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*Research Article*

# Finite-Time Robust $H_\infty$ Control for Uncertain Linear Continuous-Time Singular Systems with Exogenous Disturbances

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Singular systems arise in a great deal of domains of engineering and can be used to solve problems which are more difficult and more extensive than regular systems to solve. Therefore, in this paper, the definition of finite-time robust $H_\infty$ control for uncertain linear continuous-time singular systems is presented. The problem we address is to design a robust state feedback controller which can deal with the singular system with time-varying norm-bounded exogenous disturbance, such that the singular system is finite-time robust bounded (FTRB) with disturbance attenuation $\gamma$. Sufficient conditions for the existence of solutions to this problem are obtained in terms of linear matrix equalities (LMIs). When these LMIs are feasible, the desired robust controller is given. A detailed solving method is proposed for the restricted linear matrix inequalities. Finally, examples are given to show the validity of the methodology.

## 1. Introduction

Singular system (known as well descriptor system or algebraic differential system) was introduced to model a large class of systems in many domains, such as physical, biological, and economic ones, to which the standard representation sometimes cannot be applied. The $H_\infty$ control problem for singular systems has attracted much attention due to its both practical and theoretical importance since 2000. Various approaches have been developed, and a great number of results for continuous singular systems and discrete singular systems have been reported in the literatures; see, for instance, [1–9].

On the other hand, most of the results in this field related to stability and performance criteria were defined over an infinite-time interval. In many practical applications, the main concerns are the behavior of the system over a fixed finite-time interval. It has shown that in [10] that a sufficient condition for robust finite-time stabilization for linear systems is provided. Moreover, in [11], the assumption that the state is available for feedback is removed and the output feedback problem is investigated. The corresponding results for discrete linear systems can be found in [12]. In [13], the design of time-varying state feedback controller guaranteeing that the finite-time closed-loop stability is presented. And some finite-time control problems for uncertain discrete-time linear systems subject to exogenous disturbance was dealt with in [14]. Furthermore it appears reasonable to provide a kind of stabilization definition for a system whose state remains within prescribed bounds in the fixed finite-time interval with some given initial conditions. For example, the main aim of [15] is focused on the design a state feedback controller which ensures that the closed-loop system is finite-time bounded (FTB) and reduces the effect of the disturbance input on the controlled output to a prescribed level. Recently, Feng et al. [16–18] extended the definition of finite-time stable (FTS) and the definition of finite-time bounded (FTB) of regular systems to ones of singular systems.

In this paper, we extend the definition of $H_\infty$ control, and a new definition of finite-time $H_\infty$ control for uncertain linear continuous singular systems (ULCTSS) is presented. Our main propose is to design a state feedback controller which guarantees that the closed-loop system is regular and impulse-free and FTB with a prescribed level of disturbance attenuation. A sufficient condition is presented for the solvability of this problem, which can be reduced to a feasibility problem involving linear matrix inequalities (LMIs). As a corollary, the existence condition and design method of the finite-time $H_\infty$ controller for continuous-time singular systems are given. Finally, examples are given to show the validity of the proposed approach.

*Notation 1.* Throughout this paper, for real symmetric matrices $X$ and $Y$, the notation $X \geq Y$ ($X > Y$, resp.) means that the matrix $X - Y$ is positive semidefinite (positive definite, resp.). $I$ is the identity matrix with appropriate dimension. The notation $N^T$ represents the transpose of the matrix $N$. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The notation of rank $X$ represents the rank of matrix $X$. $\|\cdot\|$ is the Euclidean matrix norm. $\mathrm{Re}(\cdot)$ is real part of a complex. $\lambda(\cdot)$ is the eigenvalue of a real symmetric matrix. $\lambda_{\max}(\cdot)$ is maximum the eigenvalue of a real symmetric matrix.

