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Research Article

Minimum-Energy Multiwavelet Frame on the Interval [0, 1]

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Drawing inspiration from the idea of combining multiwavelets on the interval with frame theory organically, we study minimum-energy multiwavelet frame on the interval $[0, 1]$ (MEMWFI). Firstly, left boundary multiscaling functions, right boundary multiscaling functions, and the definition of MEMWFI are put forward, and the equivalent characterizations of MEMWFI are given. Then, two algorithms of constructing MEMWFI are proposed. Finally, the decomposition formula, reconstruction formulas, and numerical examples are given.

1. Introduction

The notion of frame was introduced by Duffin and Schaeffer [1]. It is a generalization of basis and can be viewed as some kind of "overcomplete basis." A basis in a Hilbert space allows each element to be written as a linear combination of the elements in the basis, and the combination is unique. An overcomplete frame also allows one to represent each element via it, but the representation is not unique. This property plays an important role in mathematics, signal analysis, time-frequency analysis, and so on [2–20]. In nearly thirty years, frame theory has been growing rapidly. Wavelet frames are a class of important frames among all kinds of frames.

Let $\mathbb{H}$ be a complex, separable Hilbert space. A sequence $\{f_k : k \in \mathbb{Z}\}$ of elements in $\mathbb{H}$ is a frame for $\mathbb{H}$ if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \le B \|f\|^2, \quad \forall f \in \mathbb{H}. \tag{1}$$

The numbers A, B are called frame bounds. A frame $\{f_k : k \in \mathbb{Z}\}$ is tight if we can choose $A = B$ as frame bounds. A frame $\{f_k : k \in \mathbb{Z}\}$ is exact if it ceases to be a frame when an arbitrary element is removed. When $A = B = 1$, then

$$f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle f_k, \quad \forall f \in \mathbb{H}. \tag{2}$$

Chui and He [6] proposed the concept of minimum-energy wavelet frames, which can reduce the computational complexity and maintain the numerical stability, and do not need to search dual frames in the decomposition and reconstruction of functions. Therefore, many people paid a lot of attention to the study of minimum-energy wavelet frames. Petukhov [8] studied minimum-energy wavelet frames with symmetry. Huang and Cheng [16] studied minimum-energy wavelet frames with arbitrary integer dilation factor. Gao and Cao [17] researched minimum-energy wavelets frames on the interval and its applications. Huang and Li [18] studied the algorithms of constructing minimum-energy wavelet frames on the interval. Recently, Huang et al. [19] and Liang and Zhao [20] have been concerned with the algorithms of constructing minimum-energy multiwavelet frames. In [19], the authors gave the definitions of multiwavelet frame multiresolution analysis for $L^2(\mathbb{R})$ and minimum-energy multiwavelet frame. But, in many practical applications, one is often interested in the problems which are confined to a finite interval, such as the differential equation solving, image processing, and signal analysis. Therefore, we should pay more attention to minimum-energy multiwavelet frames on the interval; under this case, we can combine the features (symmetry, compact support, and regularity) of multiwavelets on the interval with the superiority (redundancy, good numerical stability, and lower computational complexity) of minimum-energy wavelet frames.

Throughout this paper, let $\mathbb{Z}, \mathbb{R}$ denote the set of integers and real numbers, respectively. $\mathbb{N}$ denotes the set of positive integers. Let $a, r \geq 2$, and $a, r \in \mathbb{N}$.

The organization of this paper is as follows. In Section 2, we give the main results; left boundary multiscaling functions, right boundary multiscaling functions, and the definition of MEMWFI are put forward; the equivalent characterizations of MEMWFI, two algorithms for constructing MEMWFI, and the decomposition and reconstruction formulas of MEMWFI are proposed. Finally, numerical examples are given in Section 3.

2. Main Results

In order to give the definitions of MEMWFI, left boundary multiscaling functions, and right boundary multiscaling functions, we let $\gamma > 0$, $j_0 = \min\{j : a^j \geq \gamma, j \in \mathbb{Z}\}$.

Definition 1. If a closed subspaces sequence $V_j([0,1])$, $j \geq j_0$ in $L^2([0,1])$ satisfies

(1) $V_j([0,1]) \subset V_{j+1}([0,1])$, $j \geq j_0$;

(2) $\bigcup_{j\geq j_0} V_j([0,1])$ is dense in $L^2([0,1])$;

(3) $\bigcap_{j\geq j_0} V_j([0,1]) = V_{j_0}([0,1])$;

(4) $f(x) \in V_j([0,1]) \Leftrightarrow f(ax) \in V_{j+1}([0,1])$, $j \geq j_0$, $j \in \mathbb{Z}$;

(5) $f(x) \in V_j([0,1]) \Leftrightarrow f(x + a^{-j}k) \in V_j([0,1])$, $j \geq j_0$, $j, k \in \mathbb{Z}$;

(6) there exists a vector-valued function $\phi(x) = (\phi_1, \phi_2, \ldots, \phi_r)^T, \phi_1, \phi_2, \ldots, \phi_r \in V_{j_0}([0,1])$ with $\operatorname{supp}\phi = [0,\gamma]$ such that

$$\left\{\phi_\tau\left(a^{j_0}x - k\right),\ 1 \leq \tau \leq r,\ 0 \leq k \leq K_{j_0}\right\} \tag{3}$$

forms a frame of $V_{j_0}([0,1])$, where K_{j_0} is the biggest integer number to make $\operatorname{supp}\phi(a^{j_0}x) \cup \operatorname{supp}\phi(a^{j_0}x - K_{j_0}) \subset [0,1]$,

then we say $V_j([0,1])\,(j \geq j_0)$ is a multiwavelet frame multiresolution analysis on the interval with dilation factor a and multiplicity r or a multiwavelet frame multiresolution analysis on the interval $[0,1]$ and denote it as MWFMRA ($V_j[0,1]$). The function $\phi(x)$ is vector-valued scaling function (multiscaling function).

For a MWFMRA ($V_j[0,1]$), we let $V_{j+1}([0,1]) = V_j([0,1]) + W_j([0,1])$. If we can find vector-valued functions $\psi^1(x), \psi^2(x), \ldots, \psi^M(x)$ such that $\{\psi_\tau^i(a^{j_0}x - k) : 1 \leq \tau \leq r, 1 \leq i \leq M, 0 \leq k \leq K_{j_0}\}$ is a frame for $W_{j_0}([0,1])$, then we say $\{\psi^1(x), \psi^2(x), \ldots, \psi^M(x)\}$ is a multiwavelet frame on the interval, where $\psi^i(x) = (\psi_1^i(x), \psi_2^i(x), \ldots, \psi_r^i(x))^T$, $i = 1, 2, \ldots, M$.

2.1. Left and Right Boundary Multiscaling Functions. Let $\phi(x)$ be a vector-valued scaling function and generate a multiwavelet frame multiresolution analysis for $L^2(\mathbb{R})$ with corresponding multiwavelet frame $\Psi = \{\psi^1, \psi^2, \ldots, \psi^M\}$. Suppose $\phi(x)$ satisfies the refinable equation

$$\phi(x) = \sum_{k=0}^{\gamma_1} P_k \phi(ax - k) \tag{4}$$

with $\operatorname{supp}\phi = [0,\gamma]$.

Let $\gamma_1 = (a-1)\gamma$, $N = [\gamma/a]$.

Theorem 2. *Define left boundary multiscaling functions $\phi_{j,k}^L$ and right boundary multiscaling functions $\phi_{j,k}^{R,s}$ as follows:*

$$\begin{aligned} \phi_{j,k}^L(x) &:= \sum_{n=-\gamma+1}^{k-N} C_{k,n}\phi_{j,n}(x)\Big|_{[0,1]}, \quad 0 \leq k \leq N-1, \\ \phi_{j,k}^{R,s}(x) &:= \sum_{n=a^j-\gamma+1+k}^{a^j-1} C_{k,n}^s \phi_{j,n}(x)\Big|_{[0,1]}, \\ &\qquad 0 \leq k \leq N-1,\ 1 \leq s \leq a-1, \end{aligned} \tag{5}$$

where $C_{k,n}, C_{k,n}^s$ are $r \times r$ coefficient matrices. Denote

$$\Phi_j := \left\{\phi_{\tau,j,k}^L, k = 0, 1, \ldots, N-1; \phi_{\tau,j,k}, k = 0, 1, \ldots, a^j - \gamma; \phi_{j,k}^{\tau,R,s}, k = 0, 1, \ldots, N-1, 1 \leq s \leq a-1 : 1 \leq \tau \leq r\right\}. \tag{6}$$

If $\{C_{k,n}\}$ and $\{C_{k,n}^s\}$ make $\{\phi_{j,k}^L, k = 0, 1, \ldots, N-1\}$ and $\{\phi_{j,k}^{R,s}, k = 0, 1, \ldots, N-1, s = 1, 2, \ldots, a-1\}$ satisfy frame condition, respectively, then there exist coefficient matrices $P_{k,n}^L, P_{k,n}^{R,s}, p_{k,n}^L, p_{k,n}^{R,s}$ such that the matrix-valued refinable equations of Φ_j are expressed as follows:

$$\begin{aligned} \sqrt{a}\phi_{j,k}^L &= \sum_{n=0}^{N-1} P_{k,n}^L \phi_{j+1,n}^L + \sum_{n=0}^{ak+N_1} p_{k,n}^L \phi_{j+1,n}, \\ &\qquad 0 \leq k \leq N-1, \\ \sqrt{a}\phi_{j,k} &= \sum_{n=ak}^{ak+\gamma_1} P_{n-ak}\phi_{j+1,n}, \quad 0 \leq k \leq a^j - \gamma, \\ \sqrt{a}\phi_{j,k}^{R,s} &= \sum_{n=0}^{ak+N_2} p_{k,n}^{R,s}\phi_{j+1,a^{j+1}-\gamma-n} + \sum_{n=0}^{N-1} P_{k,n}^{R,s}\phi_{j+1,n}^{R,s}, \\ &\qquad 0 \leq k \leq N-1,\ 1 \leq s \leq a-1, \end{aligned} \tag{7}$$

where $N_1 = \gamma_1 - aN$, $N_2 = (a-1)\gamma - aN$, and Φ_j can form a frame of $V_j([0,1])$ when $j \geq j_0$.

