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LIMIT THEOREMS FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATION PROBLEMS

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ABSTRACT. In this paper linear differential equations with random processes as coefficients and as inhomogeneous term are regarded. Limit theorems are proved for the solutions of these equations if the random processes are weakly correlated processes.

Limit theorems are proved for the eigenvalues and the eigenfunctions of eigenvalue problems and for the solutions of boundary value problems and initial value problems.

KEY WORDS AND PHRASES. Limit Theorem, Stochastic Eigenvalue Problem, Stochastic Boundary Value Problem, Differential Equation.

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1. INTRODUCTION.

At the research of physical and engineering problems it is of great importance the approach to the differential equations with stochastic processes as coefficients respectively with stochastic boundary or initial conditions. There is a series of papers which deal with such a problem. The first moments of the solution are often calculated from the first moments of the stochastic processes involving the problem. For the applications this is an interesting and important problem (see [4]). The calculation of the distributions of the solution processes from the distributions of the involving processes is often more difficult. This problem contains already many difficulties for very simple problems (e.g. for the initial value problem of a linear ordinary differential equation of the first order with stochastic coefficients). If one specializes the stochastic processes involving that problem then one succeeds in a few cases in obtaining statements on the distribution of the solution. As an example of such a result we refer to the paper of G.E. Uhlenbeck und L.S. Ornstein [8]. They regard such stochastic inputs which do not possess a "distant effect", i.e. the values of the process do not possess a correlation if the distance between the observation points is large. As a result they can show that the solution of a special initial value problem with such a process without "distant effect" as the right side is approximativly a so-called Ornstein-Uhlenbeck-process, i.e. a process for which the first distribution function is a Gaussian distribution function.

These processes without "distant effect" were defined exactly by the authors in the paper [6] through the process class of the "weakly correlated processes" and they were applied in this paper at the consideration of stochastic eigenvalue problems and boundary value problems. A limit theorem is obtained for the eigenvalues, eigenfunctions of stochastic eigenvalue problems respectively for the solutions of stochastic boundary problems with weakly correlated coefficients. This limit theorem shows the approximate Gaussian distribution of the first distribution function of the solutions of eigenvalue problems and boundary value problems.

In the present paper the conception of the weakly correlated process is defined more generally than in the paper [6] (i.e. the stationarity of the process falls out the supposition). The correlation length ε denotes the minimum distance between observation points of a weakly correlated process that the values of the process do not affect in observation points which possess a distance larger than ε.

In section 2 a few theorems will be proved about functionals of weakly correlated processes which are important for the applications at eigenvalue problems, boundary value problems and initial value problems in the following sections.

Section 3 deals with stochastic eigenvalue problems for ordinary differential equations with deterministic boundary conditions where the coefficients of the differential operator are independent, weakly correlated processes of the correlation length ε. We prove that the eigenvalues and the eigenfunctions as $\varepsilon \downarrow 0$ possess a Gaussian distribution. For instance the eigenfunctions of the stochastic eigenvalue problem converge in the distribution as $\varepsilon \downarrow 0$ to Gaussian processes. Methods of the perturbation theory are essentially used. In a general example it is referred to a few remarkable appearances.

In section 4 we deal with stochastic boundary problems and we obtain similar results as for stochastic eigenvalue problems. The case of a Sturm-Liouville-operator with a stochastic inhomogeneous term (weakly correlated) which was dealt with by W.E. Boyce in [1] is included in the result of this section. Some limitations relative to the smallness of the stochastic coefficients of the operator but not of the inhomogeneous term are assumed as at the eigenvalue problems, too. We show a calculation of the correlation function by the Ritz-method to eliminate the Green function of the averaged problem from the correlation function of the limit process of the solution of the boundary value problem.

At last, section 5 deals with stochastic initial value problems of ordinary differential equations. The inhomogeneous terms are weakly correlated processes. The results in this theory of the weakly correlated processes as $\varepsilon \downarrow 0$ resemble the results of the Itô-theory if the inhomogeneous terms are replaced by Gaussian white noise according to the Itô-theory and the formed Itô-equation is solved according to the Itô-theory. The practice by the help of the weakly correlated processes differs principally from the Itô-theory. One obtains the limit theorems by use of the weakly correlated processes if at first a formula for the a.s. continuous differentiable sample functions of the solutions is derived ($\varepsilon \neq 0$) and then we go to the limit ($\varepsilon \downarrow 0$). With it we get an approximation of the solution of the initial value problem with weakly correlated processes ($\varepsilon \ll 1, \varepsilon \neq 0$). In the Itô--theory one goes to the limit in the differential equation and this equation is solved by a well worked out mathematical theory. We get different results at this different practice in problems of differential equations in which the coefficients are weakly correlated processes and not the inhomogeneous term. We do not deal with such problems in this paper.

It is principally no distinction in the proof of limit theorems for the initial value problems and boundary value respectively eigenvalue problems with weakly correlated processes. A widening of the Itô-theory on boundary value respectively eigenvalue problems with white noises for the coefficients seems to contain a few fundamental difficulties.

2. WEAKLY CORRELATED PROCESSES

DEFINITION 1. Let $(x_1,x_2,\dots,x_n)$ be a finite set of real numbers and $\varepsilon>0$. A subset $(x_{i_1},x_{i_2},\dots,x_{i_k}) \subseteq (x_1,x_2,\dots,x_n)$ is called ε-adjoining if

$$|x_{r_1}-x_{r_2}| \leq \varepsilon,\ |x_{r_2}-x_{r_3}| \leq \varepsilon\ ,\ \dots\ ,\ |x_{r_{k-1}}-x_{r_k}| \leq \varepsilon$$

is fulfilled for the x_{i_j}, $j=1,\dots,k$, which are arranged after the quality (these we have termed as $x_{r_1},x_{r_2},\dots,x_{r_k}$). A subset of one number is always called ε-adjoining. A subset $(x_{i_1},\dots,x_{i_k}) \subseteq (x_1,\dots,x_n)$ is called maximum ε-adjoining (relative to $(x_1,\dots,x_n)$) if it is ε-adjoining but the subset $(x_{i_1},\dots,x_{i_k},x_r)$ is not ε-adjoining for $x_r \in (x_1,\dots,x_r)\setminus(x_{i_1},\dots,x_{i_k})$.

Every finite set $(x_1,\dots,x_n)$ split unique in disjoint maximum ε-adjoining subsets.

DEFINITION 2. A stochastic process $f(x,\omega)$ with $\langle f(x,\omega)\rangle \doteq E\{f(x,\omega)\}=0$ is called weakly correlated of the correlation length ε when the relation

$$\langle f(x_1)\dots f(x_n)\rangle=\langle f(x_{11})\dots f(x_{1p_1})\rangle\langle f(x_{21})\dots f(x_{2p_2})\rangle\cdot\ \dots\ \cdot \\ \cdot\langle f(x_{k1})\dots f(x_{kp_k})\rangle$$

is satisfied for the nth moments (for all $n \geq 1$) if the set $(x_1,\dots,x_n)$ splits in the maximum ε-adjoining subsets $(\sum_{i=1}^{k} p_i=n)$

$$\{(x_{11},\dots,x_{1p_1}),(x_{21},\dots,x_{2p_2}),\dots,(x_{k1},\dots,x_{kp_k})\}$$

If the process $f(x,\omega)$ is weakly correlated with the correlation length ε, then we get for its correlation function

$$\langle f(x)f(y)\rangle = \begin{cases} R_\varepsilon(x,y) & \text{for } y\in K_\varepsilon(x) \\ 0 & \text{for } y\notin K_\varepsilon(x) \end{cases}$$

where $K_\varepsilon(x) \doteq \{y\in R^1 : |x-y|\leq\varepsilon\}$.

The existence of especially stationary weakly correlated processes has been proved in the paper [6]. In this paper it is also proved that weakly correlated processes with smooth sample functions exist.

In the following a few theorems will be proved about weakly correlated processes. These theorems will be used essentially in the applications at equations of the mathematical physics.