## 2. Preliminaries and Problem Formulation

In this paper, we consider the following uncertain linear continuous-time singular system (ULCTSS):

$$\begin{aligned} E\dot{x}(t) &= (A + \triangle A)\, x(t) + (B + \triangle B)\, u(t) + G\omega(t), \\ z(t) &= Cx(t) + D_1 u(t) + D_2 \omega(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathfrak{R}^n$ is the state vector; $u(t) \in \mathfrak{R}^m$ is the control input; $z(t) \in \mathfrak{R}^l$ is the control output; $\omega(t) \in \mathfrak{R}^q$ is the exogenous disturbance; matrices $E \in \mathfrak{R}^{n\times n}$, $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $G \in \mathfrak{R}^{n\times q}$, $C \in \mathfrak{R}^{l\times n}$, $D_1 \in \mathfrak{R}^{l\times m}$, and $D_2 \in \mathfrak{R}^{l\times q}$ are known mode-dependent constant matrices with appropriate dimensions, and rank $E = r < n$. $\triangle A$ and $\triangle B$ are unknown time-invariant matrix uncertainty, respectively, modeled as

$$\left[\triangle A \quad \triangle B\right] = MF(\sigma)\left[N_a \quad N_b\right], \tag{2}$$

where $M, N_a, N_b$ are known mode-dependent matrices with appropriate dimensions. $F(\sigma)$ is the time-invariant unknown matrix function with Lebesgue norm measurable elements satisfying

$$F^T(\sigma)\, F(\sigma) \leq I, \tag{3}$$

and $\sigma \in \Theta$, where $\Theta$ is a compact set. The uncertain matrices $\triangle A$ and $\triangle B$ are said to be admissible if both (2) and (3) hold. In this paper, the following assumptions, definitions, and lemmas play an important role in our later proof.

*Assumption 1.* The external disturbance $\omega(t)$ is time-variant and satisfies

$$\int_0^{+\infty} \omega^T(t)\, \omega(t)\, dt \leq d. \tag{4}$$

*Assumption 2.* There exist two orthogonal matrices $U$ and $V$ such that $E$ has the decomposition as

$$E = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^T, \tag{5}$$

where $\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ with $\sigma_i > 0$ for $i = 1, 2, \ldots, r$. Partition

$$\begin{aligned} U &= \left[U_1 \quad U_2\right], \\ V &= \left[V_1 \quad V_2\right] \end{aligned} \tag{6}$$

conformably with (5). From (5), it can be seen that $V_2$ spans the right null space of $E$, and $U_2^T$ spans the left null space of $E$; that is, $EV_2 = 0$ and $U_2^T E = 0$.

*Definition 3* (see [16]). The linear continuous-time singular system (LCTSS) (7) with $\omega(t) = 0$,

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + G\omega(t), \\ x(0) &= x_0, \end{aligned} \tag{7}$$

is said to be regular, if $\det(sE - A)$ is not identically zero.

*Definition 4* (see [16]). The LCTSS (7) with $\omega(t) = 0$ is said to be impulse-free, if $\deg(\det(sE - A)) = \mathrm{rank}\, E$.

*Definition 5.* The LCTSS (7) subject to an exogenous disturbance $\omega(t)$ satisfies (4) and is said to be finite-time bounded (FTB) with respect to $(c_1, c_2, T, R, d)$ ($0 < c_1 < c_2$ and $R > 0$), if

(i) the CTLSS (7) is said to be regular and impulse-free, when $\omega(t) = 0$;

(ii) $x_0^T E^T R E x_0 \leq c_1 \Rightarrow x^T(t) E^T R E x(t) < c_2$, $\forall t \in [0, T]$.

*Definition 6.* The uncertain linear continuous-time singular systems (ULCTSS),

$$\begin{aligned} E\dot{x}(t) &= \left[A + \triangle A\right] x(t) + G\omega(t), \\ x(0) &= x_0, \end{aligned} \tag{8}$$

subject to an exogenous disturbance $\omega(t)$ satisfy (4) and $\triangle A$ satisfies (2) and is said to be finite-time robust bounded (FTRB) with respect to $(c_1, c_2, T, R, d)$ ($0 < c_1 < c_2$ and $R > 0$), if

(i) the ULCTSS (8) is said to be regular and impulse-free, when $\omega(t) = 0$;

(ii) $x_0^T E^T R E x_0 \leq c_1 \Rightarrow x^T(t) E^T R E x(t) < c_2$, $\forall t \in [0, T]$.

**Lemma 7** (Desoer and Vidyasagar, 1975). *The matrix measure $\mu(X)$ of the matrix $X$ has following properties:*

(i) $-\|X\| \leq \mathrm{Re}\lambda(X) \leq \mu(X) \leq \|X\|$.

(ii) $\mu(X) = (1/2)\lambda_{\max}(X + X^T)$.