Proof. The proof is similar to Theorem 2.2 in [17]. □

Theorem 3. *Let $\phi_{j,k}^L$ and $\phi_{j,k}^{R,s}$ be the left boundary multiscaling functions and right boundary multiscaling functions defined in*

Theorem 2. *Let vector-valued functions* $\{\psi_{j,k}^{L,i}, 1 \le i \le M, k = 0,1,\ldots,N-1\}$, $\{\psi_{j,k}^{i}, 1 \le i \le M, k = 0,1,\ldots,a^j-\gamma\}$, $\{\psi_{j,k}^{R,s,i}, 1 \le i \le M, k = 0,1,\ldots,N-1, s = 1,2,\ldots,a-1\}$ *be expressed as follows:*

$$\sqrt{a}\psi_{j,k}^{L,i} = \sum_{n=0}^{N-1} Q_{k,n}^{L,i}\phi_{j+1,n}^{L} + \sum_{n=0}^{ak+N_1} q_{k,n}^{L,i}\phi_{j+1,n}, \quad 0 \le k \le N-1,$$

$$\sqrt{a}\psi_{j,k}^{i} = \sum_{n=ak}^{ak+\gamma_1} Q_{n-ak}^{i}\phi_{j+1,n}, \quad 0 \le k \le a^j-\gamma, \tag{8}$$

$$\sqrt{a}\psi_{j,k}^{R,s,i} = \sum_{n=0}^{ak+N_2} q_{k,n}^{R,s,i}\phi_{j+1,a^{j+1}-\gamma-n} + \sum_{n=0}^{N-1} Q_{\mu:k,n}^{R,s,i}\phi_{j+1,n}^{R,s}, \quad 0 \le k \le N-1, \; s = 1,2,\ldots,a-1,$$

where $N_1 = \gamma_1 - aN$, $N_2 = (a-1)\gamma - aN$. *Denote*

$$\Psi_j := \left\{\psi_{\tau,j,k}^{L,i}, k=0,1,\ldots,N-1; \psi_{\tau,j,k}^{i}, k=0,1,\ldots,a^j - \gamma; \psi_{\tau,j,k}^{R,s,i}, k=0,1,\ldots,N-1, 1\le s\le a-1 : 1\le i \le M, 1\le\tau\le r\right\}. \tag{9}$$

If coefficient matrices $Q_{k,n}^{L,i}, q_{k,n}^{L,i}, Q_{k,n}^{R,s,i}, q_{k,n}^{R,s,i}$ *such that* $\{\psi_{j,k}^{L,i}, k = 0,1,\ldots,N-1, i = 1,2,\ldots,M\}$ *and* $\{\psi_{j,k}^{R,s,i}, k = 0,1,\ldots,N-1, s = 1,2,\ldots,a-1, i = 1,2,\ldots,M\}$ *satisfy frame condition, respectively, then* Ψ_j *is a frame of* $W_j([0,1])$ *for any* $j \ge j_0$.

Proof. The proof is similar to Theorem 3.2 in [17] and Theorem 3.2 in [18]. □

Definition 4. Let Φ_j, Ψ_j, $j \in \mathbb{Z}$ be defined in Theorems 2 and 3, respectively. If any $f \in L^2([0,1])$, one has

$$\sum_{\theta\in\Phi_{j_0+1}} \left|\langle f,\theta\rangle\right|^2 = \sum_{\eta_\phi\in\Phi_{j_0}} \left|\langle f,\eta_\phi\rangle\right|^2 + \sum_{\eta_\psi\in\Psi_{j_0}} \left|\langle f,\eta_\psi\rangle\right|^2, \tag{10}$$

and then the family of functions Ψ_{j_0} are called a minimum-energy multiwavelet frame on the interval [0, 1] (MEMWFI) associated Φ_{j_0} with dilation factor a and multiplicity r.

By Parseval identity, (10) is equivalent to

$$\sum_{\theta\in\Phi_{j_0+1}} \langle f,\theta\rangle\theta = \sum_{\eta_\phi\in\Phi_{j_0}} \langle f,\eta_\phi\rangle\eta_\phi + \sum_{\eta_\psi\in\Psi_{j_0}} \langle f,\eta_\psi\rangle\eta_\psi, \quad f \in L^2([0,1]). \tag{11}$$

2.2. Construction of MEMWFI. For convenience, we denote the following:

(1) $\ell_j = a^j - \gamma$, $\lambda = a(N-1)$.

(2) $S_j = \{k : -N \le k \le a^j - \gamma + (a-1)N\}$.

(3) $\Phi_j = (\ldots, \phi_{j,k}{}^T, \ldots)^T$ is $|S_j|r$ dimension column vector; and $\phi_{j,k}$ are left boundary multiscaling functions and right boundary multiscaling functions defined in Theorem 2:

$$\phi_{j,k} = \begin{cases} \phi_{j,N+k}^{L}, & k = -N,\ldots,-1, \\ \phi_{j,k}, & k = 0,\ldots,a^j-\gamma, \\ \phi_{j,k-\ell_j-1}^{R,1}, & k = \ell_j+1,\ldots,\ell_j+N, \\ \phi_{j,k-\ell_j-N-1}^{R,2}, & k = \ell_j+N+1,\ldots,\ell_j+2N, \\ \vdots & \\ \phi_{j,k-\ell_j-(a-2)N-1}^{R,a-1}, & k = \ell_j+(a-2)N+1,\ldots,\ell_j+(a-1)N. \end{cases} \tag{12}$$

(4) $\Psi_j^i = (\ldots, \psi_{j,k}^{i}{}^T, \ldots)^T$ are $|S_j|r$ dimension column vector; and $\psi_{j,k}^i$ are multiwavelet frames defined in Theorem 3:

$$\psi_{j,k}^{i} = \begin{cases} \psi_{j,N+k}^{L,i}, & k = -N,\ldots,-1, \\ \psi_{j,k}^{i}, & k = 0,\ldots,a^j-\gamma, \\ \psi_{j,k-\ell_j-1}^{R,1,i}, & k = \ell_j+1,\ldots,\ell_j+N, \\ \psi_{j,k-\ell_j-N-1}^{R,2,i}, & k = \ell_j+N+1,\ldots,\ell_j+2N, \\ \vdots & \\ \psi_{j,k-\ell_j-(a-2)N-1}^{R,a-1,i}, & k = \ell_j+(a-2)N+1,\ldots,\ell_j+(a-1)N, \end{cases} \tag{13}$$

where $i = 1,2,\ldots,M$. If Ψ^i is seen as a set of functions, then $\Psi_j = \{\Psi_j^1,\ldots,\Psi_j^M\}$.

Define

$$P := \begin{pmatrix} P_{11} & P_{12} & \\ & P_{22} & \\ & P_{32} & P_{33} \end{pmatrix}, \tag{14}$$

where $P_{11}, P_{12}, P_{22}, P_{32}, P_{33}$ are block matrices as follows:

$$P_{11} = \begin{pmatrix} P_{0,0}^{L} & P_{0,1}^{L} & \cdots & P_{0,N-1}^{L} \\ P_{1,0}^{L} & P_{1,1}^{L} & \cdots & P_{1,N-1}^{L} \\ \vdots & \vdots & & \vdots \\ P_{N-1,0}^{L} & P_{N-1,1}^{L} & \cdots & P_{N-1,N-1}^{L} \end{pmatrix},$$

$$P_{12} = \begin{pmatrix} p_{0,0}^{L} & \cdots & p_{0,N_1}^{L} & & \\ p_{1,0}^{L} & \cdots & \cdots & p_{1,a+N_1}^{L} & \\ \vdots & & & \ddots & \\ p_{N-1,0}^{L} & \cdots & \cdots & \cdots & p_{N-1,\lambda+N_1}^{L} \end{pmatrix},$$

$$P_{22}=\begin{pmatrix} P_0 & \cdots & P_a & \cdots & P_{\gamma_1} & & & \\ & & P_0 & \cdots & P_a & \cdots & P_{\gamma_1} & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & P_0 & \cdots & P_a & \cdots & P_{\gamma_1} \end{pmatrix},$$

$$P_{32}=\begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \vdots \\ \mathbb{P}_{a-1} \end{pmatrix},$$

$$P_{33}=\operatorname{diag}\left(\mathbb{P}_1,\mathbb{P}_2,\ldots,\mathbb{P}_{a-1}\right),$$

$$\mathbb{p}_i=\begin{pmatrix} & & p^{R,i}_{0,N_2} & \cdots & p^{R,i}_{0,0} \\ & p^{R,i}_{1,a+N_2} & \cdots & \cdots & h^{R,i}_{1,0} \\ & \vdots & & & \vdots \\ p^{R,i}_{N-1,\lambda+N_2} & \cdots & \cdots & \cdots & p^{R,i}_{N-1,0} \end{pmatrix},$$

$$\mathbb{P}_i=\begin{pmatrix} P^{R,i}_{0,0} & \cdots & P^{R,i}_{0,N-1} \\ \vdots & & \vdots \\ P^{R,i}_{N-1,0} & \cdots & P^{R,i}_{N-1,N-1} \end{pmatrix},$$

$$i=1,\ldots,a-1. \tag{15}$$

Similarly, we can define

$$Q^i:=\begin{pmatrix} Q^i_{11} & Q^i_{12} & \\ & Q^i_{22} & \\ & Q^i_{32} & Q^i_{33} \end{pmatrix},\quad i=1,2,\ldots,M, \tag{16}$$

where $Q^i_{11},Q^i_{12},Q^i_{22},Q^i_{32},Q^i_{33}$ have the same form as $P_{11},P_{12},P_{22},P_{32},P_{33}$, respectively.

Then, by Theorems 2 and 3, we have

$$\sqrt{a}\Phi_j=P\Phi_{j+1},\qquad \sqrt{a}\Psi^i_j=Q^i\Phi^i_{j+1}. \tag{17}$$

The Fourier transform of (17) is

$$\sqrt{a}\widehat{\Phi}_j=P(z)\widehat{\Phi}_{j+1},\qquad \sqrt{a}\widehat{\Psi}^i_j=Q^i(z)\widehat{\Phi}^i_{j+1}, \tag{18}$$

where

$$P(z)=\frac{1}{a}D_j^{-1}PD_{j+1},\qquad Q^i(z)=\frac{1}{a}D_j^{-1}Q^iD_{j+1},\qquad D_j=\operatorname{diag}\left(e^{i\omega(-N)/a^j}I_r,\ldots,e^{i\omega(\ell_j+\lambda)/a^j}I_r\right). \tag{19}$$

With these $P(z),Q^i(z),\Phi_j,\Psi_j$, $i=1,2,\ldots,M$, $j\ge j_0$, we will give the characterizations of MEMWFI.

Theorem 5. *The following statements are equivalent:*

(1) Ψ_{j_0} *is a MEMWFI.*

(2) *For any* $|z|=1$,

$$a\left(P(z)^*P(z)+\sum_{i=1}^{M}Q^i(z)^*Q^i(z)\right)=I_{|s_{j+1}|r}. \tag{20}$$

(3) *For any* $m,l\in S_{j+1}$,

$$\sum_{k\in S_j}\left(P^*_{k,m}P_{k,l}+\sum_{i=1}^{M}Q^{i*}_{k,m}Q^i_{k,l}\right)-a\delta_{m,l}I_r=0_r, \tag{21}$$

where $P_{k,m},Q^i_{k,m}$ *denote the* $r\times r$ *block matrices from* $((k-1)r+1)$*th row to* kr*th row and* $((m-1)r+1)$*th row to* mr*th column in* P,Q^i*, respectively, and* $P(z)^*$ *denotes the complex conjugate of the transpose of* $P(z)$*.*

Proof. By using Theorems 2 and 3, (11) can be rewritten as

$$\sum_{k\in S_{j+1}}\sum_{\tau=1}^{r}\left\langle f,\phi_{\tau,j+1,k}\right\rangle\phi_{\tau,j+1,k}=\sum_{k\in S_j}\sum_{\tau=1}^{r}\left\langle f,\phi_{\tau,j,k}\right\rangle\phi_{\tau,j,k}+\sum_{i=1}^{M}\sum_{k\in S_j}\sum_{\tau=1}^{r}\left\langle f,\psi^i_{\tau,j,k}\right\rangle\psi^i_{\tau,j,k}. \tag{22}$$

