau^{2}\tau' - 3\tau'^{2}\right)\Big] + \kappa\Big[2\tau\kappa''\\ &\quad + v\tau^{2}\left(-2\tau\kappa' - \tau'' + \tau^{3}\right) + 2\kappa'\tau'\end{aligned}$$

$$+ \nu^2\tau\left(2\tau^2\kappa'' + \tau\left(\tau^{(3)} - 4\kappa'\tau'\right) - 3\tau'\tau''\right) + \tau^{(3)}$$
$$- 3\tau^2\tau'\Big] + \nu\kappa^3\tau\left(\nu^4\tau^4\tau' + \nu^2\tau^2\tau' + 6\tau' - \nu^3\tau^5\right.$$
$$\left.- 3\nu\tau^3\right)\Big\}$$
$$\nabla_2\nabla_2 V^3 = \frac{\tau}{\kappa\lambda^{5/2}}\left\{3\nu\tau\lambda_4 - \kappa\left(\nu^2\tau^2 + 3\nu\tau - 2\right)\tau'\right.$$
$$\left. + \kappa^2\left(3\tau - 3\nu^2\tau^3\right)\right\},$$
$$\nabla_1 h_{11} = \frac{1}{\lambda}\left\{\nu^2\tau\tau'\left(\kappa\lambda - \nu\tau'\right) - \nu\tau^3 - \lambda\kappa' - 2\nu^2\kappa\tau\tau'\right.$$
$$\left. + \nu\tau''\right\},$$
$$\nabla_2 h_{11} = \frac{1}{\lambda}\left\{\nu^2\tau\tau'\left(\kappa\lambda - \nu\tau'\right) - \nu\tau^3 + \tau'\lambda\right\},$$
$$\nabla_1 h_{12} = \frac{1}{\lambda}\left\{\tau' - \kappa\left(\nu^3\tau^4 + \nu\tau^2\right)\right\},$$
$$\nabla_2 h_{12} = \frac{1}{\lambda}\left\{\nu\tau^2\left(\kappa\lambda - \nu\tau'\right)\right\},$$
$$\nabla_1 h_{21} = \tau' + \nu\tau^3,$$
$$\nabla_2 h_{21} = \nu\tau^3,$$
$$\nabla_1 h_{22} = \nabla_2 h_{22} = 0. \tag{37}$$

Using (23) and combining (31), ((34) and (35)) with (37), we have the following.

Corollary 5. *The evolution equations for h_{ij} of $\overline{\mathbf{X}}$ are given by*

$$\frac{\partial h_{11}}{\partial t} = \frac{1}{\kappa^3\lambda^{5/2}}\left\{\nu^2\left(-\nu^3\tau'\tau^5 + \nu^4\left(\tau\tau'' - \tau'^2\right)\tau^4\right.\right.$$
$$\left. - 3\nu\tau'\tau^3 + 2\nu^2\left(2\tau\tau'' - 3\tau'^2\right)\tau^2 + 3\left(\tau'^2 + \tau\tau''\right)\right)$$
$$+ \tau^4\nu^6\left(\kappa''\tau^2 + \left(\kappa'\tau' + \tau^{(3)}\right)\tau - \tau'\tau''\right)$$
$$- \tau^2\nu^5\left(\kappa'\tau^4 + \tau''\tau^3 - \tau^{(3)}\tau^2 + 4\tau'\tau''\tau - 3\tau'^3\right)$$
$$+ \tau\nu^4\left(\kappa'' - \tau''\right)\tau^3 + \left(\tau'^2 + 8\kappa'\tau' + 2\tau^{(3)}\right)\tau^2$$
$$+ \left(2\tau'\tau''\tau - 4\tau'^3\right)\nu^4 - \left(\tau''\tau^3 + \left(\tau'^2 - 2\tau^{(3)}\right)\right)\tau^2$$
$$+ \left(4\tau'\tau''\tau + \tau'^3\right)\nu^3 - \left(\kappa'' + \tau''\right)\tau^2 + \left(7\kappa'\tau'\right.$$
$$\left.\left. + \tau^{(3)}\right)\tau + 3\tau'\tau''\nu^2 + \left(\kappa'\tau^2 + \tau^{(3)}\right)\nu - \kappa''\right\},$$
$$\frac{\partial h_{22}}{\partial t} = \frac{\tau}{\kappa\lambda^{5/2}}\left\{\nu\tau\left(\nu^2\tau^2 + 4\right)\lambda_4 - \kappa\left(\nu^4\tau^4 + \nu^3\tau^3\right.\right.$$
$$\left.\left. + 2\nu^2\tau^2 + 4\nu\tau - 2\right)\tau' - \tau\kappa^2\left(\nu^4\tau^4 + 3\nu^2\tau^2 - 4\right)\right\},$$
$$\frac{\partial h_{12}}{\partial t} = \frac{1}{\kappa^2\lambda^{5/2}}\left\{\lambda\kappa'\lambda_4 + \kappa^2\left[\nu^5\tau^5\left(\tau\kappa' + \tau''\right)\right.\right.$$
$$+ \nu^3\tau^2\left(4\tau^2\kappa' + 3\tau\tau'' - \tau'^2\right)$$
$$+ \nu\left(\tau^2\left(3\kappa' - \tau'\right) + 2\tau\tau'' + 2\tau'^2\right) + \tau'' - \nu^4\tau^5\tau'$$
$$\left. + \nu^2\tau\left(\tau\tau'' - 2\tau^2\tau' - 3\tau'^2\right)\right] + \kappa\tau\tau'\lambda\left(\kappa\lambda - \nu\tau'\right)$$
$$\left.\cdot\left(\nu\lambda_4 + \nu\kappa\left(\nu\tau + 1\right)\right) + \kappa^2\left(\nu^2\tau^2 - 1\right)\right\}. \tag{38}$$