**Lemma 8** (see [19]). *The following items are true.*

*(i) All $P$ satisfying $E^T P = P^T E \geq 0$ can be parameterized as $P = U_1 W U_1^T E + U_2 S$, where $0 \leq W \in \mathfrak{R}^{r\times r}$ and $S \in \mathfrak{R}^{(n-r)\times n}$ are parameter matrices; furthermore, when $P$ is nonsingular, $W > 0$.*

*(ii) All $X$ satisfying $XE^T = EX^T \geq 0$ can be parameterized as $X = EV_1\widehat{W}V_1^T + \widehat{S}V_2^T$, where $0 \leq \widehat{W} \in \mathfrak{R}^{r\times r}$ and $\widehat{S} \in \mathfrak{R}^{(n-r)\times n}$ are parameter matrices; furthermore, when $X$ is nonsingular, $\widehat{W} > 0$.*

*(iii) If $U_1 W U_1^T E + U_2 S$ is nonsingular with $W > 0$, then there exist $\widehat{W}$ and $\widehat{S}$ such that*

$$U_1 W U_1^T E + U_2 S = \left(EV_1\widehat{W}V_1^T + \widehat{S}V_2^T\right)^{-1} \tag{9}$$

*with $\widehat{W} = \Sigma_r^{-1} W^{-1} \Sigma_r^{-1}$.*

**Lemma 9** (see [20]). *Let $D$, $H$, $F$ be real matrices of appropriate dimensions such that $F^T(t)F(t) \leq I$. For any scalar $\varepsilon > 0$, then we have the following:*

$$DF(t)H + (DF(t)H)^T \leq \varepsilon DD^T + \frac{1}{\varepsilon}H^T H. \tag{10}$$

Consider the following state feedback controller:

$$u(t) = Kx(t), \tag{11}$$

where $K$ is the controller gain to be designed. Then, the uncertain closed-loop systems is as follows:

$$\begin{aligned} E\dot{x}(t) &= \left[A_K + \triangle A_K\right] x(t) + G\omega(t), \\ z(t) &= C_K x(t) + D_2\omega(t), \end{aligned} \tag{12}$$

where $A_K = A + BK$, $\triangle A_K = \triangle A + \triangle BK$, $C_K = C + D_1 K$.

The finite-time robust $H_\infty$ control problem to be addressed in this paper can be formulated as finding a state feedback controller in the form of (11) such that

(i) the uncertain closed-loop system (12) is FTRB;

(ii) under the zero-initial condition, the controlled output $z$ satisfies

$$\int_0^T z^T(t)z(t)\,dt < \gamma^2 \int_0^T \omega^T(t)\omega(t)\,dt, \tag{13}$$

for any nonzero $\omega(t)$ satisfies (4), where $\gamma$ is a prescribed scalar.

## 3. Main Results

The following lemma states a sufficient condition for the FTB of system (7), which is the fundament to obtain the main results.

**Lemma 10.** *The LCTSS (7) with $\omega(t) = 0$ is regular and impulse-free, if there exist a scalar $\alpha \geq 0$ and an invertible matrix $P$, such that the following conditions hold:*

$$E^T P = P^T E \geq 0, \tag{14}$$

$$A^T P + P^T A < \alpha E^T P. \tag{15}$$

*Proof.* Let $\widetilde{M}, \widetilde{N} \in \mathfrak{R}^{n\times n}$ be nonsingular matrices such that

$$\widetilde{M}E\widetilde{N} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{16}$$

New partitions $\widetilde{M}^{-T}P\widetilde{N}$ and $\widetilde{M}A\widetilde{N}$ conform to $\widetilde{M}E\widetilde{N}$; that is,

$$\begin{aligned} \widetilde{M}^{-T}P\widetilde{N} &= \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}, \\ \widetilde{M}A\widetilde{N} &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}. \end{aligned} \tag{17}$$

From (14), (16), and (17), it is easy to show that $P_1 > 0$ and $P_2 = 0$. By using (15) together with (16) and (17), we have

$$\begin{bmatrix} A_1^T P_1 + P_1^T A_1 + A_3^T P_3 + P_3^T A_3 - \alpha P_1 & A_3^T P_4 + P_1^T A_2 + P_3^T A_4 \\ A_2^T P_1 + A_4^T P_3 + P_4^T A_3 & A_4^T P_4 + P_4^T A_4 \end{bmatrix} < 0. \tag{18}$$

By Lemma 7,

$$\begin{aligned} \operatorname{Re}\lambda\left(P_4^T A_4\right) \leq \mu\left(P_4^T A_4\right) &= \frac{1}{2}\lambda_{\max}\left(A_4^T P_4 + P_4^T A_4\right) \\ &< 0. \end{aligned} \tag{19}$$