THEOREM 1. Let $f_\varepsilon(x,\omega)$ be a sequence of weakly correlated processes as $\varepsilon\downarrow 0$ with the correlation functions

$$\langle f_\varepsilon(x)f_\varepsilon(y)\rangle = \begin{cases} R_\varepsilon(x,y) & \text{for } y\in K_\varepsilon(x) \\ 0 & \text{for } y\notin K_\varepsilon(x) \end{cases}$$

where is fulfilled

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_\varepsilon(x,x+y)dy \doteq a(x) \qquad \text{and } a(x) \not\equiv 0$$

uniformly in x. Further on let $g_i(x,y)$, $i=1,2$, be in $[a-\eta,b_i+\eta]$ differentiable functions relative to x ($\eta>0$, $b_i>a$) and

$$\sup_{i,x,y}\left\{|g_i(x,y)|, \left|\frac{\partial}{\partial x}g_i(x,y)\right|\right\} \leq C.$$

When $G_2=\{(x_1,x_2): |x_1-x_2|\leq\varepsilon,\ a\leq x_i\leq b_i\}$ then we get for

$$r_{i\varepsilon}(x,\omega) = \frac{1}{\sqrt{\varepsilon}}\int_a^{b_i} f_\varepsilon(y,\omega)g_i(y,x)dy$$

the relation

$$\lim_{\varepsilon\downarrow 0}\langle r_{1\varepsilon}(x_1)r_{2\varepsilon}(x_2)\rangle = \int_a^{\min(b_1,b_2)} a(y)g_1(y,x_1)g_2(y,x_2)dy.$$

PROOF. We can set $b_1\leq b_2$ and $\varepsilon<b_1$. We substitute in

$$I_{12} \doteq \langle r_{1\varepsilon}(x_1)r_{2\varepsilon}(x_2)\rangle = \frac{1}{\varepsilon}\int_a^{b_1}\int_a^{b_2}\langle f_\varepsilon(y_1)f_\varepsilon(y_2)\rangle g_1(y_1,x_1)g_2(y_2,x_2)dy_1dy_2$$

$$= \frac{1}{\varepsilon}\iint_{G_2} R_\varepsilon(y_1,y_2)g_1(y_1,x_1)g_2(y_2,x_2)dy_1dy_2$$

$z_1=y_1$, $z_2=y_2-y_1$ and obtain (see Fig.1)

$$I_{12} = \frac{1}{\varepsilon}\left[\iint\limits_{Q} h(z_1,z_2)dz_1dz_2 - \iint\limits_{Q_1} h(z_1,z_2)dz_1dz_2\right]$$

with $h(z_1,z_2)=R_\varepsilon(z_1,z_1+z_2)g_1(z_1,x_1)g_2(z_1+z_2,x_2)$

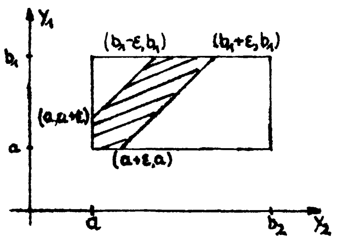
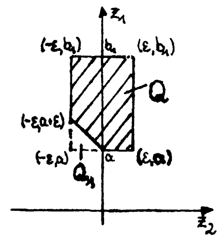


Fig.1



and $Q=\{(z_1,z_2):\ a \le z_1 \le b_1, -\varepsilon \le z_2 \le \varepsilon\}$. By $\lim\limits_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \iint\limits_{Q_1} dz_1dz_2=0$ and $|h(z_1,z_2)| \le (R_\varepsilon(z_1,z_1)R_\varepsilon(z_1+z_2,z_1+z_2))^{1/2}C^2$ it follows

$$\lim_{\varepsilon\downarrow 0} I_{12} = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_a^{b_1} g_1(z_1,x_1) \int_{-\varepsilon}^{\varepsilon} R_\varepsilon(z_1,z_1+z_2)g_2(z_1+z_2,x_2)dz_2dz_1.$$

At last we obtain

$$\lim_{\varepsilon\downarrow 0} I_{12} = \lim_{\varepsilon\downarrow 0} \int_a^{b_1} g_1(z_1,x_1)\ \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_\varepsilon(z_1,z_1+z_2)\{g_2(z_1,x_2)+O(z_2)\}\,dz_2dz_1$$

$$= \int_a^{b_1} g_1(z_1,x_1)g_2(z_1,x_2)\ \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_\varepsilon(z_1,z_1+z_2)dz_2dz_1$$

if the function $g_2(z_1+z_2,x_2)$ is developed relative to z_2 at $z_2=0$. The theorem is proved.

Before we denote the more important theorem 3, we prove a simple theorem.

THEOREM2.It is $W_n=\{(x_1,\dots,x_n):a \le x_i \le b_i, i=1,2,\dots,n\}$ and $G_n=\{(x_1,\dots,x_n)\in W_n:(x_1,\dots,x_n)\ \ \varepsilon\text{-adjoining}\}$. Let $g(x_1,\dots,x_n)$ be a function which is in $\{(x_1,\dots,x_n):a-\eta \le x_i \le b_i+\eta, i=1,\dots,n\}$ limited: $|g(x_1,\dots,x_n)| \le C$. Then the integral $\int\limits_{G_n} g(x_1,\dots,x_n)dx_1\dots dx_n$ is at least of the order ε^{n-1} for $\varepsilon\downarrow 0$.

PROOF. By $\bar b=\min(b_1,b_2,\dots,b_n)$ it is

$$\left|\int\limits_{G_n} g(x_1,\dots,x_n)dx_1\dots dx_n\right| \le Cn!\int_a^{\bar b}dx_1\int_{x_1}^{x_1+\varepsilon}dx_2\dots\int_{x_{n-1}}^{x_{n-1}+\varepsilon}dx_n=Cn!\,\varepsilon^{n-1}(\bar b-a)$$

where the term $\int_a^{\bar b} dx_1\int_{x_1}^{x_1+\varepsilon}dx_2\dots\int_{x_{n-1}}^{x_{n-1}+\varepsilon}dx_n$ denotes the volume of

$H_\varepsilon^{1\ldots n} = \{(x_1,\ldots,x_n): a \le x_1 \le b, x_1 \quad x_2 \le x_1+\varepsilon,\ldots,x_{n-1} \le x_n \le x_{n-1}+\varepsilon\}$ and we get $G_\varepsilon^{1\ldots n} \subseteq H_\varepsilon^{1\ldots n}$. $G_\varepsilon^{i_1\ldots i_n}$ marks the set of the ε-adjoining points in G_n with $x_{i_1} \le x_{i_2} \le \ldots \le x_{i_n}$. Then the above given inequality results from

$$G_n = \bigcup_{\pi\in\sigma(1,\ldots,n),\, \pi=\binom{1\ldots n}{i_1\ldots i_n}} G_\varepsilon^{i_1\ldots i_n}$$

and with it the statement of the theorem.

THEOREM 3. Let $f_\varepsilon(x,\omega)$ be a sequence of weakly correlated processes as $\varepsilon\downarrow 0$. The absolute moments $\langle|f_\varepsilon(x,\omega)|^j\rangle = c_j(x)$ are to exist and $c_j(x) \le C_j$. Let $g_i(x,y)$, $i=1,2,\ldots,n$, be in $[a-\eta, b_i+\eta]$ differentiable functions relativ to x $(\eta > 0, b_i > a)$ and

$$\sup_{i,x,y} \left\{ |g_i(x,y)| \; , \left|\frac{\partial}{\partial x} g_i(x,y)\right| \right\} \le C.$$

Then we have for

$$r_{i\varepsilon}(x,\omega) = \frac{1}{\sqrt{\varepsilon}} \int_a^{b_i} f_\varepsilon(y,\omega) g_i(y,x)\,dy$$

the relation

$$\lim_{\varepsilon\downarrow 0}\langle r_{1\varepsilon}(x_1) r_{2\varepsilon}(x_2)\ldots r_{n\varepsilon}(x_n)\rangle =$$

$$= \begin{cases} \sum\limits_{(i_1,i_2),\ldots,(i_{n-1},i_n)} \lim\limits_{\varepsilon\downarrow 0}\langle r_{i_1\varepsilon} r_{i_2\varepsilon}\rangle \lim\limits_{\varepsilon\downarrow 0}\langle r_{i_3\varepsilon} r_{i_4\varepsilon}\rangle \ldots \lim\limits_{\varepsilon\downarrow 0}\langle r_{i_{n-1}\varepsilon} r_{i_n\varepsilon}\rangle & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

The sum extends for all splittings of $(1,\ldots,n)$ in pairs (i_1,i_2), $\ldots,(i_{n-1},i_n)$ with $i_r < i_{r+1}$ for every pair (i_r,i_{r+1}). Splittings are equal which differ through the sequence of the pairs.

The proof of this theorem submits similar to the proof of theorem 7 in [6] by the help of the above given theorem 1 and it also submits the proof of the following theorem 4 from the proof of the theorem 8 in [6].