Taking (15) into account and using the above results one could have the evolution of the Gaussian and mean curvatures of $\overline{\mathbf{X}}$ as follows:

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\operatorname{Det}\left(h_{ij}\right)}{\operatorname{Det}\left(g_{ij}\right)}\right],$$
$$\frac{\partial H}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left[\operatorname{tr}\left(g^{ij}h_{jk}\right)\right], \tag{39}$$

respectively.

4. Motion of Bishop Frenet Offset of Ruled Surface

In this section, the fundamental quantities g^*_{ij} and h^*_{ij} of Bishop Frenet offset $\mathbf{X}^*$ are obtained. The velocities $V^{*\gamma}$ of $\overline{\mathbf{X}^*}$ are obtained. Thus the formulas of the evolution of the 1st and 2nd fundamental quantities of $\overline{\mathbf{X}^*}$ are derived. For this purpose, let a Bishop Frenet offset of ruled surface generated by binormal vector $\mathbf{B}^*$ of the type 2 Bishop frame, moving in 3-dimensional Euclidean space E^3, be given at time t by the parametrization [1]:

$$\overline{\mathbf{X}^*}(s, \nu, t) = \boldsymbol{\alpha}^*(s, t) + \nu\mathbf{B}^*(s, t), \tag{40}$$

where $\overline{\mathbf{X}^*}(s, \nu, 0) = \mathbf{X}^*(s, \nu)$, $\boldsymbol{\alpha}^*(s, 0) = \boldsymbol{\alpha}^*(s)$, and $\mathbf{B}^*(s, 0) = \mathbf{B}^*(s)$.

4.1. Curvatures of $\mathbf{X}^*$. From (27) and using Bishop formulas (2) it is easily checked that the coefficients of the first fundamental form g^*_{ij} of Bishop Frenet offset are given by

$$g^*_{11} = \lambda^*,$$
$$g^*_{22} = 1,$$
$$g^*_{12} = 0,$$
$$g^* = \operatorname{Det}\left(g_{ij}\right) = \lambda^*, \tag{41}$$

where $\lambda^* = \{(1 + \nu k_1)^2 + \nu^2 k_2^2\} \succ 0$ and k_1, k_2 are the type 2 Bishop curvatures of the curve $\boldsymbol{\alpha}^*$ given from (4).

Using (13), the unit normal vector field of the Bishop Frenet offset $\mathbf{X}^*$ is given by

$$E_3^* = \frac{1}{\sqrt{\lambda^*}} \left\{ \nu k_2 \mathbf{T}^* - \left(1 + \nu k_1\right) \mathbf{N}^* \right\}. \quad (42)$$

This leads to the coefficients of the second fundamental form h_{ij}^* of Bishop Frenet offset given by

$$\begin{aligned} h_{11}^* &= \frac{1}{\sqrt{\lambda^*}} \left\{ \nu^2 \left(k_1' k_2 - k_1 k_2'\right) - \nu k_2' \right\}, \\ h_{22}^* &= 0, \\ h_{12}^* &= -\frac{k_2}{\sqrt{\lambda^*}}, \\ h^* &= \operatorname{Det}\left(h_{ij}^*\right) = -\frac{k_2^2}{\lambda^*}. \end{aligned} \quad (43)$$

According to (15) one can get the Gaussian and mean curvatures of Bishop Frenet offset $\mathbf{X}^*$, respectively, as follows.