Then it can be easily shown that $P_4^T A_4$ is invertible, which implies that $A_4$ is invertible, too. Hence, in the light of definition and the results of Xu [1], we have that the LCTSS (7) is regular and impulse-free. The proof is completed. □

**Lemma 11.** *The unforced ULCTSS (1) ($u(t) = 0$) is said to be FTRB with respect to $(c_1, c_2, T, R, d)$, if there exist scalars $\varepsilon > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha \geq 0$, invertible matrix $P$, and symmetric positive definite matrix $Q$ such that*

$$\begin{bmatrix} \Pi & P^T G & N_a^T \\ G^T P & -Q & 0 \\ N_a & 0 & -\varepsilon I \end{bmatrix} < 0, \tag{20}$$

$$E^T P = P^T E \geq 0, \tag{21}$$

$$\lambda_1 E^T P < E^T R E < \lambda_2 E^T P, \quad (22)$$

$$\lambda_2 e^{\alpha T}\left[\frac{c_1}{\lambda_1} + d\lambda_{\max}(Q)\right] < c_2, \quad (23)$$

*hold, where* $\Pi = A^T P + P^T A - \alpha E^T P + \varepsilon P^T M M^T P$.

*Proof.* Using Schur complements formula, from (20), it is easy to show that

$$\Pi + \frac{1}{\varepsilon} N_a^T N_a + P^T G Q^{-1} G^T P < 0. \quad (24)$$

By Lemma 9,

$$\begin{aligned}[\triangle A]^T P + P^T [\triangle A] &= \left(M F(\sigma) N_a\right)^T P \\ &\quad + P^T \left(M F(\sigma) N_a\right) \\ &\leq \varepsilon P^T M M^T P + \frac{1}{\varepsilon} N_a^T N_a.\end{aligned} \quad (25)$$

Hence,

$$\begin{aligned}&[A + \triangle A]^T P + P^T [A + \triangle A] - \alpha E^T P \\ &\quad + P^T G Q^{-1} G^T P \leq \Pi + \frac{1}{\varepsilon} N_a^T N_a + P^T G Q^{-1} G^T P.\end{aligned} \quad (26)$$

By noting (24) and (26), (20) implies that

$$\begin{aligned}&[A + \triangle A]^T P + P^T [A + \triangle A] - \alpha E^T P \\ &\quad + P^T G Q^{-1} G^T P < 0.\end{aligned} \quad (27)$$

Or equivalently

$$\begin{bmatrix} [A + \triangle A]^T P + P^T [A + \triangle A] - \alpha E^T P & P^T G \\ G^T P & -Q \end{bmatrix} < 0. \quad (28)$$

By noting that (27) implies that $[A + \triangle A]^T P + P^T [A + \triangle A] - \alpha E^T P < 0$, (21) and Lemma 10, then the unforced ULCTSS (1) ($u(t) = 0$) is said to be regular and impulse-free when $\omega(t) = 0$.

On the other hand, (22) is equivalent to

$$\frac{1}{\lambda_2} E^T R E < E^T P < \frac{1}{\lambda_1} E^T R E. \quad (29)$$

Let $V(x(t)) = x^T(t) E^T P x(t) \geq 0$, and $\dot{V}(x(t))$ denotes the derivative of $V(x(t))$ along the solution of the unforced ULCTSS (1) ($u(t) = 0$). We have

$$\begin{aligned}\dot{V}(x(t)) &= [(A + \triangle A) x(t) + G\omega(t)]^T P x(t) + x^T(t) \\ &\quad \cdot P^T [(A + \triangle A) x(t) + G\omega(t)] = \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix}^T \\ &\quad \cdot \begin{bmatrix} [A + \triangle A]^T P + P^T [A + \triangle A] & P^T G \\ G^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix}.\end{aligned} \quad (30)$$

From (21), (28), and (30), we have

$$\dot{V}(x(t)) < \alpha V(x(t)) + \omega^T(t) Q \omega(t). \quad (31)$$

Multiplying (31) by $e^{-\alpha t}$, we can obtain

$$e^{-\alpha t} \dot{V}(x(t)) - \alpha e^{-\alpha t} V(x(t)) < e^{-\alpha t} \omega^T(t) Q \omega(t). \quad (32)$$

Furthermore,

$$\frac{d}{dt}\left(e^{-\alpha t} V(x(t))\right) < e^{-\alpha t} \omega^T(t) Q \omega(t). \quad (33)$$