For convenience, for every $f\in L^2([0,1])$, we denote $\langle f,\phi_{j+1,k}\rangle$ as r dimension column vector $\mathbf{c}_{j+1,k}$, where the τth component of $\mathbf{c}_{j+1,k}$ is $\langle f,\phi_{\tau,j+1,k}\rangle$. Similarly, we denote $\langle f,\psi^i_{j+1,k}\rangle$ as r dimension column vectors $\mathbf{d}^i_{j+1,k}$. Then, the former formula is equivalent to

$$\begin{aligned}
&\sum_{k\in S_{j+1}}\left\langle f,\phi_{j+1,k}\right\rangle^T\phi_{j+1,k}\\
&=\sum_{k\in S_j}\left\langle f,\phi_{j,k}\right\rangle^T\phi_{j,k}+\sum_{i=1}^{M}\sum_{k\in S_j}\left\langle f,\psi^i_{j,k}\right\rangle^T\psi^i_{j,k}\\
&=\sum_{k\in S_j}\left\langle f,\frac{1}{\sqrt{a}}\sum_{l\in S_{j+1}}P_{k,l}\phi_{j+1,l}\right\rangle^T\frac{1}{\sqrt{a}}\sum_{l\in S_{j+1}}P_{k,l}\phi_{j+1,l}\\
&\quad+\sum_{i=1}^{M}\sum_{k\in S_j}\left\langle f,\frac{1}{\sqrt{a}}\sum_{l\in S_{j+1}}Q^i_{k,l}\phi_{j+1,l}\right\rangle^T\frac{1}{\sqrt{a}}\sum_{l\in S_{j+1}}Q^i_{k,l}\phi_{j+1,l}\\
&=\frac{1}{a}\sum_{l\in S_{j+1}}\sum_{m\in S_{j+1}}\sum_{k\in S_j}\left\langle f,\phi_{j+1,m}\right\rangle^T\left(P^*_{k,m}P_{k,l}+\sum_{i=1}^{M}Q^{i*}_{k,m}Q^i_{k,l}\right)\phi_{j+1,l},
\end{aligned} \tag{23}$$

which is equivalent to

$$\sum_{l\in S_{j+1}}\sum_{m\in S_{j+1}}\left\langle f,\phi_{j+1,m}\right\rangle^{T} \cdot\left\{\sum_{k\in S_j}\left(P_{k,m}^{*}P_{k,l}+\sum_{i=1}^{M}Q_{k,m}^{i*}Q_{k,l}^{i}\right)-a\delta_{m,l}I_r\right\}\phi_{j+1,l} = 0 \tag{24}$$

or

$$\sum_{l\in S_{j+1}}\sum_{m\in S_{j+1}}\left\langle f,\phi_{j+1,m}\right\rangle^{T}\alpha_{m,l}\phi_{j+1,l}=0, \quad \forall f\in L^2([0,1]), \tag{25}$$

where $\alpha_{m,l}=\sum_{k\in S_j}(P_{k,m}^{*}P_{k,l}+\sum_{i=1}^{M}Q_{k,m}^{i*}Q_{k,l}^{i})-a\delta_{m,l}I_r$.

We multiply the same identities in (20) by $\widehat{\Phi}_{j+1}(\omega/a)$, to get

$$a\sqrt{a}\left(P(z)^{*}\widehat{\Phi}_j(\omega)+\sum_{i=1}^{M}Q^{i}(z)^{*}\widehat{\Psi}_j^{i}(\omega)\right) =\widehat{\Phi}_{j+1}\left(\frac{\omega}{a}\right), \tag{26}$$

and this is equivalent to

$$\begin{aligned}&\sqrt{a}\left(\left(D_j^{-1}PD_{j+1}\right)^{*}\widehat{\Phi}_j(\omega)+\sum_{i=1}^{M}\left(D_j^{-1}Q^{i}D_{j+1}\right)^{*}\widehat{\Psi}_j^{i}(\omega)\right)=\widehat{\Phi}_{j+1}\left(\frac{\omega}{a}\right)\\&\Longleftrightarrow P^{*}\sqrt{a}\Phi_j(x)+\sum_{i=1}^{M}\left(Q^{i}\right)^{*}\sqrt{a}\Phi_j(x)=a\Phi_{j+1}(x)\Longleftrightarrow P^{*}P\Phi_{j+1}(x)\\&+\sum_{i=1}^{M}\left(Q^{i}\right)^{*}Q^{i}\Phi_{j+1}(x)=a\Phi_{j+1}(x)\Longleftrightarrow\left(P^{*}P+\sum_{i=1}^{M}\left(Q^{i}\right)^{*}Q^{i}\right)\Phi_{j+1}(x)=a\Phi_{j+1}(x)\end{aligned} \tag{27}$$

or

$$\sum_{m\in S_{j+1}}\alpha_{m,l}\phi_{j+1,l}=\mathbf{0}, \quad \forall l\in S_{j+1}. \tag{28}$$

In other words, the proof of theorem can be reduced to the proof of the equivalence of (21) and (25). It is obvious that (21) $\Rightarrow$ (28) $\Rightarrow$ (25). To show (25) $\Rightarrow$ (21), take any $f\in L^2([0,1])$ and define

$$\beta_l(f):=\sum_{m\in S_{j+1}}\left\langle f,\phi_{j+1,m}\right\rangle^{T}\alpha_{m,l}, \quad \forall l\in S_{j+1}. \tag{29}$$

The Fourier transform of (25) is

$$\sum_{l\in S_{j+1}}\beta_l(f)\,e^{-il\omega/a^{j+1}}=\mathbf{0}. \tag{30}$$

Then $\beta_l(f)=\mathbf{0}$, $\forall l\in S_{j+1}$; that is,

$$\left\langle f,\sum_{m\in S_{j+1}}\alpha_{m,l}^{*}\phi_{j+1,m}\right\rangle=\mathbf{0}, \quad l\in S_{j+1}. \tag{31}$$

Because $f\in L^2([0,1])$ is a compact support function,

$$\beta_l(f)=\mathbf{0}, \quad \forall l\in S_{j+1},\ \forall f\in L^2([0,1]); \tag{32}$$

that is,

$$\left\langle f,\sum_{m\in S_{j+1}}\alpha_{m,l}^{*}\phi_{j+1,m}\right\rangle=\mathbf{0}, \quad l\in S_{j+1},\ \forall f\in L^2([0,1]). \tag{33}$$

Then, we can get

$$\sum_{m\in S_{j+1}}\alpha_{m,l}\phi_{j+1,m}=\mathbf{0}, \quad l\in S_{j+1}, \tag{34}$$

and this is equivalent to $\alpha_{m,l}=0_r$, $\forall m,l\in S_{j+1}$.

Related to the proof, $\mathbf{0}$ is zero vector with appropriate size, and 0_r is $r\times r$ zero matrix. □

Theorem 5 characterizes the necessary and sufficient condition for the existence of the MEMWFI associated with ϕ. But it is not a good choice to use this theorem to construct MEMWFI with arbitrary dilation a. For convenience, we need to present some algorithms of constructing MEMWFI.

2.3. Algorithms of Constructing MEMWFI. With matrices P,Q^{i}, $i=1,2,\ldots,M$, we formulate the following block matrix:

$$\mathscr{M}=\left(P^{*},Q^{1*},Q^{2*},\ldots,Q^{M*}\right), \tag{35}$$

and then, (21), the characterization of MEMWFI, can be rewritten as

$$\mathscr{M}\mathscr{M}^{*}=aI_{|S_{j_0}|r}. \tag{36}$$

Now we present two algorithms of constructing MEMWFI starting from a multiscaling function and a scalar scaling function, respectively.

Algorithm 1. (1) Take a proper multiscaling function $\phi(x)$.

(2) Calculate the parameters $\gamma,\gamma_1,N=\lceil\gamma/a\rceil$, $j_0=\lceil\log_a\gamma\rceil$, N_1,N_2.

(3) Choose proper coefficient matrices $C_{k,n}$, $0\le k\le N-1$, $-\gamma+1\le n\le k-N$ and $C_{k,n}^{s}$, $0\le k\le N-1$, $a^{j_0}-\gamma+1+k\le n\le a^{j_0}-1$, $1\le s\le a-1$, and construct boundary multiscaling functions using (5); in addition, we can get Φ_{j_0}.

(4) Use Theorem 2, ascertain coefficient matrices $P_{k,n}^{L}, 0 \le k, n \le N-1$; $p_{k,n}^{L}, 0 \le k \le N-1, 0 \le n \le ak+N_1$ and $p_{k,n}^{R,s}, 0 \le k \le N-1, 0 \le n \le ak+N_2$; $P_{k,n}^{R,s}$, $0 \le k, n \le N-1$; $1 \le s \le a-1$ to get the block matrix P.

(5) Take coefficient matrices $Q_{k,n}^{L,i}, 0 \le k, n \le N-1$; $q_{k,n}^{L,i}, 0 \le k \le N-1, 0 \le n \le ak+N_1$; $Q_n^i, 0 \le n \le \gamma_1$ and $q_{k,n}^{R,s,i}, 0 \le k \le N-1, 0 \le n \le ak+N_2$; $Q_{k,n}^{R,s,i}, 0 \le k, n \le N-1$; $1 \le s \le a-1, 1 \le i \le M$ to construct the block matrixes $Q^i, i = 1, \ldots, M$, and ensure that they satisfy (21) with P.

(6) Use the coefficient matrices Q^i which we have got in step (5) to construct MEMWFI $\Psi_{j_0} = \{\Psi_{j_0}^1, \ldots, \Psi_{j_0}^M\}$, by Theorem 3.

(7) Draw the graphs of Φ_{j_0} and Ψ_{j_0} by Matlab.

Algorithm 2. (1) Take a proper scalar scaling function $\varphi(x)$ (the quadratic sum of recursion coefficients can not be larger than 1).

(2) Use $\varphi(x)$ to get refinable vector-valued function $\phi(x)$; for example,

$$\begin{aligned} \phi(x) &= (\varphi(x), \varphi(x), \ldots, \varphi(x))_r^T, \\ \phi(x) &= (\varphi(x), \varphi(x-1), \ldots, \varphi(x-r+1))^T, \\ \phi(x) &= (\varphi(ax), \varphi(ax-1), \ldots, \varphi(ax-r+1))^T, \end{aligned} \tag{37}$$

and so on; we can find the sequence of matrices $\{P_k\}$ which satisfies refinable equation with ϕ. We should know that the necessary and sufficient conditions for MEMWFI in Theorem 5 reduce to sufficient conditions when the set of components in the vector-valued function $\phi(x)$ is linearly dependent.

(3) Turn to steps (3)–(7) of Algorithm 1.

We should note that the refinable vector-valued function ϕ in Algorithm 2 can not be a multiscaling function.