Lemma 6. *Consider*

$$\begin{aligned} G^* &= -\frac{k_2^2}{\lambda^{*2}}, \\ H^* &= \frac{1}{2\lambda^{*3/2}} \left\{ \nu \left(\nu k_2' k_1' - \left(\nu k_1 + 1\right) k_2' \right) \right\}. \end{aligned} \quad (44)$$

From (17), one can obtain the Christoffel symbols Γ_{ij}^{*k} of Bishop Frenet offset $\mathbf{X}^*$ as follows.

Lemma 7. *Consider*

$$\begin{aligned} \Gamma_{11}^{*1} &= \frac{1}{\lambda^*} \left\{ \nu \left(\left(\nu k_1 + 1\right) k_1' + \nu k_2 k_2' \right) \right\}, \\ \Gamma_{12}^{*1} &= \Gamma_{21}^{*1} = \frac{1}{\lambda^*} \left\{ \nu \left(k_1^2 + k_2^2\right) + k_1 \right\}, \\ \Gamma_{11}^{*2} &= -\lambda^* \Gamma_{12}^{*1}, \end{aligned} \quad (45)$$

and other components equal zero.

4.2. Curvatures' Evolution of $\overline{\mathbf{X}^*}$. By a similar manner to Section 3.1, to calculate $\partial g_{ij}^*/\partial t$ and $\partial h_{ij}^*/\partial t$, we have to compute the velocities $V^{*\gamma}$ of $\overline{\mathbf{X}^*}$. Thus, using (9), (10), and (20) with the assumption that velocities of the curve $\boldsymbol{\alpha}^*$ are $v^{*1} = 0$, $v^{*2} = k_1$, and $v^{*3} = k_2$, one can get the tangential velocities V^{*i} and the normal velocity V^3 of $\overline{\mathbf{X}^*}$ as the following.

Corollary 8. *Consider*

$$\begin{aligned} V^{*1} &= -\frac{1}{\lambda^*} \left[\nu^2 k_2 \lambda_1^* + \nu \lambda_2^* \lambda_3^* - \nu k_1 k_2 \right], \\ V^{*2} &= k_2, \\ V^{*3} &= \frac{1}{\sqrt{\lambda^*}} \left[\nu \lambda_1^* \lambda_3^* - \nu^2 k_2 \lambda_2^* - k_1 \lambda_3^* \right], \end{aligned} \quad (46)$$

where $\lambda_1^* = (1/k_1)(k_2'' + k_2^3 - k_1 k_2')$, $\lambda_2^* = k_2' - k_1 k_2$, and $\lambda_3^* = (1 + \nu k_1)$.

Having (19) and (33) in mind and taking into account the above corollary and after straightforward computations we get the following:

$$\begin{aligned} \nabla_1 V^{*1} &= \frac{\nu}{k_1^2 \lambda^{*2}} \left\{ \left[-\nu^2 k_2^5 + 4\nu^3 k_1' k_2^4 + \left(k_2'' \nu^3 \right.\right.\right. \\ &\quad + 2\left(k_1' - 4k_2'\right)\nu^2 - 1\big) k_2^3 - \nu^2 \left(\nu k_2' \left(k_1' + 2k_2'\right) \right. \\ &\quad \left. + k_2''\right) k_2^2 + k_1' \left(4k_2'' \nu^3 - 3k_2' \nu^2 + 2\right) + \nu \left(k_2'' \right. \\ &\quad \left. + \nu \left(3k_2'^2 - 2k_2^{(3)}\right)\right) k_2 + \nu k_2' \left(2k_1' + k_2'\right) - 2\nu^2 k_2' k_2'' \\ &\quad \left. - k_2''\right] k_1^2 - 4\nu k_2' k_2^3 + \nu^2 \left(k_2' k_2^3 + k_2^{(3)} k_2 - 2k_2' k_2''\right) k_2^2 \\ &\quad - 5\nu k_1' \left(k_2^3 + k_2''\right) k_2 + k_2^{(3)} k_2 + k_1 k_2' k_2'' \\ &\quad \left. + \nu^2 k_2 k_1' \left(\nu^2 k_2^2 + 1\right)\left(k_2^3 + k_2''\right) \right\}, \\ \nabla_1 V^{*2} &= \frac{1}{k_1 \lambda^{*2}} \left\{ \nu^2 k_1^4 \left(\nu k_2' - 3k_2\right) - \nu^2 k_1^3 \left(\nu k_2 \left(k_2' \right.\right.\right. \\ &\quad \left. + k_2^2\right) - k_2'\big) + k_1^2 \left(\nu^3 k_2^2 k_2' - \nu k_2' + k_2 \left(\nu^3 k_2'' + \nu^2 k_2' \right.\right. \\ &\quad \left.\left. + 2\right) + \nu^3 k_2^4 - 4\nu^2 k_2^3\right) - k_1 \left(\nu^3 k_2^3 k_2' - 2\nu k_2 k_2' + k_2' \right. \\ &\quad \left. + \nu^2 k_2 \left(k_2'' - 2k_2 k_2' + k_2^3\right)\right) - \nu^3 k_1^5 k_2 + \nu k_2 \left(\nu^2 k_2^2 \right. \\ &\quad \left.\left. - 2\right)\left(k_2'' + k_2^3\right) \right\}, \\ \nabla_2 V^{*1} &= \frac{1}{k_1 \lambda^*} \left\{ \nu \left[\left(\nu k_1^2 + k_1 + \nu k_2^2\right)\left(k_1^2 \left(2k_2 - \nu k_2'\right) \right.\right.\right. \\ &\quad \left.\left.\left. + k_1 \left(\nu k_2 - 1\right) k_2' + \nu k_1^3 k_2 - \nu k_2 \left(k_2'' + k_2^3\right)\right)\right]\right\}, \\ \nabla_2 V^{*2} &= 0. \end{aligned} \quad (47)$$