Integrating (33) from 0 to $t$ with $t \in [0, T]$, we have

$$e^{-\alpha t} V(x(t)) - V(x(0)) < \int_0^t e^{-\alpha \tau} \omega^T(\tau) Q \omega(\tau)\, d\tau. \quad (34)$$

Noting that $\alpha \geq 0$, we can obtain

$$V(x(t)) < e^{\alpha t}\left[V(x(0)) + \int_0^t e^{-\alpha \tau} \omega^T(\tau) Q \omega(\tau)\, d\tau\right] \quad (35)$$

$$< e^{\alpha t}\left[x^T(0) E^T P x(0) + \int_0^t \omega^T(\tau) Q \omega(\tau)\, d\tau\right], \quad t \in [0, T]. \quad (36)$$

Noting that (29), we have

$$V(x(t)) = x^T(t) E^T P x(t) > \frac{1}{\lambda_2} x^T(t) E^T R E x(t). \quad (37)$$

Noting that (36) and Assumption 1, from (29), it follows that

$$V(x(t)) < e^{\alpha T}\left[\frac{1}{\lambda_1} x^T(0) E^T R E x(0) + \lambda_{\max}(Q) d\right]. \quad (38)$$

Combining (37) with (38), we have

$$\begin{aligned}x^T(t) E^T R x(t) &< \lambda_2 V(x(t)) \\ &< \lambda_2 e^{\alpha T}\left[\frac{1}{\lambda_1} x^T(0) E^T R E x(0) + \lambda_{\max}(Q) d\right].\end{aligned} \quad (39)$$

Condition (23) implies that $x^T(t) E^T R E x(t) < c_2$ with $t \in [0, T]$, if $x_0^T E^T R E x_0 \leq c_1$. The proof is completed. □

**Theorem 12.** *The unforced ULCTSS (1) ($u(t) = 0$) is said to be FTRB with respect to $(c_1, c_2, T, R, d)$, and (13) is satisfied for any admissible $\triangle A$, if there exist scalars $\varepsilon > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha \geq 0$ and invertible matrix $P$ such that (21), (22), (40), and (41) hold:*

$$\begin{bmatrix} \Pi & P^T G & N_a^T & C^T \\ G^T P & -\gamma^2 e^{-\alpha T} I & 0 & D_2^T \\ N_a & 0 & -\varepsilon I & 0 \\ C & D_2 & 0 & -I \end{bmatrix} < 0, \quad (40)$$

$$\lambda_2 e^{\alpha T}\left[\frac{c_1}{\lambda_1} + d\gamma^2 e^{-\alpha T}\right] < c_2, \quad (41)$$

*where* $\Pi = A^T P + P^T A - \alpha E^T P + \varepsilon P^T M M^T P$.

*Proof.* Note that condition (40) implies that

$$\begin{bmatrix} \Pi & P^T G & N_a^T \\ G^T P & -\gamma^2 e^{-\alpha T} I & 0 \\ N_a & 0 & -\varepsilon I \end{bmatrix} < 0. \quad (42)$$

From Lemma 11, let $Q = \gamma^2 e^{-\alpha T} I$, then it is guaranteed by conditions (21), (22), (40), and (41) that the ULCTSS (1) ($u(t) = 0$) is FTRB. Note that

$$\begin{aligned} &\begin{bmatrix} [A+\triangle A]^T P + P^T [A+\triangle A] - \alpha E^T P & P^T G \\ G^T P & -\gamma^2 e^{-\alpha T} I \end{bmatrix} \\ &+ \begin{bmatrix} C^T \\ D_2^T \end{bmatrix} [C \;\; D_2] \\ &\leq \begin{bmatrix} \Pi & P^T G \\ G^T P & -\gamma^2 e^{-\alpha T} I \end{bmatrix} \\ &+ \begin{bmatrix} N_a^T & C^T \\ 0 & D_2^T \end{bmatrix} \begin{bmatrix} \frac{1}{\varepsilon} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} N_a & 0 \\ C & D_2 \end{bmatrix}. \end{aligned} \quad (43)$$

Using Schur complements formula, it is easy to know that (40) implies

$$\begin{aligned} &\begin{bmatrix} [A+\triangle A]^T P + P^T [A+\triangle A] - \alpha E^T P & P^T G \\ G^T P & -\gamma^2 e^{-\alpha T} I \end{bmatrix} \\ &+ \begin{bmatrix} C^T \\ D_2^T \end{bmatrix} [C \;\; D_2] < 0. \end{aligned} \quad (44)$$