2.4. Decomposition Formula and Reconstruction Formula of MEMWFI. The decomposition formula and reconstruction formula of MEMWFI are obtained by Theorems 2, 3, and 5, which are similar to those of orthogonal wavelets or multiwavelets on the interval. Firstly, for $\forall f \in L^2([0,1])$, denote

$$\begin{aligned} \mathbf{c}_{j,k} &= \langle f, \phi_{j,k} \rangle, \\ \mathbf{c}_{j,k}^{L} &= \langle f, \phi_{j,k}^{L} \rangle, \\ \mathbf{c}_{j,k}^{R,s} &= \langle f, \phi_{j,k}^{R,s} \rangle, \\ \mathbf{d}_{j,k}^{i} &= \langle f, \psi_{j,k}^{i} \rangle, \\ \mathbf{d}_{j,k}^{L,i} &= \langle f, \psi_{j,k}^{L,i} \rangle, \\ \mathbf{d}_{j,k}^{R,s,i} &= \langle f, \psi_{j,k}^{R,s,i} \rangle \\ i &= 1, \ldots, M;\ s = 1, \ldots, a-1. \end{aligned} \tag{38}$$

Theorem 6. *The decomposition formula of MEMWFI is*

$$\begin{aligned} \sqrt{a}\mathbf{c}_{j,k}^{L} &= \sum_{n=0}^{N-1} P_{k,n}^{L}\mathbf{c}_{j+1,n}^{L} + \sum_{n=0}^{ak+N_1} p_{k,n}^{L}\mathbf{c}_{j+1,n}^{L}, \\ &\qquad 0 \le k \le N-1 \\ \sqrt{a}\mathbf{c}_{j,k} &= \sum_{n=ak}^{ak+\gamma_1} P_{n-ak}\mathbf{c}_{j+1,n}, \quad 0 \le k \le a^j - \gamma \\ \sqrt{a}\mathbf{c}_{j,k}^{R,s} &= \sum_{n=0}^{ak+N_2} p_{k,n}^{R,s}\mathbf{c}_{j+1,a^{j+1}-\gamma-n} + \sum_{n=0}^{ak+N_1} P_{k,n}^{R,s}\mathbf{c}_{j+1,n}^{R,s}, \\ &\qquad 0 \le k \le N-1,\ 1 \le s \le a-1 \\ \sqrt{a}\mathbf{d}_{j,k}^{L,i} &= \sum_{n=0}^{N-1} Q_{k,n}^{L,i}\mathbf{c}_{j+1,n}^{L} + \sum_{n=0}^{ak+N_1} q_{k,n}^{L,i}\mathbf{c}_{j+1,n}, \\ &\qquad 0 \le k \le N-1 \\ \sqrt{a}\mathbf{d}_{j,k}^{i} &= \sum_{n=ak}^{ak+\gamma_1} Q_{n-ak}^{i}\mathbf{c}_{j+1,n}, \quad 0 \le k \le a^j - \gamma \\ \sqrt{a}\mathbf{d}_{j,k}^{R,s,i} &= \sum_{n=0}^{ak+N_2} q_{k,n}^{R,s,i}\mathbf{c}_{j+1,a^{j+1}-\gamma-n} + \sum_{n=0}^{N-1} Q_{k,n}^{R,s,i}\mathbf{c}_{j+1,n}^{R,s}, \\ &\qquad 0 \le k \le N-1,\ 1 \le s \le a-1,\ i = 1, \ldots, M, \end{aligned} \tag{39}$$

and the reconstruction formula of MEMWFI is

$$\begin{aligned} \sqrt{a}\mathbf{c}_{j+1,n}^{L} &= \sum_{k=0}^{N-1} P_{k,n}^{L}\mathbf{c}_{j,k}^{L} + \sum_{i=1}^{M}\sum_{k=0}^{N-1} Q_{k,n}^{L,i}\mathbf{d}_{j,k}^{L,i}, \quad 0 \le n \le N-1 \\ \sqrt{a}\mathbf{c}_{j+1,n}^{R,s} &= \sum_{k=0}^{N-1} P_{k,n}^{R,s}\mathbf{c}_{j,k}^{R,s} + \sum_{i=1}^{M}\sum_{k=0}^{N-1} Q_{k,n}^{L,s,i}\mathbf{d}_{j,k}^{L,s,i}, \quad 0 \le n \le N-1,\ 1 \le s \le a-1 \\ \sqrt{a}\mathbf{c}_{j+1,n} &= \sum_{k=0}^{N-1} p_{k,n}^{L}\mathbf{c}_{j,k}^{L} + \sum_{i=0}^{M}\sum_{k=0}^{N-1} q_{k,n}^{L}\mathbf{d}_{j,k}^{L,i} + \sum_{k} P_{n-ak}\mathbf{c}_{j,k} + \sum_{i=1}^{M}\sum_{k} Q_{n-ak}^{i}\mathbf{d}_{j,k}^{i}, \quad 0 \le n \le N_1 \end{aligned}$$

$$\sqrt{a}\mathbf{c}_{j+1,n} = \sqrt{a}\mathbf{c}_{j+1,N_1+al+n'} = \sum_{k=l+1}^{N-1} p^{L}_{k,N_1+al+n'}\mathbf{c}^{L}_{j,k} + \sum_{i=1}^{M}\sum_{k=l+1}^{N-1} q^{L}_{k,N_1+al+n'}\mathbf{d}^{L,i}_{j,k} + \sum_{k} P_{N_1+n'+a(l-k)}\mathbf{c}_{j,k} + \sum_{i=1}^{M}\sum_{k} Q^{i}_{N_1+n'+a(l-k)}\mathbf{d}^{i}_{j,k},$$
$$0 \le l \le N-2,\ 1 \le n' \le a,\ N_1+1 \le n \le a(N-1)+N_1$$

$$\sqrt{a}\mathbf{c}_{j+1,n} = \sum_{k} P_{n-ak}\mathbf{c}_{j,k} + \sum_{i=1}^{M}\sum_{k} Q^{i}_{n-ak}\mathbf{d}^{i}_{j,k}, \quad (a-1)\gamma - a + 1 \le n \le a^{j+1} - a\gamma + a - 1 \tag{40}$$

$$\begin{aligned}\sqrt{a}\mathbf{c}_{j+1,n} &= \sqrt{a}\mathbf{c}_{j+1,(N-1-l)a+N_2-n'+(a^{j+1}-a\gamma+a)}\\ &= \sum_{s=1}^{a-1}\sum_{k=l+1}^{N-1} p^{R,s}_{k,(N-1-l)a+N_2-n'}\mathbf{c}^{R,s}_{j,k} + \sum_{i=1}^{M}\sum_{s=1}^{a-1}\sum_{k=l+1}^{N-1} q^{R,s,i}_{k,(N-1-l)a+N_2-n'}\mathbf{d}^{R,s,i}_{j,k} + \sum_{k} P_{(N-1-l-k)a+N_2-n'+(a^{j+1}-a\gamma+a)}\mathbf{c}_{j,k}\\ &\quad + \sum_{i=1}^{M}\sum_{k} Q^{i}_{(N-1-l-k)a+N_2-n'+(a^{j+1}-a\gamma+a)}\mathbf{d}^{i}_{j,k},\end{aligned} \tag{41}$$
$$0 \le l \le N-2,\ 0 \le n' \le a-1,\ a^{j+1} - a\gamma + a \le n \le a^{j+1} - a\gamma + aN - 1$$

$$\sqrt{a}\mathbf{c}_{j+1,n} = \sum_{s=1}^{a-1}\sum_{k=0}^{N-1} p^{R,s}_{k,a^{j+1}-\gamma-n}\mathbf{c}^{R,s}_{j,k} + \sum_{i=1}^{M}\sum_{s=1}^{a-1}\sum_{k=0}^{N-1} q^{R,s,i}_{k,a^{j+1}-\gamma-n}\mathbf{d}^{R,s,i}_{j,k} + \sum_{k} P_{n-ak}\mathbf{c}_{j,k} + \sum_{i=1}^{M}\sum_{k} Q^{i}_{n-ak}\mathbf{d}^{i}_{j,k}, \tag{42}$$
$$a^{j+1} - a\gamma + aN \le n \le a^{j+1} - \gamma.$$

Proof. The decomposition formula can be obtained by Theorems 2 and 3, and the reconstruction formula is gotten from Theorems 2, 3, and 5. □

3. Numerical Examples

In this section, we present some numerical examples to show the effectiveness of the proposed algorithm with dilator factors $a = 2$, $a = 3$.

$N^{a}_{m}(x)$ is mth order cardinal B-spline with dilation factor a; the refinement equation of $N^{a}_{m}(x)$ is the following:

$$\begin{aligned}\widehat{N}^{a}_{m}(\omega) &= h^{a}_{m}(z)\,\widehat{N}^{a}_{m}\left(\frac{\omega}{a}\right),\\ h^{a}_{m}(z) &= \left(\frac{1 + z + \cdots + z^{a-1}}{a}\right)^{m},\end{aligned} \tag{43}$$

where $z = e^{-i\omega/a}$.

3.1. $a = 2$

Example 1. With $a = 2$, the symbol of the B-spline $N^{2}_{2}(x)$ is

$$h^{2}_{2}(z) = \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^{2}. \tag{44}$$

Take $\phi_1(x) = N_2(x), \phi_2(x) = N_2(x-1), \phi(x) = (\phi_1(x), \phi_2(x))^{T}$, and $\phi(x)$ satisfies

$$\begin{aligned}\phi(x) = \frac{1}{2}\left\{\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\phi(2x) + \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\phi(2x-1)\right.\\ \left. + \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\phi(2x-2) + \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\phi(2x-3)\right\}.\end{aligned} \tag{45}$$

In fact, for $\phi(x)$, we can choose different set of coefficient matrices such that it satisfies the above equation.

Let $\gamma = 3 = \gamma_1, N = [\gamma/a] = 1, j_0 = \lceil \log_a \gamma \rceil = 2$. Taking $C_{0,-2} = C^{1}_{0,3} = 0_2, C_{0,-1} = C^{1}_{0,2} = I_2, M = 3$. Then, we can obtain a set of multiscaling functions on the interval by Theorem 2 and coefficient matrices by Theorem 5,

$$P = \frac{1}{2}\begin{pmatrix} 1&1&0&0&0&0&&&&&&&&&& \\ 0&0&1&1&1&1&&&&&&&&&& \\ &&1&1&1&1&0&0&0&0&&&&&& \\ &&0&0&0&0&1&1&1&1&&&&&& \\ &&&&&&1&1&1&1&0&0&0&0&& \\ &&&&&&0&0&0&0&1&1&1&1&& \\ &&&&&&&&&&1&1&1&1&0&0 \\ &&&&&&&&&&0&0&0&0&1&1 \end{pmatrix},$$

$$Q^1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & & & & & & & & & & \\ 0 & 0 & 1 & -1 & 1 & -1 & & & & & & & & & & \\ & & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & & & & & & \\ & & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & & & & & & \\ & & & & & & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & & \\ & & & & & & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & & \\ & & & & & & & & & & -1 & 1 & -1 & 1 & 0 & 0 \\ & & & & & & & & & & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$Q^2 = \frac{1}{2}\begin{pmatrix} \sqrt{3}I_2 & & & & & & & & & & & \\ & -2 & 0 & 2 & 0 & & & & & & & \\ & 0 & -\sqrt{2} & 0 & \sqrt{2} & & & & & & & \\ & & & & & -2 & 0 & 2 & 0 & & & \\ & & & & & 0 & -\sqrt{2} & 0 & \sqrt{2} & & & \\ & & & & & & & & & -2I_2 & 2I_2 & 0I_2 \end{pmatrix},$$

$$Q^3 = \frac{1}{2}\begin{pmatrix} \sqrt{3}I_2 & & & & & & & & & & & \\ & 0 & 0 & 0 & 0 & & & & & & & \\ & 0 & -\sqrt{2} & 0 & \sqrt{2} & & & & & & & \\ & & & & & 0 & 0 & 0 & 0 & & & \\ & & & & & 0 & -\sqrt{2} & 0 & \sqrt{2} & & & \\ & & & & & & & & & 0I_2 & 0I_2 & \sqrt{6}I_2 \end{pmatrix}. \tag{46}$$

The graphs of ϕ, left (right) boundary multiscaling functions, and MEMWFI are showed in Figure 1.