Considering the above obtained results, we can formulate the following.

Corollary 9. *The evolution equations for the metric tensor g_{ij}^* of $\overline{\mathbf{X}^*}$ are given by*

$$\begin{aligned} \frac{\partial g_{11}^*}{\partial t} &= \frac{2}{k_1^2 \lambda^{*2}} \left\{ 2\nu^3 k_2'^2 + 3\nu^4 k_2 k_2'^2 k_1^3 - k_2 k_1^3 \right. \\ &\quad - 4\nu^5 k_2^3 k_1' k_1^3 + 2\nu^4 k_2^3 k_1' - 4\nu^3 k_1' k_1^3 + \nu^6 k_2^4 k_2' k_1^3 \\ &\quad - 4\nu^4 k_2' k_1^3 + 5\nu^2 k_2' k_2 k_1^3 + 6\nu^4 k_2' k_1^3 - 4\nu^3 k_2^2 k_2' k_1^3 \\ &\quad + 2\nu^3 k_2' k_1^3 + 4\nu^3 k_1' k_2' - \nu^4 k_2 k_1' k_2' - 3\nu^2 k_2'' k_1^3 \\ &\quad - \nu^4 k_2^2 k_2'' k_1^3 + \nu^3 k_2 k_2'' k_1^3 - \nu^4 k_2' k_2'' k_1^3 - \nu^4 k_2 k_1^3 \end{aligned}$$

$$
\begin{aligned}
&- v^3 k_2^5 k_1^2 - v k_2^3 k_1^2 + v^2 k_2'^2 k_1^2 - 2v^4 k_2^2 k_2'^2 \\
&+ 3v^3 k_2 k_2'^2 k_1^2 - v^6 k_1' k_1^2 + 4v^4 k_2^4 k_1' k_1^2 - 2v^4 k_1' k_1^2 \\
&+ 2v^3 k_2^3 k_1' k_1^2 - v^2 k_2 k_1' k_1^2 + 2v k_2 k_1' k_1^2 + v^5 k_2^4 k_2' k_1^2 \\
&- 8v^3 k_2^3 k_2' k_1^2 + 2v^3 k_2^2 k_2' k_1^2 + 2v^2 k_1' k_2' k_1^2 \big\}, \\
\frac{\partial g_{12}^*}{\partial t} &= \frac{1}{k_1 \lambda^{*2}} \big\{ k_1^2 \big[-v^3 k_2^4 \left(v^2 k_2' + 2 \right) + v^3 k_2^2 k_2' + 2v k_2' \\
&+ v^2 k_2^3 \left(-2v^3 k_2'' + 3v^2 k_2' \right) \\
&+ k_2 \left(-2v^3 k_2'' + 2v^2 k_2' + 8v + 2 \right) - 2v^5 k_2^6 + 2v^4 k_2^5 \big] \\
&+ k_1 k_2 \big[v^4 \left(-3k_2^2 k_2'' - 3k_2^5 \right) + v^5 k_2^4 k_2' + 2v k_2' \\
&+ v^2 \left(-2k_2'' + 3k_2 k_2' - 2k_2^3 + 4k_2^2 \right) + 2k_2 \big] \\
&+ v^4 \left(v k_1^7 k_2 - k_2^4 \right) \left(k_2'' + k_2^3 \right) \big\}, \\
\frac{\partial g_{22}^*}{\partial t} &= 0.
\end{aligned}
\tag{48}
$$