Let $V(x(t)) = x^T(t) E^T P x(t)$; we have

$$\begin{aligned} \dot{V}(x(t)) &= [(A+\triangle A) x(t) + G\omega(t)]^T P x(t) + x^T(t) \\ &\cdot P^T [(A+\triangle A) x(t) + G\omega(t)] = \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix}^T \\ &\cdot \begin{bmatrix} [A+\triangle A]^T P + P^T [A+\triangle A] & P^T G \\ G^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix}. \end{aligned} \quad (45)$$

From (40) and (44), we have

$$\begin{aligned} \dot{V}(x(t)) &< \alpha V(x(t)) + \gamma^2 e^{-\alpha T} \omega^T(t) \omega(t) \\ &- z^T(t) z(t). \end{aligned} \quad (46)$$

The above equation implies that

$$\begin{aligned} \frac{d}{dt}\left(e^{-\alpha t} V(x(t))\right) &< \gamma^2 e^{-\alpha(t+T)} \omega^T(t) \omega(t) \\ &- e^{-\alpha t} z^T(t) z(t). \end{aligned} \quad (47)$$

Integrating (47) from 0 to $T$, and noting that $x(0) = 0$, we have

$$\begin{aligned} &e^{-\alpha T} V(x(T)) \\ &< \int_0^T \left[\gamma^2 e^{-\alpha(t+T)} \omega^T(t) \omega(t) - e^{-\alpha t} z^T(t) z(t)\right] dt, \end{aligned} \quad (48)$$

which implies that

$$\int_0^T e^{-\alpha t} z^T(t) z(t)\, dt \leq \gamma^2 e^{-\alpha T} \int_0^T e^{-\alpha t} \omega^T(t) \omega(t)\, dt. \quad (49)$$

Noting that

$$\begin{aligned} &e^{-\alpha T} \int_0^T z^T(t) z(t)\, dt < \int_0^T e^{-\alpha t} z^T(t) z(t)\, dt, \\ &\gamma^2 e^{-\alpha T} \int_0^T e^{-\alpha t} \omega^T(t) \omega(t)\, dt \\ &\quad < \gamma^2 e^{-\alpha T} \int_0^T \omega^T(t) \omega(t)\, dt. \end{aligned} \quad (50)$$

From (49)-(50), we can obtain

$$\int_0^T z^T(t) z(t)\, dt < \gamma^2 \int_0^T \omega^T(t) \omega(t)\, dt. \quad (51)$$

The proof is completed. □

*Remark 13.* In Theorem 12, a sufficient condition of FTRB and (13) with respect to $(c_1, c_2, T, R, d)$ is satisfied, but condition (21) is difficult to determine due to the nonlinear constraints of $P$.

**Theorem 14.** *The unforced ULCTSS (1) ($u(t) = 0$) is said to be FTRB with respect to $(c_1, c_2, T, R, d)$, and (13) is satisfied for any admissible $\triangle A$, if there exist scalars $\varepsilon > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha \geq 0$, symmetric positive definite matrix $\widehat{W}$, and matrix $\widehat{S}$ such that*

$$\begin{bmatrix} \Phi & G & X N_a^T & X C^T \\ G^T & -\gamma^2 e^{-\alpha T} I & 0 & D_2^T \\ N_a X^T & 0 & -\varepsilon I & 0 \\ C X^T & D_2 & 0 & -I \end{bmatrix} < 0, \quad (52)$$

$$\lambda_1 \left(\Sigma_r U_1^T R U_1 \Sigma_r\right)^{-1} < \widehat{W} < \lambda_2 \left(\Sigma_r U_1^T R U_1 \Sigma_r\right)^{-1}, \quad (53)$$

$$\lambda_1 \gamma^2 e^{-\alpha T} < 1, \quad (54)$$

$$\lambda_2 e^{\alpha T} \left(c_1 + d\right) < \lambda_1 c_2, \quad (55)$$

*hold, where $\Phi = XA^T + AX^T - \alpha X E^T + \varepsilon M M^T$, $X = E V_1 \widehat{W} V_1^T + \widehat{S} V_2^T$.*

*Proof.* From (52), we can obtain $\Phi < 0$, and $X$ is invertible. According to Lemma 8, there exist $W$ and $S$ such that

$$U_1WU_1^TE + U_2S = \left(EV_1\widehat{W}V_1^T + \widehat{S}V_2^T\right)^{-1}, \quad (56)$$

where $\widehat{W} = \Sigma_r^{-1}W^{-1}\Sigma_r^{-1}$.