The graphs of the first row in Figure 1 are the multiscaling function ϕ and its boundary multiscaling functions, and the rest of rows belong to MEMWFI associated with ϕ.

We can discover from Figure 1 that each component of multiscaling function and MEMWFI is (anti)symmetrical. The boundary multiscaling functions have advantages of simple structure and some of them are (anti)symmetric.

Example 2. With $a = 2$, the symbol of 3rd order B-spline $N_3^2(x)$ is

$$h_3^2(z) = \frac{1}{8} + \frac{3}{8}z + \frac{3}{8}z^2 + \frac{1}{8}z^3. \tag{47}$$

Take $\phi_1(x) = N_3(2x), \phi_2(x) = N_3(2x-1), \phi(x) = (\phi_1(x), \phi_2(x))^T$, and ϕ satisfies

$$\phi(x) = \frac{1}{4}\left\{\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}\phi(2x) + \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\phi(2x-1) + \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}\phi(2x-2)\right\}. \tag{48}$$

Certainly, for $\phi(x)$, we can choose different set of coefficient matrices such that it satisfies the above equation.

Let $\gamma = 2 = \gamma_1, N = [\gamma/a] = 1$, $j_0 = \lceil \log_a \gamma \rceil = 1$. Taking $C_{0,-1} = C_{0,1}^1 = (1/3)I_2$, $M = 4$. By Theorems 2 and 5, we can obtain that

$$P = \frac{1}{4}\begin{pmatrix} 3 & 1 & 0 & 0 & & & & & & \\ 1 & 3 & 1 & \frac{1}{3} & & & & & & \\ & & 1 & 3 & 3 & 1 & 0 & 0 & & \\ & & 0 & 0 & 1 & 3 & 3 & 1 & & \\ & & & & & & \frac{1}{3} & 1 & 3 & 1 \\ & & & & & & 0 & 0 & 1 & 3 \end{pmatrix},$$

$$Q^1=\frac{1}{4}\begin{pmatrix} 3 & -2 & -\frac{1}{3} & -\frac{1}{9} & & & & & & \\ 0 & \frac{11}{3} & -1 & -\frac{1}{3} & & & & & & \\ & & 1 & -3 & 3 & -1 & 0 & 0 & & \\ & & 0 & 0 & -1 & 3 & -3 & 1 & & \\ & & & & & & -\frac{1}{3} & -1 & \frac{11}{3} & 0 \\ & & & & & & -\frac{1}{9} & -\frac{1}{3} & -2 & 3 \end{pmatrix},$$

$$Q^2=\frac{1}{4}\begin{pmatrix} \sqrt{6} & 0 & \sqrt{\frac{286}{27}} & 0 & & & & & & \\ 0 & 0 & \sqrt{\frac{19}{27}} & -\sqrt{\frac{19}{27}} & & & & & & \\ & & \sqrt{3} & 0 & -\sqrt{12} & 0 & \sqrt{3} & 0 & & \\ & & 0 & \sqrt{3} & 0 & -\sqrt{12} & 0 & \sqrt{3} & & \\ & & & & & & -\sqrt{\frac{19}{27}} & \sqrt{\frac{19}{27}} & 0 & 0 \\ & & & & & & 0 & \sqrt{\frac{286}{27}} & 0 & \sqrt{6} \end{pmatrix},$$

$$Q^3=\frac{1}{4}\begin{pmatrix} \sqrt{6} & 0 & -\sqrt{\frac{286}{27}} & 0 & & & & & & \\ 0 & \sqrt{\frac{32}{9}} & 0 & 0 & & & & & & \\ & & \sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 & & \\ & & 0 & -\sqrt{3} & 0 & 0 & 0 & \sqrt{3} & & \\ & & & & & & 0 & 0 & \sqrt{\frac{32}{9}} & 0 \\ & & & & & & 0 & -\sqrt{\frac{286}{27}} & 0 & \sqrt{6} \end{pmatrix},$$

$$Q^4=\frac{1}{4}\begin{pmatrix} 1 & 0 & & & & & & & & \\ 0 & 1 & & & & & & & & \\ & & 0 & \sqrt{\frac{286}{81}} & 0 & 0 & \sqrt{\frac{286}{81}} & 0 & & \\ & & 0 & \sqrt{\frac{286}{81}} & 0 & 0 & -\sqrt{\frac{286}{81}} & 0 & & \\ & & & & & & & & 1 & 0 \\ & & & & & & & & 0 & 1 \end{pmatrix}. \tag{49}$$

The graphs of ϕ, left (right) boundary multiscaling functions, and MEMWFI are showed in Figure 2.

The graphs of the first row in Figure 2 are multiscaling function ϕ and left (right) boundary multiscaling functions, and the rest of graphs belong to MEMWFI.

We can discover from Figure 2 that each component of multiscaling function and MEMWFI is (anti)symmetric and smooth in this example. Left (right) boundary multiscaling functions are smooth and some of them are (anti)symmetric. Every pair of left boundary multiscaling functions and right boundary multiscaling functions are mutually symmetric.

Example 3. The previous two examples are constructed when $r = 2$; we will give an example based on B-spline with $r = 3$ here.

With $a = 2$, the symbol of 4th order B-spline $N_4^2(x)$ is

$$h_4^2(z)=\frac{1}{16}+\frac{4}{16}z+\frac{6}{16}z^2+\frac{4}{16}z^3+\frac{1}{16}z^4. \tag{50}$$

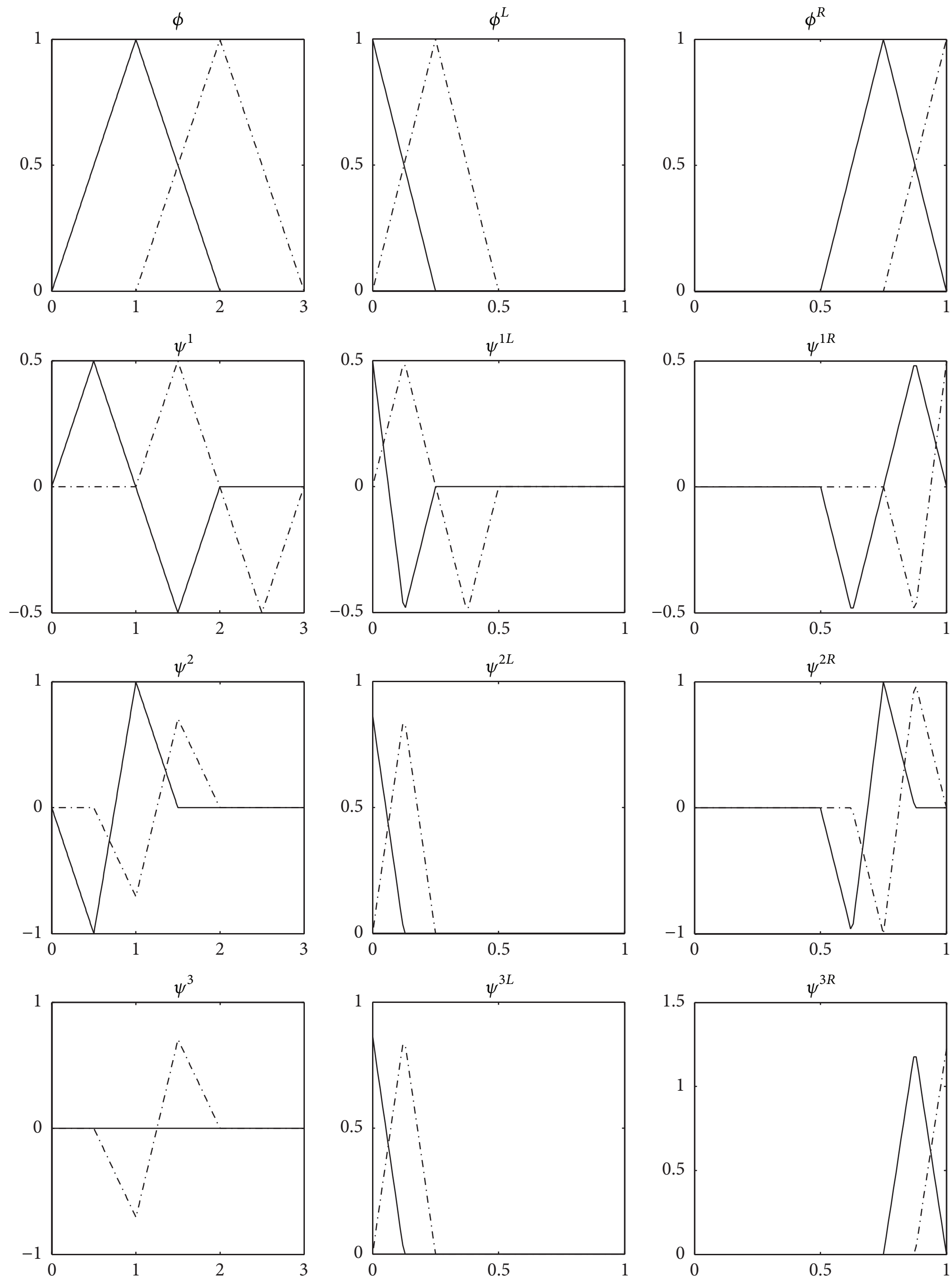

Figure 1: Graphs of multiscaling function and left (right) boundary multiscaling functions and MEMWFI with dilation factor 2 and multiplicity 2.

Take $\phi_1(x) = N_4(3x), \phi_2(x) = N_4(3x-1), \phi_3(x) = N_4(3x-2), \phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))^T$, and ϕ satisfies

$$\phi(x) = \frac{1}{8}\left\{\begin{pmatrix}1 & 4 & 6\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\phi(2x) + \begin{pmatrix}4 & 1 & 0\\ 4 & 6 & 4\\ 0 & 1 & 4\end{pmatrix}\phi(2x-1) + \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 6 & 4 & 1\end{pmatrix}\phi(2x-2)\right\}. \quad (51)$$