THEOREM 4. Let $f_{1\varepsilon}(x,\omega),\ldots,f_{n\varepsilon}(x,\omega)$ be sequences of independent weakly correlated processes as $\varepsilon\downarrow 0$. Let $g_{ij}(x,y)$, $i=1,\ldots,k$, $j=1,\ldots,n$, be in $[a-\eta, b_i+\eta]$, $i=1,\ldots,k$, differentiable functions relativ to x and

$$\sup_{i,j,x,y} \left\{ |g_{ij}(x,y)|,\ \left|\frac{\partial}{\partial x} g_{ij}(x,y)\right| \right\} \leq C.$$

Then we have for

$$r_{i\varepsilon}^{(j)}(x,\omega) = \frac{1}{\sqrt{\varepsilon}} \int_a^{b_i} f_j(y,\omega) g_{ij}(y,x)\,dy$$

the relation $(r_{i\varepsilon}^{(j)}(x_i,\omega) \doteq r_{i\varepsilon}^{(j)})$

$$\lim_{\varepsilon\downarrow 0} \langle (r_{1\varepsilon}^{(1)}+\dots+r_{1\varepsilon}^{(n)})\dots(r_{k\varepsilon}^{(1)}+\dots+r_{k\varepsilon}^{(n)})\rangle = \begin{cases} \lim\limits_{\varepsilon\downarrow 0} A_\varepsilon & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

with

$$\lim_{\varepsilon\downarrow 0} A_\varepsilon = \sum_{\substack{v_1=0\\ \vdots \\ v_n=0 \\ \sum_{j=1}^{n} v_j=\frac{k}{2}}}^{\frac{k}{2}} \ \sum_{\substack{\text{all splittings}\\ \text{of } (1,\dots,k) \text{ in}\\ (i_1^1,\dots,i_{2v_1}^1)\\ \vdots \\ (i_1^n,\dots,i_{2v_n}^n)}} A^1_{i_1^1\dots i_{2v_1}^1} \dots A^n_{i_1^n\dots i_{2v_n}^n}$$

and

$$A^p_{i_1^p\dots i_{2v_p}^p} = \sum_{(i^p_{l_1},i^p_{l_2}),\dots,(i^p_{l_{2v_p-1}},i^p_{l_{2v_p}})} \lim_{\varepsilon\downarrow 0}\langle r^{(p)}_{i^p_{l_1}\varepsilon}\ r^{(p)}_{i^p_{l_2}\varepsilon}\rangle \dots \lim_{\varepsilon\downarrow 0}\langle r^{(p)}_{i^p_{l_{2v_p-1}}\varepsilon}\ r^{(p)}_{i^p_{l_{2v_p}}\varepsilon}\rangle .$$

The sum in the term for $A^p_{i_1^p\dots i_{2v_p}^p}$ extends for all splittings of $(i_1^p,\dots,i_{2v_p}^p)$ in pairs like in theorem 3.

As an application of the theorems 1 to 4 we will prove the theorem 5.

THEOREM 5. Let $f_{1\varepsilon}(x,\omega),\dots,f_{n\varepsilon}(x,\omega)$ be sequences of independent, weakly correlated processes with continuous sample functions as the correlation length $\varepsilon\downarrow 0$. Let $g_{ij}(x,y)$ be differentiable functions in $[a-\eta,b+\eta]\times[a,b]$ relative to x with the conditions of the limited like in theorem 4. Then the stochastic vector process

$$\underline{r}_\varepsilon(x,\omega) = (r_{i\varepsilon}(x,\omega))^T_{1\leq i\leq m} \ {}^{1)}$$

$$\text{with } r_{i\varepsilon}(x,\omega) = \frac{1}{\sqrt{\varepsilon}} \sum_{j=1}^{n} \int_a^x g_{ij}(y,x) f_{j\varepsilon}(y,\omega)\,dy \qquad (1)$$

converges in the distribution as $\varepsilon\downarrow 0$ to a Gaussian vector process

1) Matrices are denoted by underlining.

$\underline{\xi}(x,\omega)=(\xi_i(x,\omega))^T_{1\le i\le m}$ with $\xi_i(x,\omega)=\sum_{j=1}^{n}\xi_{ij}(x,\omega)$ where the Gaussian processes $\underline{\xi}^j(x,\omega)=(\xi_{ij}(x,\omega))^T_{1\le i\le m}$, $j=1,\dots,n$, are independent and we have $\langle\underline{\xi}^j(x)\rangle=0$,

$$\langle\underline{\xi}^j(x_1)\underline{\xi}^j(x_2)^T\rangle = \Big(\int_a^{\min(x_1,x_2)} a_j(y)g_{pj}(y,x_1)g_{qj}(y,x_2)dy\Big)_{1\le p,q\le m},$$

i.e. all the distribution functions of $\underline{r}_\varepsilon(x,\omega)$ converge as $\varepsilon\downarrow 0$ to the adequate distribution functions of the Gaussian vector process $\underline{\xi}(x,\omega)$. It is for the weakly correlated processes $f_{i\varepsilon}(x,\omega)$, $i=1,\dots,n$,

$$\langle f_{i\varepsilon}(x)f_{i\varepsilon}(y)\rangle = \begin{cases} R_{i\varepsilon}(x,y) & \text{for } y\in K_\varepsilon(x)\\ 0 & \text{for } y\notin K_\varepsilon(x)\end{cases}$$

and $a_i(x) \doteq \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_{i\varepsilon}(x,x+y)dy \quad (a_i\ne 0)$.

PROOF. At first we calculate the limit of the k-th moment with the theorems 3 and 4

$$\langle r_{a_1\varepsilon}(x_1)r_{a_2\varepsilon}(x_2)\dots r_{a_k\varepsilon}(x_k)\rangle \quad \text{with } a_p\in\{1,2,\dots,m\},\ x_p\in[a,b]$$

and we obtain by $r_{i\varepsilon}(x,\omega)=\sum_{j=1}^{n} r_{ij\varepsilon}(x,\omega)$,

$$r_{ij\varepsilon}(x,\omega) = \frac{1}{\sqrt{\varepsilon}}\int_a^x g_{ij}(y,x)f_{j\varepsilon}(y,\omega)dy$$

then

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\langle r_{a_1\varepsilon}(x_1)\dots r_{a_k\varepsilon}(x_k)\rangle &= \lim_{\varepsilon\downarrow 0}\Big\langle\Big(\sum_{j=1}^{n} r_{a_1 j\varepsilon}(x_1)\Big)\dots\Big(\sum_{j=1}^{n} r_{a_k j\varepsilon}(x_k)\Big)\Big\rangle\\ &= \sum_{\substack{v_1=0\\ \vdots\\ v_n=0\\ \sum_{j=1}^{n} v_j=\frac{k}{2}}}\ \sum_{\substack{\text{all splittings}\\ \text{of } (1,\dots,k) \text{ in}\\ (i_1^1,\dots,i_{2v_1}^1)\\ \vdots\\ (i_1^n,\dots,i_{2v_n}^n)}} A^1_{i_1^1\dots i_{2v_1}^1}\cdots A^n_{i_1^n\dots i_{2v_n}^n}.\end{aligned} \tag{2}$$

It is

$$A^p_{i_1^p\dots i_{2v_p}^p} = \sum_{(i^p_{l_1},i^p_{l_2}),\dots,(i^p_{l_{2v_p-1}},i^p_{l_{2v_p}})} \lim_{\varepsilon\downarrow 0}\langle r_{a_{i^p_{l_1}}p\varepsilon}(x_{i^p_{l_1}})r_{a_{i^p_{l_2}}p\varepsilon}(x_{i^p_{l_2}})\rangle\cdot$$

$$\cdot\ \dots\ \cdot \lim_{\varepsilon\downarrow 0}\langle r_{a_{i^p_{l_{2v_p-1}}}p\varepsilon}(x_{i^p_{l_{2v_p-1}}})r_{a_{i^p_{l_{2v_p}}}p\varepsilon}(x_{i^p_{l_{2v_p}}})\rangle$$

and from theorem 1

$$\lim_{\varepsilon\downarrow 0}\langle r_{a_i p\varepsilon}(x_i) r_{a_j p\varepsilon}(x_j)\rangle = \int_a^{\min(x_i,x_j)} a_p(y) g_{a_i p}(y,x_i) g_{a_j p}(y,x_j)\,dy.$$

We introduce the Gaussian process $\xi^p(x,\omega)=(\xi_{ip}(x,\omega))^T_{1\le i\le m}$ with the moments $\langle\xi^p(x)\rangle=0$,

$$\langle\xi^p(x_1)\xi^p(x_2)^T\rangle = \Big(\int_a^{\min(x_1,x_2)} a_p(y) g_{ip}(y,x_1) g_{jp}(y,x_2)\,dy\Big)_{1\le i,j\le m}$$

and obtain

$$A^p_{i_1^p\ldots i_{2v_p}^p} = \langle \xi_{a_{i_1^p}p}(x_{i_1^p}) \ldots \xi_{a_{i_{2v_p}^p}p}(x_{i_{2v_p}^p})\rangle.$$

Further on it follows by $\xi(x,\omega)\doteq\sum_{j=1}^{n}\xi^j(x,\omega)$ if the processes $\xi^1(x,\omega),\ldots,\xi^n(x,\omega)$ are supposed independent from (2) with $\xi(x,\omega)=(\xi_i(x,\omega))^T_{1\le i\le m}$ the formula

$$\lim_{\varepsilon\downarrow 0}\langle r_{a_1\varepsilon}(x_1)\ldots r_{a_k\varepsilon}(x_k)\rangle = \langle\xi_{a_1}(x_1)\ldots\xi_{a_k}(x_k)\rangle. \tag{3}$$

Then from (3) we obtain

$$\lim_{\varepsilon\downarrow 0} F_{\varepsilon x_1\ldots x_s}(\underline{t}_1,\ldots,\underline{t}_s) = \Phi_{x_1\ldots x_s}(\underline{t}_1,\ldots,\underline{t}_s)$$

with $F_{\varepsilon x_1\ldots x_s}(\underline{t}_1,\ldots,\underline{t}_s)=P(\underline{r}_\varepsilon(x_1)<\underline{t}_1,\ldots,\underline{r}_\varepsilon(x_s)<\underline{t}_s)$ and $\Phi_{x_1\ldots x_s}(\underline{t}_1,\ldots,\underline{t}_s)$ denotes the distribution function of the Gaussian vector $(\xi(x_1),\ldots,\xi(x_s))$. The theorem 5 is proved.