Let $P = U_1WU_1^TE + U_2^TS$; then $X = EV_1\widehat{W}V_1^T + \widehat{S}V_2^T = P^{-T}$.

Premultiplying (52) by $\mathrm{diag}(P^T, I, I, I)$ and postmultiplying (52) by $\mathrm{diag}(P, I, I, I)$, we can obtain the equivalent condition (40).

Noting that

$$E^TP = P^TE = E^TU_1WU_1^TE = E^TU_1\Sigma_r^{-1}\widehat{W}^{-1}\Sigma_r^{-1}U_1^TE \geq 0, \quad (57)$$

and noting (53), we can obtain (21) and (22).

Noting (54) and (55), we have

$$\lambda_2 e^{\alpha T}\left[\frac{c_1}{\lambda_1} + d\gamma^2 e^{-\alpha T}\right] < \lambda_2 e^{\alpha T}\left[\frac{c_1}{\lambda_1} + \frac{1}{\lambda_1}d\right] < c_2. \quad (58)$$

Hence, the unforced ULCTSS (1) ($u(t) = 0$) is FTRB with respect to $(c_1, c_2, T, R, d)$, and (13) is satisfied under conditions (52)–(55). The proof is completed. □

*Remark 15.* Theorem 14 is obtained based on the results in Theorem 12, in which a sufficient condition is given to guarantee the ULCTSS (1) ($u(t) = 0$) FTRB with respect to $(c_1, c_2, T, R, d)$. Meanwhile, (13) is satisfied in terms of LMI in (52)–(55) when $\alpha$ is fixed. Therefore, they can be solved efficiently.

**Corollary 16.** *The linear continuous-time singular system (LCTSS),*

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + G\omega(t), \\ z(t) &= Cx(t) + D_1u(t) + D_2\omega(t), \end{aligned} \quad (59)$$

*is FTB with respect to $(c_1, c_2, T, R, d)$, and (13) is satisfied when $u(t) = 0$, if there exist scalars $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha \geq 0$, symmetric positive definite matrix $\widehat{W}$, and matrix $\widehat{S}$ such that (53)–(55) and (60) hold.*

$$\begin{bmatrix} \overline{\Phi} & G & XC^T \\ G^T & -\gamma^2 e^{-\alpha T}I & D_2^T \\ CX^T & D_2 & -I \end{bmatrix} < 0, \quad (60)$$

*where $\overline{\Phi} = XA^T + AX^T - \alpha XE^T$, $X = EV_1\widehat{W}V_1^T + \widehat{S}V_2^T$.*

**Theorem 17.** *There exists a state feedback controller in the form of (11) such that the uncertain closed-loop system (12) is FTRB with respect to $(c_1, c_2, T, R, d)$, and (13) is satisfied for any admissible $\triangle A$ and $\triangle B$; if there exist scalars $\varepsilon > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha \geq 0$, symmetric positive definite matrix $\widehat{W}$ and matrices $\widehat{S}$ and $Z$ such that (53)–(55) and (61) hold:*

$$\begin{bmatrix} \Upsilon & G & XN_a^T + ZN_b^T & XC^T + ZD_1^T \\ G^T & -\gamma^2 e^{-\alpha T}I & 0 & D_2^T \\ N_aX^T + N_bZ^T & 0 & -\varepsilon I & 0 \\ CX^T + D_1Z^T & D_2 & 0 & -I \end{bmatrix} < 0, \quad (61)$$

*where $\Upsilon = XA^T + AX^T + ZB^T + BZ^T - \alpha XE^T + \varepsilon MM^T$, $X = EV_1\widehat{W}V_1^T + \widehat{S}V_2^T$. In this case, a finite-time robust $H_\infty$ state feedback controller can be chosen as*

$$u(t) = Z^T\left(EV_1\widehat{W}V_1^T + \widehat{S}V_2^T\right)^{-T}x(t). \quad (62)$$

*Proof.* From $\widehat{W} > 0$, we can obtain that $X = EV_1\widehat{W}V_1^T + \widehat{S}V_2^T$ is invertible. From Theorems 12 and 14, let $A_K = A + BK$, $\triangle A_K = \triangle A + \triangle BK = MF(\sigma)[N_a + N_bK]$, $C_K = C + D_1K$, and $Z = XK^T$; then we can obtain the conclusion. The proof is completed. □