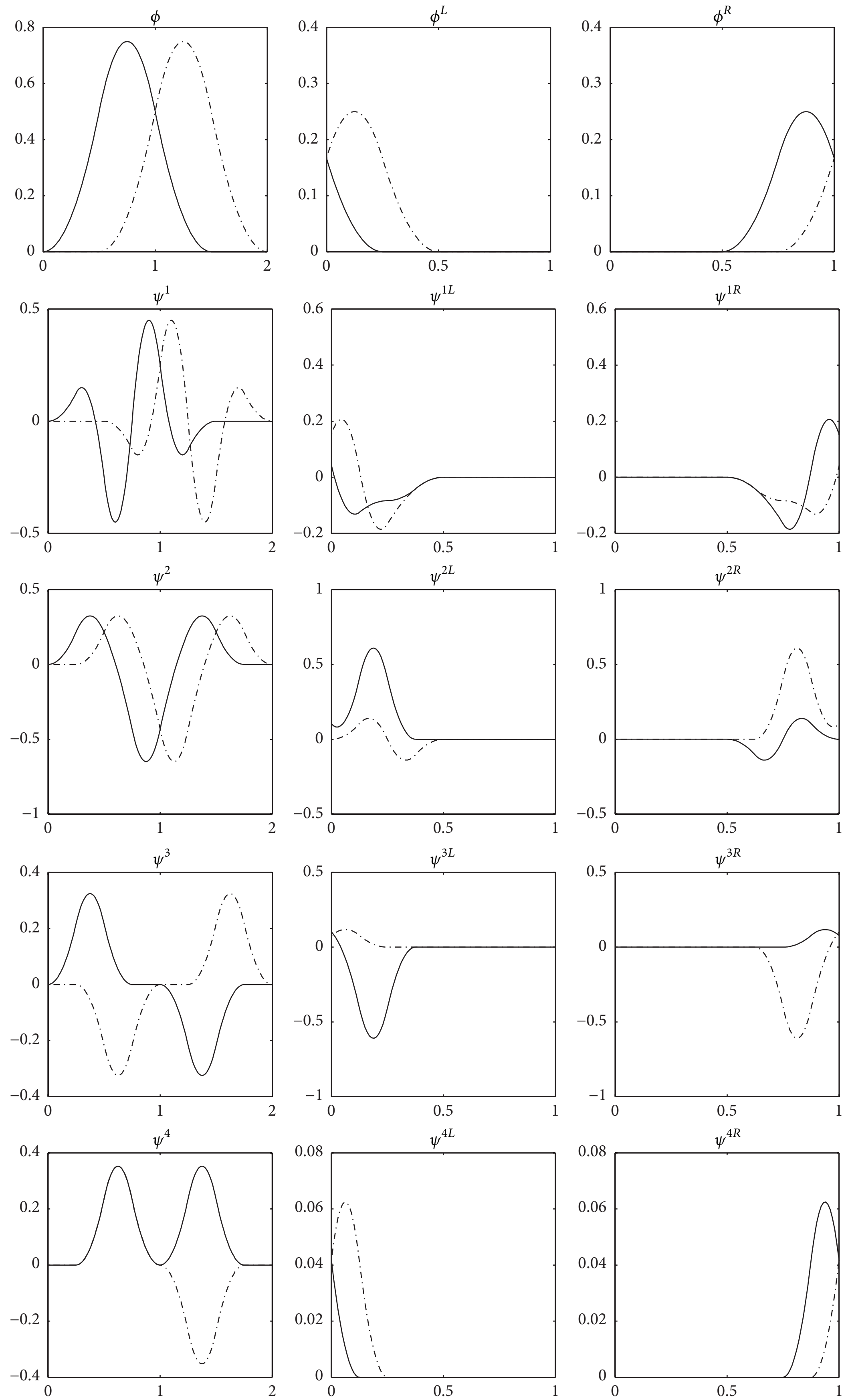

Figure 2: Graphs of multiscaling function and left (right) boundary multiscaling functions and MEMWFI with dilation factor 2 and multiplicity 2.

Let $\gamma = 2 = \gamma_1, N = [\gamma/a] = 1, j_0 = \lceil \log_a \gamma \rceil = 1$. Taking $C_{0,-1} = C_{0,1}^1 = (1/4)I_2, M = 6$. Then, by Theorems 2 and 5, we can get that

$$P = \frac{1}{8}\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & & & & & & & & & \\ 4 & 6 & 4 & \frac{1}{4} & 0 & 0 & & & & & & & & & \\ 0 & 1 & 4 & \frac{6}{4} & \frac{4}{4} & \frac{1}{4} & & & & & & & & & \\ & & & 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & & & \\ & & & 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 & 0 & & & \\ & & & 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & & & \\ & & & & & & & & & \frac{1}{4} & \frac{4}{4} & \frac{6}{4} & 4 & 1 & 0 \\ & & & & & & & & & 0 & 0 & \frac{1}{4} & 4 & 6 & 4 \\ & & & & & & & & & 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}_{9\times 15},$$

$$\left(Q_{11}^1, Q_{12}^1\right) = \frac{1}{8}\begin{pmatrix} \sqrt{19} & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 2 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 2 & -\frac{6}{2} & -\frac{4}{2} & -\frac{1}{2} \end{pmatrix},$$

$$Q_{22}^1 = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & -\sqrt{14} & \sqrt{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & -2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{14} & -\sqrt{14} & 0 & 0 & 0 \end{pmatrix},$$

$$\left(Q_{32}^1, Q_{33}^1\right) = \frac{1}{8}\begin{pmatrix} -\frac{1}{2} & -\frac{4}{2} & -\frac{6}{2} & 2 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{19} \end{pmatrix},$$

$$\left(Q_{11}^2, Q_{12}^2\right) = \frac{1}{8}\begin{pmatrix} \sqrt{34} & -\sqrt{34} & 0 & 0 & 0 & 0 \\ \sqrt{20} & 0 & -\sqrt{20} & 0 & 0 & 0 \\ 0 & -\sqrt{35} & \sqrt{35} & 0 & 0 & 0 \end{pmatrix},$$

$$Q_{22}^2 = \frac{1}{8}\begin{pmatrix} \frac{4}{7} & \frac{16}{7} & 4 & -7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7 & 4 & \frac{16}{7} & \frac{4}{7} \end{pmatrix},$$

$$\left(Q_{32}^2, Q_{33}^2\right) = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & \sqrt{35} & -\sqrt{35} & 0 \\ 0 & 0 & 0 & -\sqrt{20} & 0 & \sqrt{20} \\ 0 & 0 & 0 & 0 & -\sqrt{34} & \sqrt{34} \end{pmatrix},$$

$$\left(Q_{11}^3, Q_{12}^3\right) = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & \sqrt{\frac{15}{2}} & -\sqrt{\frac{15}{2}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{15}{8}} & 0 & -\sqrt{\frac{15}{8}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{4}} & -\sqrt{\frac{5}{4}} \end{pmatrix},$$

$$Q_{22}^3 = \frac{1}{8}\begin{pmatrix} \frac{1}{3} & \frac{4}{3} & 4 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 4 & \frac{4}{3} & \frac{1}{3} \end{pmatrix},$$

$$\left(Q_{32}^3, Q_{33}^3\right) = \frac{1}{8}\begin{pmatrix} -\sqrt{\frac{5}{4}} & \sqrt{\frac{5}{4}} & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{15}{8}} & 0 & \sqrt{\frac{15}{8}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{15}{2}} & \sqrt{\frac{15}{2}} & 0 & 0 & 0 \end{pmatrix},$$

$$\left(Q_{11}^4, Q_{12}^4\right) = \frac{1}{8}\begin{pmatrix} \sqrt{19} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{47}{4}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{33} & 0 & 0 & 0 \end{pmatrix},$$

$$Q_{22}^4 = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & \sqrt{3} & 0 & 0 & -\frac{4}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{4}{\sqrt{3}} & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \end{pmatrix},$$

$$\left(Q_{32}^4, Q_{33}^4\right) = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & \sqrt{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{47}{4}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{19} \end{pmatrix},$$

$$Q_{22}^5 = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & -\sqrt{7} & \sqrt{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{7} & \sqrt{7} & 0 & 0 & 0 \end{pmatrix},$$

$$\left(Q_{11}^5, Q_{12}^5\right) = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & \sqrt{\frac{2536}{441}} & -\sqrt{\frac{2536}{441}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{202}{21}} & 0 & -\sqrt{\frac{202}{21}} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{808}{21}} & \sqrt{\frac{808}{21}} \end{pmatrix},$$

$$\left(Q_{32}^5, Q_{33}^5\right) = \frac{1}{8}\begin{pmatrix} \sqrt{\frac{808}{21}} & -\sqrt{\frac{808}{21}} & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{202}{21}} & 0 & \sqrt{\frac{202}{21}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{2536}{441}} & \sqrt{\frac{2536}{441}} & 0 & 0 & 0 \end{pmatrix},$$

$$\left(Q_{11}^6, Q_{12}^6\right) = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & \sqrt{\frac{636841}{7056}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{82945}{1764}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{127}{112}} \end{pmatrix},$$

$$Q_{22}^6 = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & -\sqrt{7} & \sqrt{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{7} & \sqrt{7} & 0 & 0 & 0 \end{pmatrix},$$

$$\left(Q_{32}^6, Q_{33}^6\right) = \frac{1}{8}\begin{pmatrix} \sqrt{\frac{127}{112}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{82945}{1764}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{636841}{7056}} & 0 & 0 & 0 \end{pmatrix},$$

$$Q^i = \begin{pmatrix} Q_{11}^i & Q_{12}^i & \\ & Q_{22}^i & \\ & Q_{32}^i & Q_{33}^i \end{pmatrix}_{9\times 15}, \quad i = 1, 2, \ldots, 6. \tag{52}$$

The graphs of ϕ, left (right) boundary multiscaling functions, and MEMWFI are showed in Figure 3.

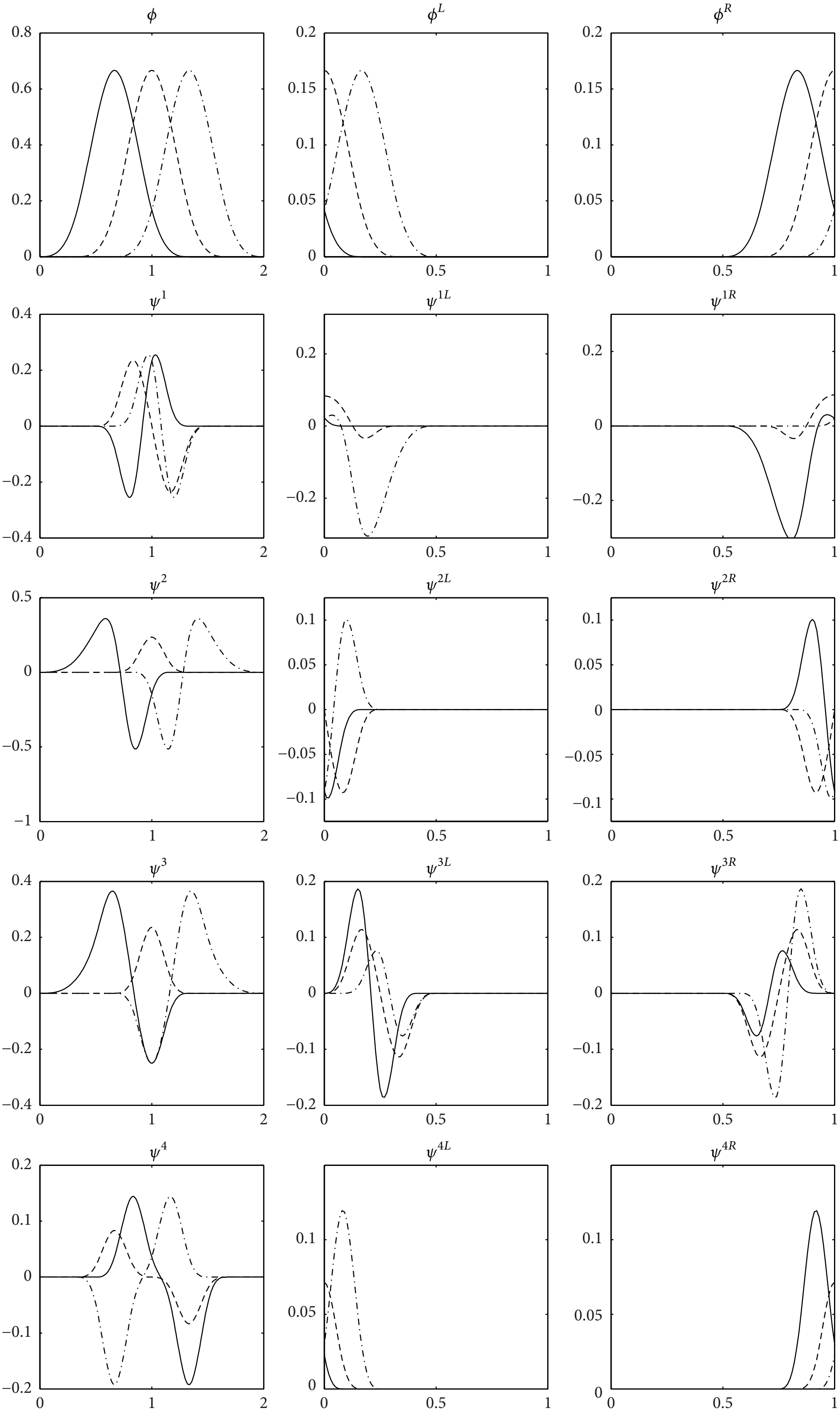

Figure 3: Continued.