3. STOCHASTIC EIGENVALUE PROBLEMS

3.1. We regard the stochastic eigenvalue problem

$$L(\omega)u \doteq (-1)^m[f_m(x)u^{(m)}]^{(m)} + \sum_{r=0}^{m-1}(-1)^r[f_r(x,\omega)u^{(r)}]^{(r)} = \lambda u$$
$$u^{(k)}(0) = u^{(k)}(1) = 0,\ k=0,1,\ldots,m-1. \tag{4}$$

Let $\bar f_r(x,\omega)\doteq f_r(x,\omega)-\langle f_r(x)\rangle$ for $0\le r\le m-1$ be independent, weakly correlated processes with the correlation length ε and a.s. all the trajectories of $\bar f_r(x,\omega)$ be continuous. The deterministic function $f_m(x)\neq 0$ has continuous m-th order derivatives. Further on $|\bar f_r^{(s)}(x,\omega)|<\eta$ is assumed to be with a small η for $s=0,1,\ldots,r$ and $r=0,1,\ldots,m-1$. The functions $c_r(x)\doteq\langle f_r(x)\rangle$, $r=0,1,\ldots,m-1$, possess such properties that the averaged problem to (4) (see [4])

$$\langle L(\omega)\rangle w \doteq (-1)^m[f_m(x)w^{(m)}]^{(m)} + \sum_{r=0}^{m-1}(-1)^r[c_r(x)w^{(r)}]^{(r)} = \mu w$$

$$w^{(k)}(0) = w^{(k)}(1) = 0, \ k=0,1,\ldots,m-1$$

is positive definite. Then $L(\omega)$ is also a.s. positive definite for small η.

Let $\lambda_l(\omega)$ be the eigenvalues of Eq.(4). Further on let μ_l the eigenvalues and $w_l(x)$ the eigenfunctions of the averaged problem. We assume that the eigenvalues μ_l are simple. Then we have the development of $\lambda_l(\omega)$

$$\lambda_l(\omega) = \mu_l + \lambda_{1l}(\omega) + \lambda_{2l}(\omega) + \ldots \qquad \text{(convergence a.s.)}$$

with $\lambda_{1l}(\omega)=b_{ll}(\omega)$ and of the eigenfunctions of Eq.(4)

$$u_l(x,\omega) = w_l(x) + u_{1l}(x,\omega) + u_{2l}(x,\omega) + \ldots$$

(convergence in $L_2(0,1)$ a.s.) with

$$u_{1l}(x,\omega) = -\sum_{i\neq l}^{\infty} \frac{1}{\mu_{il}} b_{il}(\omega)w_i(x)$$

(see [4]). It is $\mu_{ij} \doteq \mu_i - \mu_j$ and $b_{ij}(\omega)=(L_1(\omega)w_i,w_j)$ where

$$L_1(\omega)u \doteq L(\omega)u - \langle L\rangle u = \sum_{r=0}^{m-1}(-1)^r[\bar{f}_r(x,\omega)u^{(r)}]^{(r)}.$$

The following important theorem 6 is proved in the paper [6].

THEOREM 6. We get for the terms $u_{kl}(x,\omega)$ and $\lambda_{kl}(\omega)$ of the developments of $u_l(x,\omega)$ and $\lambda_l(\omega)$

$$u_{kl}(x,\omega) = \sum_{r_1,\ldots,r_k=0}^{m-1} \int_0^1\cdots\int_0^1 \bar{f}_{r_1}(y_1)\cdots\bar{f}_{r_k}(y_k)\overset{l}{H}_{r_1\cdots r_k}(x;y_1,\ldots,y_k)\,dy_1\ldots dy_k$$

$$\lambda_{kl}(\omega) = \sum_{r_1,\ldots,r_k=0}^{m-1} \int_0^1\cdots\int_0^1 \bar{f}_{r_1}(y_1)\cdots\bar{f}_{r_k}(y_k)\overset{l}{F}_{r_1\cdots r_k}(y_1,\ldots,y_k)\,dy_1\ldots dy_k$$

$(k\geq 1)$ where

$$\frac{\partial^p}{\partial x^p}\overset{l}{H}_{r_1\ldots r_k}(x;y_1,\ldots,y_k) \doteq \overset{l}{H}{}^{(p)}_{r_1\ldots r_k}(x;y_1,\ldots,y_k) \in C([0,1]^{k+1})$$

for $0\leq p\leq m-1$ and

$$\overset{l}{F}_{r_1\ldots r_k}(y_1,\ldots,y_k) \in C([0,1]^k).$$

Particulary we obtain

$$\overset{l}{F}_r(y) = (w_l^{(r)}(y))^2, \quad \overset{l}{H}_r(x;y) = -\frac{\partial^r G_l(x,y)}{\partial y^r}\, w_l^{(r)}(y)$$

if $G_I(x,y)$ is the generalized Green function to the operator $\langle L\rangle-\mu_I$ and the boundary conditions $u^{(k)}(0)=u^{(k)}(1)=0$ for $k=0,1,\dots,m-1$.

For the following we define the notation

$$\Lambda_{Ii}(\omega) \doteq \begin{cases} \lambda_I(\omega) & \text{for } i=0 \\ u_I(x_i,\omega) & \text{for } i \neq 0 \end{cases}$$

if $x_i \in [0,1]$ for $i=1,2,\dots$ are given numbers. Then we obtain, as given above,

$$\Lambda_{Ii}(\omega) = \sum_{k=0}^{\infty} \Lambda_{kIi}(\omega) \qquad \text{(convergence a.s.)}$$

when we additionally assume the convergence of $\sum_{k=0}^{\infty} u_{kI}(x,\omega)$ a.s. for each $x \in [0,1]$. Hence we have

$$\Lambda_{kIi}(\omega)=\sum_{r_1,\dots,r_k=0}^{m-1} \int_0^1\dots\int_0^1 \bar{f}_{r_1}(y_1)\dots\bar{f}_{r_k}(y_k)\overset{I}{G}_{r_1\dots r_k,i}(y_1,\dots,y_k)dy_1\dots dy_k$$

with

$$\overset{I}{G}_{r_1\dots r_k,i}(y_1,\dots,y_k) = \begin{cases} \overset{I}{F}_{r_1\dots r_k}(y_1,\dots,y_k) & \text{for } i=0 \\ \overset{I}{H}_{r_1\dots r_k}(x_i;y_1,\dots,y_k) & \text{for } i\neq 0. \end{cases}$$

With these notations we prove the following theorem 7.

THEOREM 7. Be $x_i \in [0,1]$, $\Lambda_{Ii}(\omega)=\sum_{k=0}\Lambda_{kIi}(\omega)$ and let $\bar{f}_{r\varepsilon}(x,\omega)$ be sequences of independent, weakly correlated processes with the correlation length $\varepsilon\downarrow 0$ where

$$\langle\bar{f}_{r\varepsilon}(x)\bar{f}_{r\varepsilon}(y)\rangle = \begin{cases} R_{r\varepsilon}(x,y) & \text{for } y\in K_\varepsilon(x) \\ 0 & \text{for } y\notin K_\varepsilon(x) \end{cases}$$

and $\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_{r\varepsilon}(x,x+y)dy = a_r(x)$ uniformly in x. Then the random vector

$$\frac{1}{\sqrt{\varepsilon}} (\Lambda_{I_1 i_1}-\Lambda_{oI_1 i_1}, \dots ,\Lambda_{I_s i_s}-\Lambda_{oI_s i_s})^T$$

converges in the distribution to a Gaussian random vector

$$(\xi_{I_1 i_1},\xi_{I_2 i_2},\dots,\xi_{I_s i_s})^T \quad \text{with} \quad \xi_{I_p i_p} = \sum_{t=0}^{m-1} \xi_{tI_p i_p}.$$

The random vector $\xi_t \doteq (\xi_{tI_1 i_1},\dots,\xi_{tI_s i_s})^T$, $t=0,1,\dots,m-1$, are independent and we have

$$\langle\xi_t\rangle = 0, \quad \langle\xi_t\xi_t^T\rangle = \left(\int_0^1 a_t(y)\overset{I_p}{G}_{t,i_p}(y)\overset{I_q}{G}_{t,i_q}(y)dy\right)_{1\le p,q\le s}.$$

PROOF. We calculate the k-th moment

$$\frac{1}{\sqrt{\varepsilon}^k}\left\langle(\Lambda^{(n)}_{1_{a_1}i_{a_1}}-\Lambda_{o1_{a_1}i_{a_1}})\cdots(\Lambda^{(n)}_{1_{a_k}i_{a_k}}-\Lambda_{o1_{a_k}i_{a_k}})\right\rangle$$

with $a_q\in\{1,\dots,s\}$, $\Lambda^{(n)}_{1i}(\omega)\doteq\sum_{k=o}^{n}\Lambda_{k1i}(\omega)$ and show that this moment converges to the adequate k-th moment of $\{\xi_{1_1i_1},\dots,\xi_{1_si_s}\}$.