**Corollary 18.** *There exists a state feedback controller in the form of (11) such that the closed-loop system (63),*

$$\begin{aligned} E\dot{x}(t) &= [A + BK]Ax(t) + G\omega(t), \\ z(t) &= [C + D_1K]x(t) + D_2\omega(t), \end{aligned} \quad (63)$$

*is FTB with respect to $(c_1, c_2, T, R, d)$, and (13) is satisfied, if there exist scalars $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha \geq 0$, symmetric positive definite matrix $\widehat{W}$ and matrices $\widehat{S}$ and $Z$ such that (53)–(55) and (64) hold:*

$$\begin{bmatrix} \overline{\Upsilon} & G & XC^T + ZD_1^T \\ G^T & -\gamma^2 e^{-\alpha T}I & D_2^T \\ CX^T + D_1Z^T & D_2 & -I \end{bmatrix} < 0, \quad (64)$$

*where $\overline{\Upsilon} = XA^T + AX^T + ZB^T + BZ^T - \alpha XE^T$, $X = EV_1\widehat{W}V_1^T + \widehat{S}V_2^T$. In this case, a finite-time robust $H_\infty$ state feedback controller can be chosen as*

$$u(t) = Z^T\left(EV_1\widehat{W}V_1^T + \widehat{S}V_2^T\right)^{-T}x(t). \quad (65)$$

*Remark 19.* From Theorem 17 and Corollary 18, the similar sufficient conditions are given, respectively. Noting (53)–(55) and (61), we can see that the conditions in Theorem 17 are not LMIs with respect to $c_1$, $c_2$, $T$, $d$, $\varepsilon$, $\alpha$, $\lambda_1$, $\lambda_2$, $\gamma$, $\widehat{W}$, $\widehat{S}$, $Z$. However, once we can fix $c_1$, $c_2$, $T$, $d$ and $\alpha$, they can be converted to feasibility problem based on LMIs.

## 4. Numerical Examples

In this section, a numerical example is provided to demonstrate the effectiveness of the proposed method.

*Example 1.* Consider the uncertain linear singular system (1) with

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 0.5 & 1 \\ -1 & 0 & 1 \\ 0.5 & 0.5 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0.5 & -0.5 \\ 1 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \end{bmatrix}; \tag{66}$$

$$M^T = \begin{bmatrix} 0.1 & -0.1 & 0.1 \end{bmatrix},$$

$$N_a = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix},$$

$$N_b = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix};$$

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0.5 & 1 & 0.5 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

In this paper, the finite-time robust $H_\infty$ controller is derived by using the algorithm sketch below, with the aid of Matlab LMI Toolbox.

*Step 1.* Some fixed values are given for $c_1$, $T$, $d$ and $R$.

*Step 2.* An initial value is given for $c_2$.

*Step 3.* Starting from stable the index $\alpha = 0$, we kept increasing $\alpha$ until a solution is found or maximum value for $\alpha$ is reached.

*Step 4.* If no solution is found, then the initial value for $c_2$ should be increased; otherwise $c_2$ can be decreased until its minimum is found.

We chose $c_1 = 1$, $T = 5$, $R = I$, $\gamma = 0.5$, $d = 0.1$, and the initial value for $c_2 = 10$. By solving the LMIs (53)–(55) and (61), the following finite-time robust $H_\infty$ controller is achieved:

$$u(t) = \begin{pmatrix} -17.4909 & -2.2012 & -8.1442 \\ -16.3589 & 1.4712 & -9.9141 \end{pmatrix} x(t), \tag{67}$$

which guarantees the desired close-loop properties with $c_2 = 4$ and stable index $\alpha = 0.258$.

Moreover, we can fix $c_2$ and find the admissible maximum $c_1$ to guarantee the desired close-loop finite-time property.

## 5. Conclusions

In this paper, we extended the definition of $H_\infty$ control of singular systems to finite-time $H_\infty$ control of singular systems. First, new sufficient conditions for FTRB are presented, which can decrease conservation. Then, we considered the finite-time robust $H_\infty$ control problem for ULCTSS with time-varying norm-bounded exogenous disturbance via state feedback controller. The sufficient conditions of the theorems, which ensure that the system is FTRB, are given in terms of linear matrix inequalities, and they can be solved by LMI toolbox. Numerical examples were given to demonstrate the validity of the proposed methodology.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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