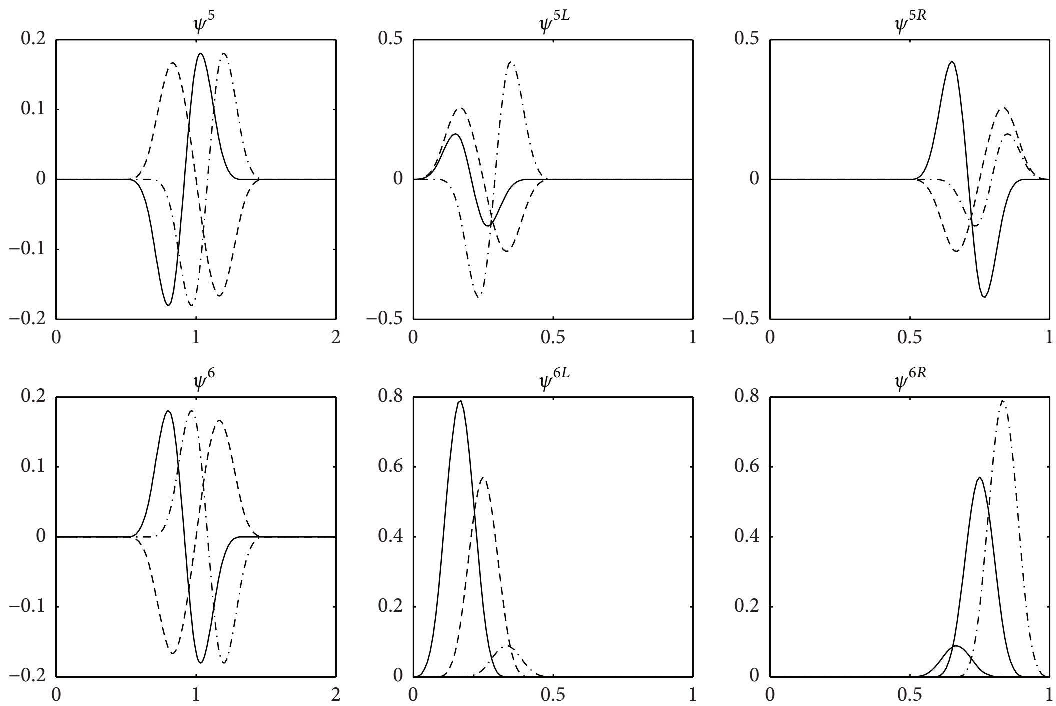

FIGURE 3: Graphs of multiscaling function and left (right) boundary multiscaling functions and MEMWFI with dilation factor 2 and multiplicity 3.

The graphs of the first row in Figure 3 are multiscaling function ϕ and left (right) boundary multiscaling functions, and the rest rows belong to MEMWFI.

We can discover from Figure 3 that each component of multiscaling function and MEMWFI is (anti)symmetric and smooth in this example. Left (right) boundary multiscaling functions are smooth and some of them are (anti)symmetric. Every pair of left boundary multiscaling functions and right boundary multiscaling functions are mutually symmetric.

3.2. a = 3

Example 4. We will construct this numerical example based on 2nd B-spline $N_2(x)$. With $a = 3$, the symbol of $N_2^3(x)$ is

$$h_2^3(z) = \frac{1 + 2z + 3z^2 + 2z^3 + z^4}{9}. \quad (53)$$

Take $\phi_1(x) = N_2(x), \phi_2(x) = N_2(x-1), \phi(x) = (\phi_1(x), \phi_2(x))^T$, and ϕ satisfies

$$\phi(x) = \frac{1}{3}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\phi(3x) + \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\phi(3x-1) + \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}\phi(3x-2) + \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\phi(3x-3) + \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\phi(3x-4) + \begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}\phi(3x-5) + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\phi(3x-6)\right\}. \quad (54)$$

In fact, we have other choices to get the refinable equations associated with ϕ.

Let $\gamma = 3, \gamma_1 = (a-1)\gamma = 6, N = [\gamma/a] = 1, j_0 = \lceil \log_a \gamma \rceil = 1$. Taking

$$C_{0,-2} = C_{0,2}^1 = C_{0,2}^2 = 0_2,$$
$$C_{0,-1} = \sqrt{2}C_{0,1}^1 = \sqrt{2}C_{0,1}^2 = \frac{1}{2}I_2, \quad (55)$$

where 0_2 denotes 2×2 zero matrix, I_2 denotes 2×2 unit matrix. Then, we can construct MEMWFI by Theorem 2 and Theorem 5 (see Figure 4). It is so lengthy that we do not write the expression of $P, Q^1, Q^2, Q^3, Q^4, Q^5, Q^6$ here. The graphs of ϕ, left (right) boundary multiscaling functions, and MEMWFI are showed in Figure 4.

The graphs of the first row in Figure 4 are multiscaling function ϕ and its left (right) boundary multiscaling functions, and the rest of rows belong to MEMWFI.

We can discover from Figure 4 that each component of multiscaling function and MEMWFI is (anti)symmetric in

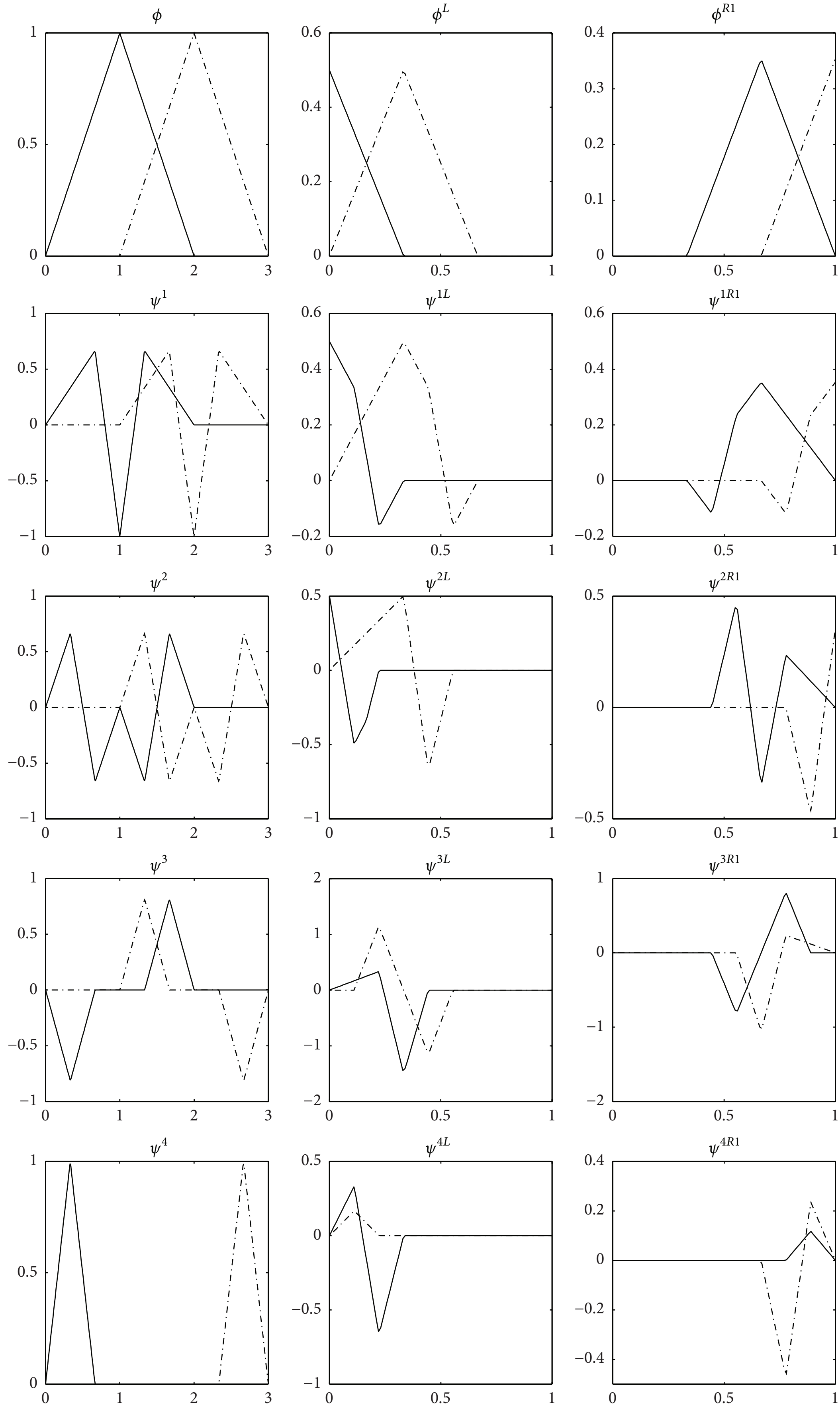

Figure 4: Continued.

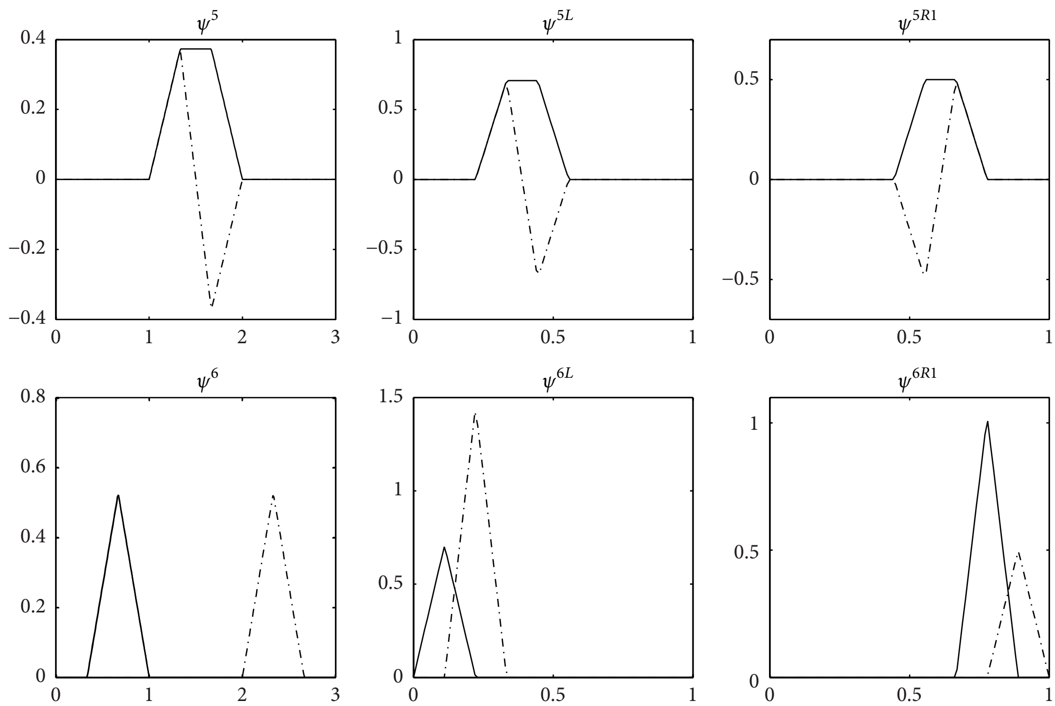

Figure 4: Graphs of multiscaling function and left (right) boundary multiscaling functions and MEMWFI with dilation factor 3 and multiplicity 2.

this example. Left (right) boundary multiscaling functions have advantages of simple structure and some of them are (anti)symmetric. The second right boundary multiscaling function has the same graph with the first.