At first we calculate the order of a term of the form

$$P_{p\varepsilon}=\sum_{r_1,\dots,r_p=o}^{m-1}\int_o^1\cdots\int_o^1\bar{f}_{r_1\varepsilon}(y_1)\cdots\bar{f}_{r_p\varepsilon}(y_p)L_{r_1\dots r_p}(y_1,\dots,y_p)dy_1\dots dy_p$$

as $\varepsilon\downarrow 0$ if $L_{r_1\dots r_p}$ satisfies the condition $|L_{r_1\dots r_p}(y_1,\dots,y_p)|\leq C$. Because of the independence of the processes $\bar{f}_{i\varepsilon}(x,\omega)$, we must deal with terms of the form

$$Q_\varepsilon=\int_o^1\cdots\int_o^1\langle\bar{f}_{o\varepsilon}(\overset{o}{y}_1)\cdots\bar{f}_{o\varepsilon}(\overset{o}{y}_{p_o})\rangle\langle\bar{f}_{1\varepsilon}(\overset{1}{y}_1)\cdots\bar{f}_{1\varepsilon}(\overset{1}{y}_{p_1})\rangle\cdot\ \dots\ \cdot \tag{5}$$
$$\cdot\langle\bar{f}_{m-1\varepsilon}(\overset{m-1}{y}_1)\cdots\bar{f}_{m-1\varepsilon}(\overset{m-1}{y}_{p_{m-1}})\rangle L_{\dots}(\overset{o}{y}_1,\dots,\overset{m-1}{y}_{p_{m-1}})d\overset{o}{y}_1\dots d\overset{m-1}{y}_{p_{m-1}}$$

when we want to calculate the order of $P_{p\varepsilon}$. It follows

$$|Q_\varepsilon|\leq C\int_o^1\cdots\int_o^1|\langle\bar{f}_{o\varepsilon}(\overset{o}{y}_1)\cdots\bar{f}_{o\varepsilon}(\overset{o}{y}_{p_o})\rangle|\,d\overset{o}{y}_1\dots d\overset{o}{y}_{p_o}\ \cdot\ \dots\ \cdot$$
$$\cdot\int_o^1\cdots\int_o^1|\langle\bar{f}_{m-1\varepsilon}(\overset{m-1}{y}_1)\cdots\bar{f}_{m-1\varepsilon}(\overset{m-1}{y}_{p_{m-1}})\rangle|\,d\overset{m-1}{y}_1\dots d\overset{m-1}{y}_{p_{m-1}}$$

since $L_{\dots}$ is bounded. By the use of

$$\int_o^1\cdots\int_o^1|\langle\bar{f}_{q\varepsilon}(\overset{q}{y}_1)\cdots\bar{f}_{q\varepsilon}(\overset{q}{y}_{p_q})\rangle|\,d\overset{q}{y}_1\dots d\overset{q}{y}_{p_q}=\begin{cases}O(\varepsilon^{p_q/2}) & \text{for } p_q \text{ even}\\ O(\varepsilon^{(p_q+1)/2}) & \text{for } p_q \text{ odd}\end{cases}$$

(see the proof of the adequate theorem in [6] to the theorem 3 of this paper) we obtain an estimation of the order of Q_ε and hence we obtain the estimate of $P_{p\varepsilon}$

$$|P_{p\varepsilon}|\leq\begin{cases}O(\varepsilon^{p/2}) & \text{for p even}\\ O(\varepsilon^{(p+1)/2}) & \text{for p odd.}\end{cases} \tag{6}$$

With this result we can now show considering theorem 6

$$\frac{1}{\sqrt{\varepsilon}^k}\langle(\Lambda^{(n)}_{1_{a_1}i_{a_1}}-\Lambda_{o1_{a_1}i_{a_1}})\cdots(\Lambda^{(n)}_{1_{a_k}i_{a_k}}-\Lambda_{o1_{a_k}i_{a_k}})\rangle$$
$$=\frac{1}{\sqrt{\varepsilon}^k}\left\langle\left(\sum_{p=1}^{n}\Lambda_{p1_{a_1}i_{a_1}}\right)\cdots\left(\sum_{p=1}^{n}\Lambda_{p1_{a_k}i_{a_k}}\right)\right\rangle$$

$$= \begin{cases} \frac{1}{\sqrt{\varepsilon}^k} \langle \Lambda_{11_{a_1} i_{a_1}} \cdots \Lambda_{11_{a_k} i_{a_k}} \rangle + O(\varepsilon) & \text{for } k \text{ even} \\ O(\sqrt{\varepsilon}) & \text{for } k \text{ odd} \end{cases}$$

and hence

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\sqrt{\varepsilon}^k} \langle (\Lambda^{(n)}_{1_{a_1} i_{a_1}} - \Lambda_{o1_{a_1} i_{a_1}}) \cdots (\Lambda^{(n)}_{1_{a_k} i_{a_k}} - \Lambda_{o1_{a_k} i_{a_k}}) \rangle$$

$$= \begin{cases} \lim_{\varepsilon \downarrow 0} \frac{1}{\sqrt{\varepsilon}^k} \langle \bar{\Lambda}_{a_1} \bar{\Lambda}_{a_2} \cdots \bar{\Lambda}_{a_k} \rangle & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$$

with $\bar{\Lambda}_{a_q} \doteq \Lambda_{11_{a_q} i_{a_q}}$. By $\Lambda_{11i} = \sum_{t=0}^{m-1} \int_0^1 \bar{f}_t(y) \overset{1}{G}_{t,i}(y)dy$ and by similar considerations as in the proof of theorem 5 we obtain the theorem for $\Lambda^{(n)}_{1i}$.

We will show in the following considerations that the theorem 7 is also right for $\frac{1}{\sqrt{\varepsilon}}(\Lambda_{1_1 i_1} - \Lambda_{o1_1 i_1}, \ldots, \Lambda_{1_s i_s} - \Lambda_{o1_s i_s})^T$ instead of $\frac{1}{\sqrt{\varepsilon}}(\Lambda^{(n)}_{1_1 i_1} - \Lambda_{o1_1 i_1}, \ldots, \Lambda^{(n)}_{1_s i_s} - \Lambda_{o1_s i_s})^T$. For this we denote the convergence

$$\lim_{n\to\infty} \frac{1}{\varepsilon} \langle (\Lambda^{(n)}_{1i} - \Lambda_{1i})^2 \rangle = 0 \tag{7}$$

uniformly in ε. We will deal with the idea of this proof at the boundary value problems because we use for the proof fundamental estimations from the perturbation theory.

Then from this convergence in L_2-mean it follows the convergence in probability uniformly in ε

$$\operatorname*{P-lim}_{n\to\infty} \frac{1}{\sqrt{\varepsilon}} (\Lambda^{(n)}_{1i} - \Lambda_{o1i}) = \frac{1}{\sqrt{\varepsilon}} (\Lambda_{1i} - \Lambda_{o1i})$$

and then

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{\varepsilon}} (\Lambda^{(n)}_{1i} - \Lambda_{o1i}) < t\right) = P\left(\frac{1}{\sqrt{\varepsilon}} (\Lambda_{1i} - \Lambda_{o1i}) < t\right) \tag{8}$$

uniformly in ε. We have shown in a first part of this proof

$$\lim_{\varepsilon\downarrow 0} F^{(n)}_{\varepsilon s}(t_1, \ldots, t_s) = \Phi_s(t_1, \ldots, t_s) \qquad \text{for all } n \geq 1 \tag{9}$$

if we set

$$F^{(n)}_{\varepsilon s}(t_1, \ldots, t_s) = P\left(\frac{1}{\sqrt{\varepsilon}}(\Lambda^{(n)}_{1_1 i_1} - \Lambda_{o1_1 i_1}) < t_1, \ldots, \frac{1}{\sqrt{\varepsilon}}(\Lambda^{(n)}_{1_s i_s} - \Lambda_{o1_s i_s}) < t_s\right)$$

$$\Phi_s(t_1, \ldots, t_s) = P(\xi_{1_1 i_1} < t_1, \ldots, \xi_{1_s i_s} < t_s).$$

Let $F_{\varepsilon s}(t_1,\dots,t_s)$ be the distribution function of $\frac{1}{\sqrt{\varepsilon}}(\Lambda_{l_1 i_1}-\Lambda_{ol_1 i_1},\dots,\Lambda_{l_s i_s}-\Lambda_{ol_s i_s})^T$. Hence we obtain from relations as in Eq.(8)

$$|F_{\varepsilon s}^{(n)}(t_1,\dots,t_s) - F_{\varepsilon s}(t_1,\dots,t_s)| < \eta$$

for all $n \geq n_o(\eta)$ uniformly in ε and with Eq.(9)

$$\phi_s(t_1,\dots,t_s)-\eta \leq \overline{\lim_{\varepsilon\downarrow o}}\, F_{\varepsilon s}(t_1,\dots,t_s) \leq \phi_s(t_1,\dots,t_s)+\eta$$

for every $\eta>0$. Then it follows

$$\lim_{\varepsilon\downarrow o} F_{\varepsilon s}(t_1,\dots,t_s) = \phi_s(t_1,\dots,t_s) \tag{10}$$

and by it the theorem 7.