Using (19), (43), and (46), one can get the following:

$$
\begin{aligned}
\nabla_1 \nabla_1 V^{*3} &= \frac{1}{k_1^3 \lambda^{*5/2}} \big\{ 3v^2 \left(\lambda_3^* k_1' + v k_2 k_2' \right)^2 + v\lambda^* \big(k_1'' \\
&+ v k_1'^2 + k_2'^2 + k_1 k_1'' + k_2 k_2'' \big) k_1^4 \\
&+ v^3 k_2' \left(\left(k_2^2 - k_2'' k_2 - k_2'^2 + k_2^{(3)} \right) - k_2'' \right) \big(k_1^2 \\
&- 2v\lambda^* \left(\left(k_1 v + 1 \right) k_1' + v k_2 k_2' \right) \big) v^6 k_2^4 + v \big(3k_2' k_2^2 \\
&+ k_2^{(3)} \big) k_1 - v k_1' \left(k_2^3 + k_2'' \right) + v^7 k_2^2 \left(\left(k_2' - k_2^2 \right) + 1 \right) \\
&\cdot k_1^2 + v \left(k_2' - v \left(k_2^3 - k_2' k_2 + k_2'' \right) \right) k_1 - \left(k_2^3 + k_2'' \right) \\
&\cdot v^5 k_2 \big\}, \\
\nabla_1 \nabla_2 V^{*3} &= \nabla_2 \nabla_1 V^{*3} = \frac{1}{k_1^2 \lambda^{*5/2}} \big\{ v^4 \big(-k_2^2 + 2v k_2' k_2 \\
&+ k_2' - v k_2'' \big) + v^3 \left(-v k_2^3 + \left(3v^2 k_2' - 4 \right) k_2^2 \right. \\
&+ v \left(11 k_2' - v k_2'' \right) k_2 - v^2 k_2'^2 + 4k_2' \\
&+ v \left(v k_2^{(3)} - 6k_2'' \right) \big) k_1^2 - v^2 \big(2v^2 k_2^4 \\
&+ v \left(\left(k_1' - 3k_2' \right) v^2 + 4 \right) k_2^3 \\
&+ \left(2k_2'' v^3 - 17 k_2' v^2 + 6 \right) k_2^2 \\
&- v \left(v^2 k_2'^2 + \left(k_1' v^2 + 24 \right) k_2' - 5v k_2'' \right) k_2 + 5v^2 k_2'^2 \big) \\
&\cdot k_1 \big\}, \\
\nabla_2 \nabla_2 V^{*3} &= -\frac{1}{k_1 \lambda^{*5/2}} \big\{ k_2 \big[v k_1^3 \left(2v k_2' - 3k_2 \right) \\
&+ k_1^2 \left(-3k_2 \left(v^2 k_2' + 1 \right) + 4v k_2' + 3v^2 k_2^3 \right) \\
&+ k_1 \left(-3v k_2 k_2' + 2k_2' + v^2 k_2 \left(3k_2'' - k_2 k_2' + 3k_2^3 \right) \right) \\
&+ 3v k_2 \left(k_2'' + k_2^3 \right) \big] \big\}, \\
\nabla_1 h_{11}^* &= \frac{1}{\lambda^*} \big\{ v \left(v k_2 k_1'' - \lambda_3^* k_2'' \right) \lambda^* - v^2 \big(v k_2 k_1' \\
&- \lambda_3^* k_2' \big) \left(\lambda_3^* k_1' + v k_2 k_2' \right) + k_2 \left(v k_1^2 + k_1 + v k_2^2 \right) \big\}, \\
\nabla_1 h_{12}^* &= -\frac{1}{\lambda^*} \big\{ v^3 \left(k_1^2 + k_2^2 \right) \left(k_1 k_2' - k_2 k_1' \right) \\
&- 3v^2 k_1 \left(k_2 k_1' - k_1 k_2' \right) + v \left(2k_2 k_1' - 3k_1 k_2' \right) + k_2' \big\}, \\
\nabla_2 h_{11}^* &= -\frac{1}{\lambda^*} \big\{ v^2 \left(v k_2 k_1' - \lambda_3^* k_2' \right) \left(\lambda_3^* k_1' + v k_2 k_2' \right) \\
&- k_2 \left(v k_1^2 + k_1 + v k_2^2 \right) + k_2' \lambda^* \big\}, \\
\nabla_2 h_{12}^* &= -\frac{1}{\lambda^*} \big\{ v \left(v k_1^2 + k_1 + v k_2^2 \right) \left(v k_2 k_1' - \lambda_3^* k_2' \right) \big\}, \\
\nabla_1 h_{21}^* &= -k_2 \left(v k_1^2 + k_1 + v k_2^2 \right) - k_2', \\
\nabla_2 h_{21}^* &= -k_2 \left(v k_1^2 + k_1 + v k_2^2 \right), \\
\nabla_2 h_{22}^* &= \nabla_1 h_{22}^* = 0.
\end{aligned}
\tag{49}
$$

From the foregoing results, using (23), and after straightforward computations we conclude the following.