Example 5. We will construct this numerical example based on 3rd B-spline $N_3(x)$. With $a = 3$, the symbol of $N_3^3(x)$ is

$$h_3^3(z) = \frac{1 + 3z + 6z^2 + 7z^3 + 6z^4 + 3z^5 + z^6}{27}. \tag{56}$$

Take $\phi_1(x) = N_3(x), \phi_2(x) = N_3(x), \phi(x) = (\phi_1(x), \phi_2(x))^T$, and ϕ satisfies

$$\begin{aligned} \phi(x) = \frac{1}{9}\Bigg\{ &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \phi(3x) + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \phi(3x-1) \\ &+ \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \phi(3x-2) + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} \phi(3x-3) \\ &+ \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \phi(3x-4) + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \phi(3x-5) \\ &+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \phi(3x-6) \Bigg\}. \end{aligned} \tag{57}$$

Certainly, we have other choices to get the refinable equations associated with ϕ. The components of the multiscaling function are linear dependence. It does not satisfy the assumption in Section 2. The sufficient and necessary conditions for minimum-energy frames in Theorem 5 reduce to sufficient conditions. We can still use Theorem 5 to get MEMWFI.

Now $\gamma = 3, \gamma_1 = (a-1)\gamma = 6, N = [\gamma/a] = 1, j_0 = \lceil \log_a \gamma \rceil = 1$. Take

$$\begin{aligned} C_{0,-2} &= C_{0,2}^1 = C_{0,2}^2 = 0_2, \\ C_{0,-1} &= \sqrt{2} C_{0,1}^1 = \sqrt{2} C_{0,1}^2 = \frac{1}{3} I_2. \end{aligned} \tag{58}$$

Then, we can construct MEMWFI by Theorem 2 and Theorem 5 (see Figure 5). It is so lengthy that we do not write the expression of $P, Q^1, Q^2, Q^3, Q^4, Q^5, Q^6, Q^7, Q^8$ here. The graphs of ϕ, left (right) boundary multiscaling functions, and MEMWFI are showed in Figure 5.

The graphs of the first row in Figure 5 are ϕ and its left (right) boundary multiscaling functions, and the rest of rows belong to MEMWFI associated with ϕ.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

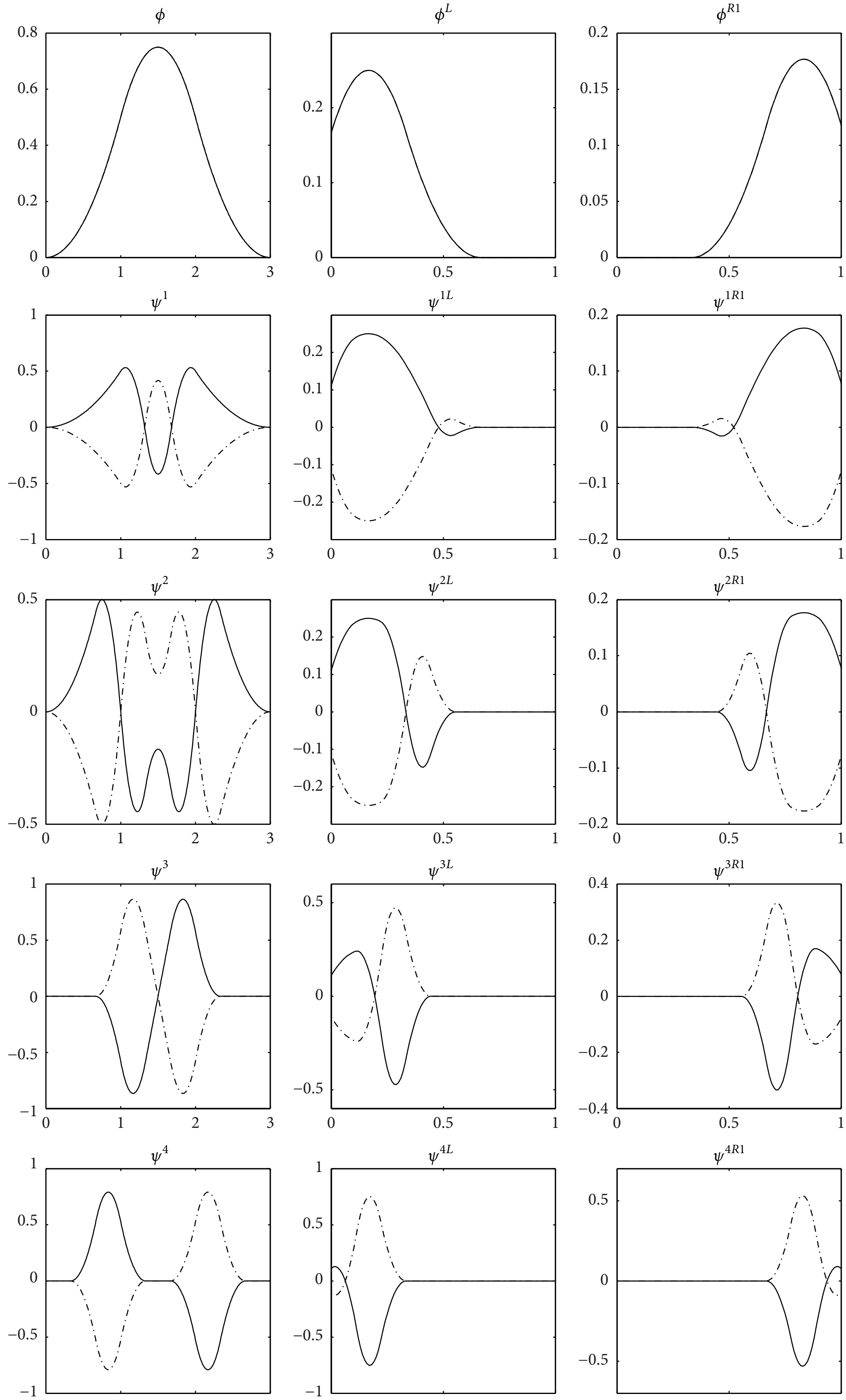

FIGURE 5: Continued.

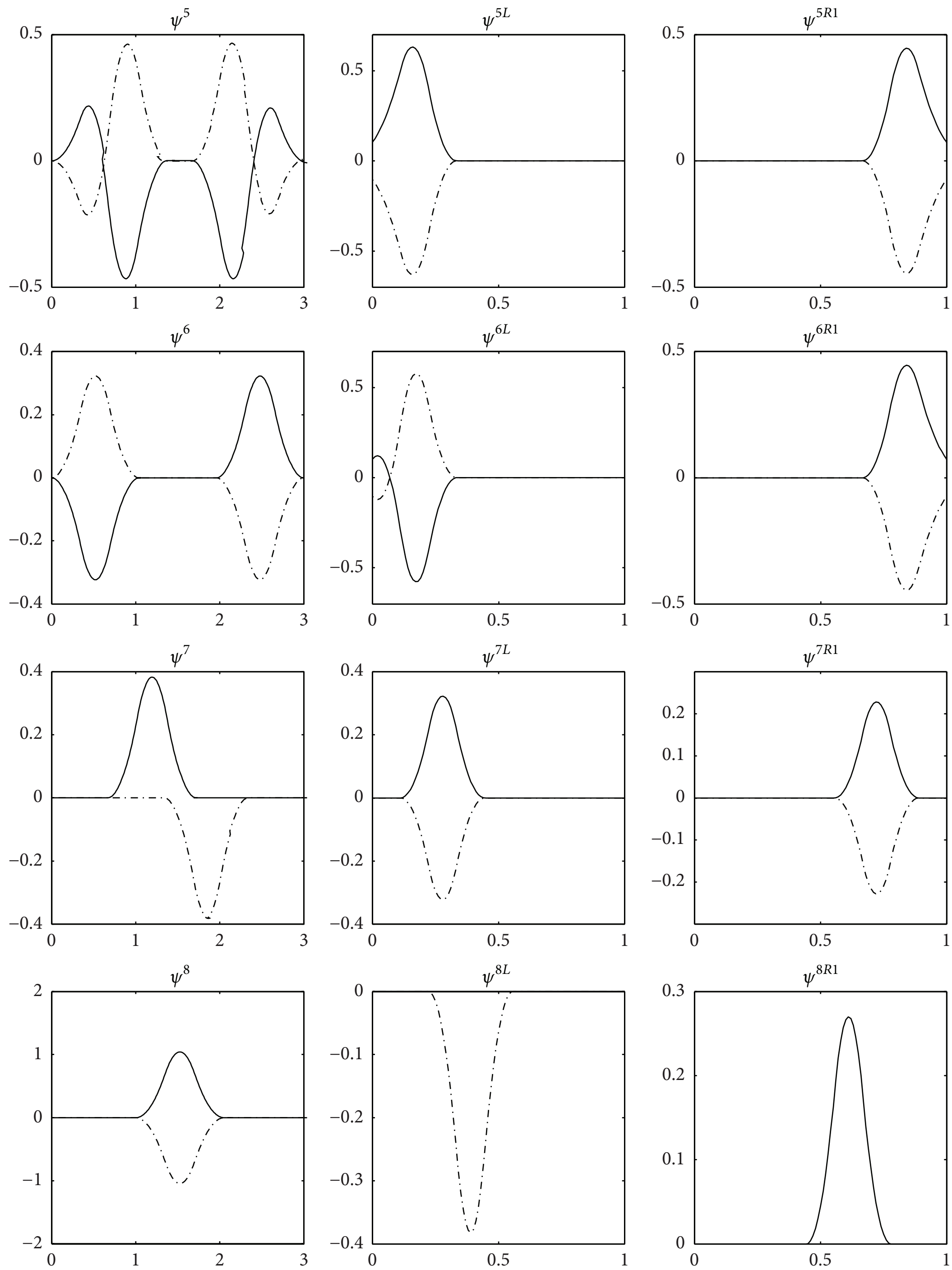

FIGURE 5: Graphs of multiscaling function and left (right) boundary multiscaling functions and MEMWFI with dilation factor 3 and multiplicity 2.

Acknowledgments

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