Now we consider a few important special cases of theorem 7:

(a) $\frac{1}{\sqrt{\varepsilon}}(\lambda_l(\omega)-\mu_l) \xrightarrow[\varepsilon\downarrow o]{\text{in the distribution}} \xi_{lo}$, where ξ_{lo} is a Gaussian random variable with the parameters

$$\langle\xi_{lo}\rangle = 0,\ \langle\xi_{lo}^2\rangle = \sum_{t=o}^{m-1}\int_o^1 a_t(y)(w_l^{(t)}(y))^4dy.$$

(b) $\frac{1}{\sqrt{\varepsilon}}(u_l(x,\omega)-w_l(x)) \xrightarrow[\varepsilon\downarrow o]{\text{in the distribution}} \eta_l(x)$, where $\eta_l(x)$ is a Gaussian process with the parameters

$$\langle\eta_l(x)\rangle = 0,$$
$$\langle\eta_l(x)\eta_l(y)\rangle = \sum_{t=o}^{m-1}\int_o^1 a_t(z)\frac{\partial^t G_l(x,z)}{\partial z^t}\,\frac{\partial^t G_l(y,z)}{\partial z^t}(w_l^{(t)}(z))^2dz.$$

By theorem 7 the distribution function of the random vector $\underline{U}=\frac{1}{\sqrt{\varepsilon}}(u_l(x_1)-w_l(x_1),\dots,u_l(x_k)-w_l(x_k))^T$ converges as $\varepsilon\downarrow 0$ to the distribution function $\phi_{x_1\dots x_k}(t_1,\dots,t_k)$ of the random vector $(\xi_{l1},\dots,\xi_{lk})^T$ which is a Gaussian random vector with

$$\langle\xi_{lp}\xi_{lq}\rangle = \sum_{t=o}^{m-1}\langle\xi_{tlp}\xi_{tlq}\rangle = \sum_{t=o}^{m-1}\int_o^1 a_t(y)\overset{l}{G}_{t,p}(y)\overset{l}{G}_{t,q}(y)dy$$
$$= \sum_{t=o}^{m-1}\int_o^1 a_t(y)\frac{\partial^t}{\partial y^t}G_l(x_p,y)\frac{\partial^t}{\partial y^t}G_l(x_q,y)(w_l^{(t)}(y))^2dy.$$

Hence the case (b) is proved.

(c) $\frac{1}{\sqrt{\varepsilon}}(\lambda_{l_1}-\mu_{l_1},\dots,\lambda_{l_r}-\mu_{l_r},u_{l_{r+1}}(x)-w_{l_{r+1}}(x),\dots,u_{l_k}(x)-w_{l_k}(x))^T$

$$\xrightarrow[\varepsilon\downarrow o]{\text{in the distribution}} (\xi_{l_1 o},\dots,\xi_{l_r o},\eta_{l_{r+1}}(x),\dots,\eta_{l_k}(x))^T,$$

where $\Sigma(x)=(\xi_{l_1 o},\dots,\xi_{l_r o},\eta_{l_{r+1}}(x),\dots,\eta_{l_k}(x))^T$ is a Gaussian process with $\langle \underline{\zeta}(x)\rangle = 0$ and

$$\langle \underline{\zeta}(x)\underline{\zeta}(y)^T\rangle =$$

$$=\sum_{t=0}^{m-1}\int_0^1 a_t(z)\begin{pmatrix} \left((w_{l_p}^{(t)}(z)w_{l_q}^{(t)}(z))^2\right)_{1\leq p,q\leq r} & (V_{qp}^t(y,z))_{\substack{1\leq p\leq r\\ r+1\leq q\leq k}} \\ (V_{pq}^t(x,z))_{\substack{r+1\leq p\leq k\\ 1\leq q\leq r}} & (W_{pq}^t(x,y,z))_{r+1\leq p,q\leq k}\end{pmatrix} dz$$

with $$V_{pq}^t(x,z) = -\frac{\partial^t G_{l_p}(x,z)}{\partial z^t}(w_{l_q}^{(t)}(z))^2 w_{l_p}^{(t)}(z)$$

$$W_{pq}^t(x,y,z) = \frac{\partial^t G_{l_p}(x,z)}{\partial z^t}\,\frac{\partial^t G_{l_q}(y,z)}{\partial z^t}\,w_{l_p}^{(t)}(z)w_{l_q}^{(t)}(z).$$

This case (c) also follows from theorem 7.

3.2. The results of this section can also be used of eigenvalue problems of the form

$$Lu = \lambda h(x,\omega)u, \quad U_i[u] = 0,\ i=1,2,\dots,2m,\ 0\leq x\leq 1. \tag{11}$$

The operator L is a deterministic differential operator of the order 2m

$$Lu \doteq \sum_{r=0}^{m}(-1)^r[f_r(x)u^{(r)}]^{(r)} \tag{12}$$

where the coefficient $f_r(x)$ is continuous differentiable of r-th order. It is for the stochastic process $h(x,\omega)$ the equation $h(x,\omega)=h_o(x)+g(x,\omega)$. Let $h_o(x)$ be positive and let the process $g(x,\omega)$ be a weakly correlated process of the correlation length ε with $|g(x,\omega)|\leq\eta$ (η sufficient small). The deterministic boundary conditions $U_i[u]=0, i=1,2,\dots,2m$, are constituted in that manner that the problem (11) is selfadjoint and positive definite.

By the made suppositions the averaged problem to (11) $Lw=\mu h_o w$, $U_i[w]=0, i=1,2,\dots,2m$, possesses enumerable many positive eigenvalues $0<\mu_1\leq\mu_2\leq\dots$ Let all eigenvalues μ_l be simple. The eigenfunctions $w_l(x)$ are assumed to be orthonormal in $L_2(0,1)_{h_o}$, i.e. we have for $k,l=1,2,\dots$

$$(w_k,w_l)_{h_o} \doteq \int_0^1 w_k(x)w_l(x)h_o(x)\,dx = \delta_{kl}.$$

We define as in section 3.1.

$$\Lambda_{1i}(\omega) = \begin{cases} \lambda_1(\omega) & \text{for } i=0 \\ u_1(x_i,\omega) & \text{for } i\neq 0 \end{cases}$$

with $x_i \in [0,1]$. Then a development exists of $\Lambda_{1i}(\omega)$ (see /5/) $\Lambda_{1i}(\omega)=\sum_{k=0}^{\infty}\Lambda_{k1i}(\omega)$ for which we assume the convergence of $\sum_{k=0}^{\infty}\langle|\Lambda_{k1i}|\rangle$. It is

$$\Lambda_{o1i}=\begin{cases} \mu_1 & \text{for } i=0 \\ w_1(x_i) & \text{for } i\neq 0 \end{cases} \text{ and } \Lambda_{11i}=\begin{cases} -\mu_1 b_{11} & \text{for } i=0 \\ \mu_1 \int_0^1 G_1(x_i,y)g(y,\omega)w_1(y)dy & \text{for } i\neq 0 \end{cases}$$

where $G_1(x,y)$ denotes the generalized Green function to $L-\mu_1 h_o$ and the boundary conditions U_i, $i=1,2,\dots,2m$, and

$b_{11}(\omega) \doteq \int_0^1 (w_1(y))^2 g(y,\omega)dy$. By

$$\Lambda_{11i} \doteq \int_0^1 g(y,\omega)\overset{1}{G}_i(y)dy, \qquad \overset{1}{G}_i(y)=\begin{cases} -\mu_1 (w_1(y))^2 & \text{for } i=0 \\ \mu_1 G_1(x_i,y)w_1(y) & \text{for } i\neq 0 \end{cases}$$

we can formulate a to theorem 7 adequate theorem.