Corollary 10. *The evolution equations for h_{ij}^* of $\overline{\mathbf{X}^*}$ are given by*

$$
\begin{aligned}
\frac{\partial h_{11}^*}{\partial t} &= \frac{1}{k_1^3 \lambda^{*5/2}} \big\{ v \left(k_2' - v \left(k_2^3 - k_2' k_2 + k_2'' \right) \right) k_1 \\
&- v \left(k_2^3 + k_2'' \right) + v^6 \big[-v k_1^3 + \left(v^2 \left(k_2^2 - k_2' \right) - 1 \right) k_1^2 \\
&+ v \left(v \left(k_2^3 - k_2' k_2 + k_2'' \right) - k_2' \right) k_1 + v \left(k_2^3 + k_2'' \right) \big] k_2^4 \\
&+ v^5 \left(k_1' \left(v^2 \left(k_2^2 - k_2' \right) - 1 \right) \right. \\
&+ v \left(v \left(3k_2' k_2^2 - k_2'' k_2 - k_2'^2 + k_2^{(3)} \right) - k_2'' \right) \big) k_2^5 \\
&+ v \left(3k_2' k_2^2 + k_2^{(3)} \right) k_1 - v k_1' \left(k_2^3 + k_2'' \right) k_1 \\
&+ v^4 k_2 \big[v k_1^3 + \left(\left(k_2' - k_2^2 \right) v^2 + 1 \right) k_1^2 \\
&+ v \left(k_2' - v \left(k_2^3 - k_2' k_2 + k_2'' \right) \right) k_1 - v \left(k_2^3 + k_2'' \right) \big]
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda^{*}v^{3}\left[k_1'\left(v^{2}\left(k_2^{2}-k_2'\right)-1\right)\right.\\
&\left.+v\left(v\left(3k_2'k_2^{2}-k_2''k_2-k_2'^{2}+k_2^{(3)}\right)-k_2''\right)\right]\\
&+\lambda^{*}v^{2}\left(k_1'\left(v^{2}\left(k_2^{2}-k_2'\right)-1\right)\right.\\
&\left.+v\left(v\left(3k_2'k_2^{2}-k_2''k_2-k_2'^{2}+k_2^{(3)}\right)-k_2''\right)\right)+\lambda^{*2}\\
&\left.-2k_1'k_2''-k_2'\left(k_1''+3k_2''\right)-\left(2k_1'^{2}+k_2^{(3)}\right)v-k_1''\right\},\\
\frac{\partial h_{22}^{*}}{\partial t}&=-\frac{k_2}{k_1\lambda^{*5/2}}\left\{vk_1^{3}\left(2vk_2'-3k_2\right)\right.\\
&+k_1^{2}\left(-3k_2\left(v^{2}k_2'+1\right)+4vk_2'+3v^{2}k_2^{3}\right)\\
&+k_1\left(-3vk_2k_2'+2k_2'+v^{2}k_2\left(3k_2''-k_2k_2'+3k_2^{3}\right)\right)\\
&-k_2v^{2}\lambda^{*}\left[k_1^{2}\left(v^{2}\left(k_2'-k_2^{2}\right)+1\right)\right.\\
&+vk_1\left(k_2'-v\left(k_2''-k_2k_2'+k_2^{3}\right)\right)+vk_1^{3}\\
&\left.\left.-v\left(k_2''+k_2^{3}\right)\right]+3vk_2\left(k_2''+k_2^{3}\right)\right\},\\
\frac{\partial h_{12}^{*}}{\partial t}&=\frac{1}{k_1^{2}\lambda^{*5/2}}\left\{v^{4}\left(-k_2^{2}+2vk_2'k_2+k_2'-vk_2''\right)k_1^{7}\right.\\
&+v^{3}\left[-vk_2^{3}+\left(3v^{2}k_2'-4\right)k_2^{2}+v\left(11k_2'-vk_2''\right)k_2\right.\\
&\left.-v^{2}k_2'^{2}+4k_2'+v\left(vk_2^{(3)}-6k_2''\right)\right]k_1^{6}-v^{2}\left[2v^{2}k_2^{4}\right.\\
&+v\left(\left(k_1'-3k_2'\right)v^{2}+4\right)k_2^{3}+\left(2k_2''v^{3}-17k_2'v^{2}+6\right)\\
&\cdot k_2^{2}-v\left(v^{2}k_2'^{2}+\left(k_1'v^{2}+24\right)k_2'-5vk_2''\right)k_2\\
&\left.+5v^{2}k_2'^{2}-6k_2'+v\left(\left(k_1'v^{2}+14\right)k_2''-5vk_2^{(3)}\right)\right]k_1^{5}\\
&+v\left[-2v^{3}k_2^{5}+v^{2}\left(v^{2}\left(k_1'+5k_2'\right)-4\right)k_2^{4}\right.\\
&-v\left(2k_2''v^{3}+\left(5k_1'-7k_2'\right)v^{2}+6\right)k_2^{3}\\
&-\left(\left(k_2'^{2}+k_1'k_2'-2k_2^{(3)}\right)v^{4}+7k_2''v^{3}-34k_2'v^{2}+3\right)\\
&\cdot k_2^{2}\\
&+v\left(5v^{2}k_2'^{2}+\left(-k_2''v^{3}+5k_1'v^{2}+23\right)k_2'-9vk_2''\right)k_2\\
&\left.-9v^{2}k_2'^{2}+4k_2'+v\left(10vk_2^{(3)}-\left(5k_1'v^{2}+16\right)k_2''\right)\right]k_1^{4}\\
&+\left(k_2'-k_1'\right)k_2^{3}-k_2''k_2^{2}+k_2'\left(k_1'+k_2'\right)k_2-k_1'k_2''v^{5}\\
&-k_2k_2^{5}-\left(5k_1'+11k_2'\right)k_2^{3}+5k_2''k_2^{2}+5\left(k_1'k_2'\right.\\
&\left.-k_2^{(3)}\right)k_2+5k_2'k_2''v^{4}-\left(4k_2^{5}-5\left(k_2'-2k_1'\right)k_2^{3}\right.\\
&\left.\left.-8k_2''k_2^{2}+7k_2'\left(k_1'+k_2'\right)k_2-10k_1'k_2''\right)v^{3}\right\}.
\end{aligned}
\tag{50}
$$