THEOREM 8. Let $x_1,x_2,\dots$ be from $[0,1]$ and let $g_\varepsilon(x,\omega)$ be a sequence of weakly correlated processes with the correlation length ε and

$$\langle g_\varepsilon(x)g_\varepsilon(y)\rangle = \begin{cases} R_\varepsilon(x,y) & \text{for } y\in K_\varepsilon(x) \\ 0 & \text{for } y\notin K_\varepsilon(x) \end{cases}, \quad \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_\varepsilon(x,x+y)dy=a(x)$$

($a(x)\not\equiv 0$) uniformly in x. Then the random vector

$$\frac{1}{\sqrt{\varepsilon}}(\Lambda_{1_1 i_1}-\Lambda_{o1_1 i_1},\dots,\Lambda_{1_s i_s}-\Lambda_{o1_s i_s})^T$$

formed from the stochastic eigenvalue problem (11) with $g_\varepsilon(x,\omega)$ converges in the distribution to a Gaussian random vector $(\xi_{1_1 i_1},\dots,\xi_{1_s i_s})^T$ with

$$\langle \xi_{1_p i_p}\rangle = 0, \ \langle \xi_{1_p i_p}\xi_{1_q i_q}\rangle = \int_0^1 a(y)\overset{1_p}{G}_{i_p}(y)\overset{1_q}{G}_{i_q}(y)dy.$$

REMARK. We can also prove such a theorem for the more general stochastic eigenvalue problem

$$Lu + L_1(\omega)u = \lambda(h_o u + M_1(\omega)u), \ U_i[u] = 0, \ i=1,2,\dots,2m, \ 0\leq x\leq 1.$$

Here the operator L is a deterministic differential operator like in Eq.(12) and

$$L_1(\omega)u = \sum_{k=0}^{m-1} a_k(x,\omega)u^{(k)}, \ M_1(\omega)u = \sum_{k=0}^{m-1} b_k(x,\omega)u^{(k)}$$

with $\langle L_1(\omega)\rangle = \langle M_1(\omega)\rangle = 0$ where the coefficients $a_k(x,\omega)$, $b_k(x,\omega)$ are independent, weakly correlated processes. Perturbation serieses relativ to $L_1(\omega)$, $M_1(\omega)$ form the basic. These serieses were deduced in [5].

3.3. We regard the stochastic eigenvalue problem

$$-u'' + a(x,\omega)u = \lambda(1 + b(x,\omega))u, \quad u(0) = u(1) = 0 \tag{13}$$

where $a(x,\omega)$, $b(x,\omega)$ denote independent, weakly correlated processes of the correlation length ε. The averaged problem to (13) is

$$-w'' = \mu w, \quad w(0) = w(1) = 0.$$

This averaged problem possesses the simple eigenvalues $\mu_l = (l\pi)^2$ and the eigenfunctions $w_l(x) = \sqrt{2}\sin(l\pi x)$. It is for the processes $a(x,\omega)$ and $b(x,\omega)$

$$\langle a(x)a(y)\rangle = R_{a\varepsilon}(x,y), \qquad \langle b(x)b(y)\rangle = R_{b\varepsilon}(x,y)$$

($R_{a\varepsilon}(x,y) = R_{b\varepsilon}(x,y) = 0$ for $y \notin K_\varepsilon(x)$) and

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_{a\varepsilon}(x,x+y)\,dy = \alpha(x), \quad \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_{b\varepsilon}(x,x+y)\,dy = \beta(x).$$

By the supposition that $|a(x,\omega)|$ and $|b(x,\omega)|$ are small we obtain the development (with the notation as above given)

$$\Lambda_{li}(\omega) = \sum_{k=o}^{\infty} \Lambda_{kli}(\omega) \qquad \text{where}$$

$$\Lambda_{oli} = \begin{cases} \mu_l & \text{for } i=0 \\ w_l(x_i) & \text{for } i\neq 0 \end{cases} \text{ is and with } \overset{l}{G}_i(y) = \begin{cases} (w_l(y))^2 & \text{for } i=0 \\ -G_l(x_i,y)w_l(y) & \text{for } i\neq 0 \end{cases}$$

the formula $\Lambda_{1li} = \int_0^1 (a(y) - \mu_l b(y))\overset{l}{G}_i(y)\,dy$ is right. It is

$$G_l(x,y) = \begin{cases} \frac{1}{l\pi}\left[\left\{\frac{1}{2l\pi}\sin(l\pi x) - x\cos(l\pi x)\right\}\sin(l\pi y) + (1-y)\cos(l\pi y)\sin(l\pi x)\right] \\ \qquad \text{for } 0 \le x < y \le 1 \\ \frac{1}{l\pi}\left[\left\{\frac{1}{2l\pi}\sin(l\pi x) + (1-x)\cos(l\pi x)\right\}\sin(l\pi y) - y\cos(l\pi y)\sin(l\pi x)\right] \\ \qquad \text{for } 0 \le y < x \le 1. \end{cases}$$

By theorem 7 the random vector $\frac{1}{\sqrt{\varepsilon}}(\Lambda_{1l_1i_1} - \Lambda_{ol_1i_1}, \ldots, \Lambda_{1l_si_s} - \Lambda_{ol_si_s})^T$ converges in the distribution to the Gaussian random vector $(\xi_{l_1i_1}, \ldots, \xi_{l_si_s})^T$ with

$$\langle \xi_{1_p i_p} \rangle = 0, \quad \langle \xi_{1_p i_p} \xi_{1_q i_q} \rangle = \int_0^1 (\alpha(y) + \mu_{1_p}\mu_{1_q}\beta(y)) G^{1_p}_{i_p}(y) G^{1_q}_{i_q}(y)\,dy.$$

Particulary we obtain:

(a) For $F_\varepsilon^{1k}(t,s) \doteq P(\frac{1}{\sqrt{\varepsilon}}(\lambda_1-\mu_1) < t, \frac{1}{\sqrt{\varepsilon}}(\lambda_k-\mu_k) < s)$ and $\phi^{1k}(t,s) =$ $P(\xi_{1o} < t, \xi_{ko} < s)$ (Gaussian distribution) we have

$$\lim_{\varepsilon \downarrow 0} F^{1k}(t,s) = \phi^{1k}(t,s)\,.$$

with $\langle \xi_{po} \rangle = 0$, $(\langle \xi_{po}\xi_{qo} \rangle)_{p,q \in \{1,k\}} = (\int_0^1 (\alpha + \mu_p\mu_q\beta) w_p^2 w_q^2 dy)_{p,q \in \{1,k\}}$

Particulary for wide-sense stationary, weakly correlated processes it is

$$(\langle \xi_{po}\xi_{qo} \rangle)_{p,q \in \{1,k\}} = \begin{pmatrix} \frac{3}{2}(\alpha + \mu_p^2\beta) & \alpha + \mu_p\mu_q\beta \\ \alpha + \mu_p\mu_q\beta & \frac{3}{2}(\alpha + \mu_q^2\beta) \end{pmatrix}.$$

The variation of the limit distribution depends on the number of the eigenvalues

$$\langle \xi_{1o}^2 \rangle = \frac{3}{2}(\alpha + \mu_1^2\beta),$$

if $b \not\equiv 0$ and $a(x,\omega)$, $b(x,\omega)$ are wide-sense stationary, weakly correlated processes. This is reflected in a melt of the eigenvalues with increasing number (see Fig.2).

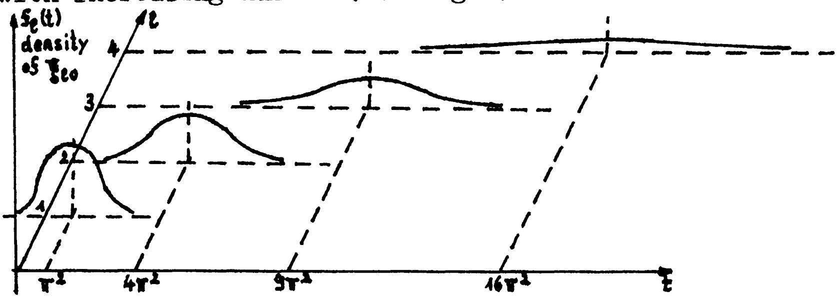


Fig.2

By $b \equiv 0$ the variation of the limit distribution does not depends on the number of the eigenvalues. In this case the correlation $\langle \xi_{1o}\xi_{ko} \rangle = \alpha$ for $k \neq 1$ is independent of k and 1. This effect is explicable from the fact that the operator $L(\omega)$ determines the eigenvalues respectively the eigenvalues $\lambda_1(\omega)$ for a ω_o determines about the random operator $L(\omega)$ the other eigenvalues $\lambda_k(\omega)$ for this ω_o.

(b) The random vector $\frac{1}{\sqrt{\varepsilon}}(u_1(x)-w_1(x),u_k(x)-w_k(x))^T$ converges in the distribution as $\varepsilon\downarrow 0$ to the Gaussian vector process $(\eta_1(x),\eta_k(x))^T$ with $\langle\eta_p(x)\rangle = 0$,

$$\langle\eta_p(x)\eta_q(y)\rangle = \int_0^1(\alpha(z)+\mu_p\mu_q\beta(z))G_p(x,z)G_q(y,z)w_p(z)w_q(z)dz$$

for $p,q\in\{k,1\}$.