Regarding the above obtained results, the evolution of the Gaussian $\partial G^{*}/\partial t$ and mean curvatures $\partial H^{*}/\partial t$ of $\overline{\mathbf{X}^{*}}$ can be obtained using the same technique as in (39).

5. Application

In this section, we give an example of how the evolving offsets base curves $(\boldsymbol{\alpha},\boldsymbol{\alpha}^{*})$ and generators $(\mathbf{B},\mathbf{B}^{*})$ of a pair $(\mathbf{X},\mathbf{X}^{*})$ of Bishop Frenet ruled surface will look like after a period of time and the effect of the evolution on curves' frames. This is given through some illustrated figures. For this purpose we consider a helix curve $\boldsymbol{\alpha}$ as a base curve of the ruled surface $\mathbf{X}$:

$$
\boldsymbol{\alpha}=\left\{\cos\left(\frac{s}{\sqrt{2}}\right),\sin\left(\frac{s}{\sqrt{2}}\right),\frac{s}{\sqrt{2}}\right\}. \tag{51}
$$

Thus, its binormal vector is given by

$$
\mathbf{B}=\frac{1}{\sqrt{2}}\left\{\sin\left(\frac{s}{\sqrt{2}}\right),-\cos\left(\frac{s}{\sqrt{2}}\right),1\right\}. \tag{52}
$$

Using relation (24), the parametrization of the base curve of the ruled surface $\mathbf{X}^{*}$ takes the following form:

$$
\begin{aligned}
\boldsymbol{\alpha}^{*}=\frac{1}{\sqrt{2}}\left\{\psi\sin\left(\frac{s}{\sqrt{2}}\right)\right.\\
+\sqrt{2}\cos\left(\frac{s}{\sqrt{2}}\right),\sqrt{2}\sin\left(\frac{s}{\sqrt{2}}\right)-\psi\cos\left(\frac{s}{\sqrt{2}}\right),\psi\\
\left.+s\right\}.
\end{aligned}
\tag{53}
$$

Thus and using the parametrizations of surfaces (26), (27), (28), and (40) we can show Figures 1, 2, and 3 as follows.

Figure 1 shows the original pair $(\mathbf{X},\mathbf{X}^{*})$ of Bishop Frenet ruled surface at $t=0$ and it is noticed that the vector $\mathbf{B}$ is in direction of the vector $\mathbf{B}^{*}$.

Figure 2 shows the evolution of the pair $(\mathbf{X},\mathbf{X}^{*})$ of Bishop Frenet ruled surface. In this case, the function $\cos t$ affects the pair $(\mathbf{X},\mathbf{X}^{*})$ and these surfaces are plotted for different values of the time t. The evolution of the pair $(\mathbf{X},\mathbf{X}^{*})$ expands and collapses under these values of the time t and it is noticed that the motion of the pair $(\mathbf{X},\mathbf{X}^{*})$ conserves the motion of the moving frames associated with the offsets base curves $(\boldsymbol{\alpha},\boldsymbol{\alpha}^{*})$, respectively; that is, the vector $\mathbf{B}$ is in direction of the vector $\mathbf{B}^{*}$.

Figure 3 shows the evolution of the pair $(\mathbf{X},\mathbf{X}^{*})$ of Bishop Frenet ruled surface. In this case, the function e^{t} affects the pair $(\mathbf{X},\mathbf{X}^{*})$ and these surfaces are plotted for different values of the time t. Also, the same result as in Figure 2 is obtained; that is, the vector $\mathbf{B}$ is in direction of the vector $\mathbf{B}^{*}$.

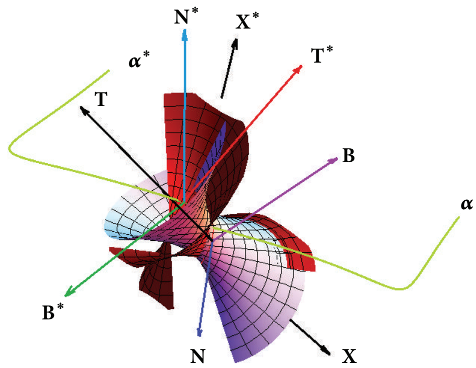

FIGURE 1: The original pair $(\mathbf{X}, \mathbf{X}^*)$, $\mathbf{B} /\!/ \mathbf{B}^*$, $t = 0$, $\psi = 1$, $s \in [-\pi, \pi]$, $v \in [-6, 6]$.

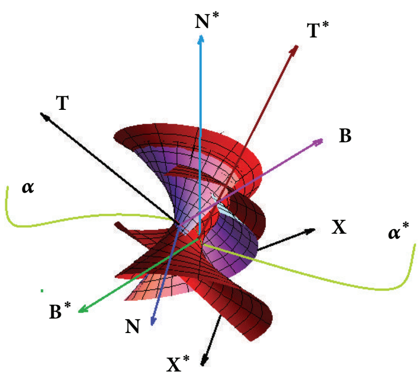

FIGURE 2: The evolution pair $(\mathbf{X}, \mathbf{X}^*)$, $\mathbf{B} /\!/ \mathbf{B}^*$, $t = 0.8$, $\psi = 1$, $s \in [-\pi, \pi]$, $v \in [-6, 6]$.

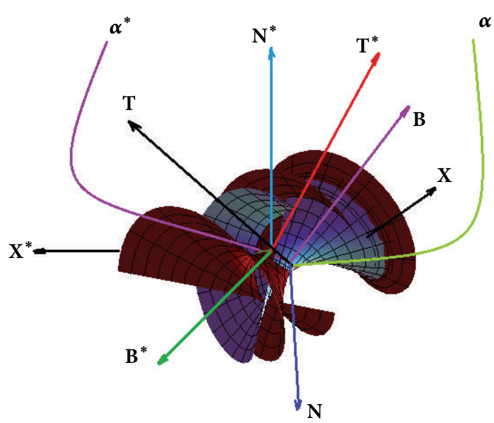

FIGURE 3: The evolution pair $(\mathbf{X}, \mathbf{X}^*)$, $\mathbf{B} /\!/ \mathbf{B}^*$, $t = 0.3$, $\psi = 1$, $s \in [-\pi, \pi]$, $v \in [-6, 6]$.

6. Conclusion

We conclude that, by changing the applied time-varying functions on the pair $(\mathbf{X}, \mathbf{X}^*)$, it is found that the moving frames associated with the offsets base curves $(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)$ describe a parallel transport frame motion. This study plays an important role in the construction and motion of new offsets of ruled surfaces by giving a new frame. We can apply this idea on many different surfaces using different methods. The field is developing rapidly, and there are a lot of problems to be solved and more work is needed to establish different results of new surfaces in different spaces. We hope that this idea will be helpful to mathematicians who are specialized in this area.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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