We can also calculate the correlation values of this Gaussian vector process from

$$\langle\eta_p(x)\eta_q(x)\rangle = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\langle u_{1p}(x)u_{1q}(y)\rangle,$$

if $u_{1p}(x,\omega)$ denotes the first term in the development of $u_p(x,\omega)-w_p(x)$ relativ to $a(x,\omega)$ and $b(x,\omega)$. One obtains

$$u_{1p}(x,\omega) = -\frac{1}{p\pi}\int_0^x\bar{a}w_p\sin(p\pi(t-x))dt + \frac{1}{2\mu_p}(xw_p'(x)-\tfrac{1}{2}w_p(x))\int_0^1\bar{a}w_p^2dt - \frac{1}{2\mu_p}w_p(x)\int_0^1(1-t)\bar{a}w_pw_p'dt \qquad (14)$$

after a few calculations with $\bar{a}(x,\omega)=a(x,\omega)-\mu_p b(x,\omega)$ and then from (14) the correlation function of the limit process of $\frac{1}{\sqrt{\varepsilon}}(u_1-w_1)$

$$\langle\eta_1(x)\eta_1(y)\rangle = \frac{\bar{\sigma}_{11}}{8\mu_1^2}\Big\{\mu_1(\tfrac{x^2}{2}+\tfrac{y^2}{2}-y+\tfrac{1}{3})w_1(x)w_1(y)+3x(1-y)w_1'(x)w_1'(y) + \left[xw_1'(x)w_1(y)-(1-y)w_1(x)w_1'(y)\right]\left[3-\tfrac{1}{2}(w_1^2(y)+w_1^2(x))\right] + \tfrac{5}{8}w_1(x)w_1(y)\left[-5+w_1^2(y)+w_1^2(x)\right]\Big\} \qquad (\bar{\sigma}_{11}=\alpha+\mu_1^2\beta)$$

and the variation

$$\langle\eta_1^2(x)\rangle = \frac{\bar{\sigma}_{11}}{4\mu_1^2}\Big\{3\mu_1 x(1-x)+\mu_1(\tfrac{1}{6}+2x(x-1))w_1^2(x)+(x-\tfrac{1}{2})(3-w_1^2(x))w_1'(x)w_1(x) + \tfrac{5}{8}(w_1^2(x)-\tfrac{5}{2})w_1^2(x)\Big\}.$$

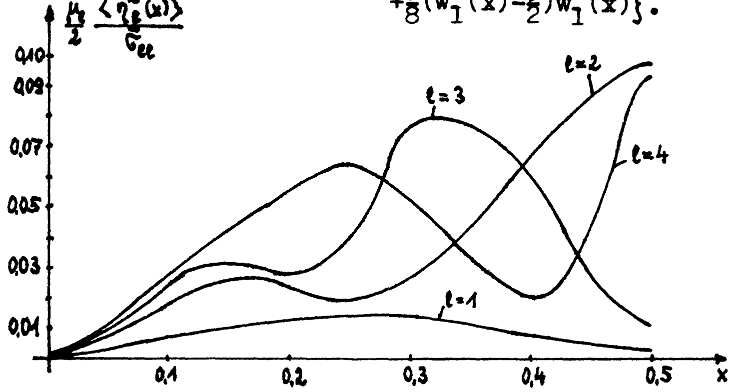


Fig.3
$\langle\eta_1^2(x)\rangle = \langle\eta_1^2(1-x)\rangle$

The figure 3 shows this variation for the parameters $l=1,2,3,4$. We can make very good statements about the behaviour of the limit process of the eigenfunctions because of the limit process $\eta_l(x)$ is a Gaussian process.

4. STOCHASTIC BOUNDARY VALUE PROBLEM.

4.1. We consider the stochastic boundary value problem in this section

$$L(\omega)u \doteq (-1)^m[f_m(x)u^{(m)}]^{(m)} + \sum_{r=0}^{m-1}(-1)^r[f_r(x,\omega)u^{(r)}]^{(r)} = g(x,\omega)$$
$$U_i[u] = 0,\ i=1,2,\dots,2m,\ 0\le x\le 1. \tag{15}$$

The boundary conditions are constituted so, that

$$(L(\omega)u,v) = \sum_{r=0}^{m}\int_0^1(-1)^r[f_r(x)u^{(r)}]^{(r)}v\,dx = \sum_{r=0}^{m}\int_0^1 f_r(x)u^{(r)}v^{(r)}dx \tag{16}$$

is right for all permissible functions u,v (i.e. functions, which possess $2m$ continuous derivatives and fulfil the boundary conditions), i.e. the boundary terms of the integration by parts must be zero. Then the stochastic operator $L(\omega)$ is symmetric relativ to all permissible functions.

$\bar{f}_r(x,\omega) \doteq f_r(x,\omega) - \langle f_r(x,\omega)\rangle$, $\bar{g}(x,\omega) \doteq g(x,\omega) - \langle g(x,\omega)\rangle$ $(0\le r\le m-1)$ are assumed to be stochastic independent and weakly correlated with the correlation length ε and $f_m(x)\neq 0$ must be a deterministic continuous function. Further on $|\bar{f}_r^{(s)}(x,\omega)|\le\eta$ is assumed to be with a small η, $s=0,1,\dots,m-1$, and the processes appearing must be almost everywhere differentiable. The boundary value problem (15) can be written in the following form:

$$L(\omega)u=\langle L(\omega)\rangle u+L_1(\omega)u=\langle g(x,\omega)\rangle+\bar{g}(x,\omega),\quad U_i[u]=0,\ i=1,2,\dots,2m$$

with $L_1(\omega)u = \sum_{r=0}^{m-1}(-1)^r[\bar{f}_r(x,\omega)u^{(r)}]^{(r)}$

$$\langle L(\omega)\rangle u = (-1)^m[f_m(x)u^{(m)}]^{(m)} + \sum_{r=0}^{m-1}(-1)^r[\langle f_r(x,\omega)\rangle u^{(r)}]^{(r)}.$$

We assume that $\langle L(\omega)\rangle w=0$, $U_i[w]=0$ possess only the trivial solution $w\equiv 0$.

We make for the solution $u(x,\omega)$ of (15) the statement

$$u(x,\omega) = \sum_{k=0}^{\infty} u_k(x,\omega),$$

when $u_k(x,\omega)$ denotes the homogeneous part of k-th order of $u(x,\omega)$ in the coefficients $\bar{f}_r$ and $\bar{g}$. Substituting this statement in (15) leads to the boundary value problem for $u_k(x,\omega)$

$$\begin{aligned} \langle L\rangle u_0 &= \langle g\rangle;\ U_i[u_0] = 0 \\ \langle L\rangle u_1 &= \bar{g} - L_1(\omega)u_0;\ U_i[u_1] = 0 \qquad (i=1,2,\dots,2m) \\ \langle L\rangle u_k &= -L_1(\omega)u_{k-1};\ U_i[u_k] = 0 \qquad \text{for } k=2,3,\dots \end{aligned} \tag{17}$$

Let $G(x,y)$ be the Green function corresponding to $\langle L\rangle$ and to the boundary conditions $U_i[.]=0$ then we obtain

$$u_0(x) = \int_0^1 G(x,y)\langle g(y,\omega)\rangle dy,$$

i.e. $u_0(x)$ is a deterministic function. With (16) for $u_1(x,\omega)$

$$\begin{aligned} u_1(x,\omega) &= \int_0^1 G(x,y)\{\bar{g}(y,\omega)-L_1(\omega)u_0(y)\}dy \\ &= \int_0^1 G(x,y)\bar{g}(y,\omega)dy - \sum_{r=0}^{m-1}\int_0^1 \bar{f}_r(y,\omega)u_0^{(r)}(y)\frac{\partial^r G(x,y)}{\partial y^r}dy \end{aligned} \tag{18}$$

is obtained.

We get the following theorem that is right analog to the theorem 6 for eigenvalue problems:

THEOREM 8. It holds

$$u_0(x) = \int_0^1 G(x,y)\langle g(y,\omega)\rangle dy$$

and for $k=1,2,\dots$

$$\begin{aligned} u_k(x,\omega) = &\sum_{r_1,\dots,r_k=0}^{m-1}\int_0^1\cdots\int_0^1 \bar{f}_{r_1}(y_1)\dots\bar{f}_{r_k}(y_k)\overset{1}{H}_{r_1\dots r_k}(x;y_1,\dots,y_k)dy_1\dots dy_k \\ &+\sum_{r_1,\dots,r_{k-1}=0}^{m-1}\int_0^1\cdots\int_0^1 \bar{f}_{r_1}(y_1)\dots\bar{f}_{r_{k-1}}(y_{k-1})\bar{g}(y_k)\ \cdot \\ &\cdot\ \overset{2}{H}_{r_1\dots r_k}(x;y_1,\dots,y_k)dy_1\dots dy_k \end{aligned}$$

for the terms $u_k(x,\omega)$ of the development of the solution $u(x,\omega)$ of the boundary value problem (15) with

$$\overset{1}{H}_r(x;y) = -u_0^{(r)}(y)\frac{\partial^r}{\partial y^r}G(x,y), \qquad \overset{2}{H}_r(x;y) = G(x,y) \qquad \text{and}$$

$$\overset{1}{H}_{r_1..r_k}(x;y_1,\dots,y_k) = \frac{\partial^{r_k}}{\partial y_k^{r_k}}(G(x,y_k))\cdot \overset{1(r_k)}{H}_{r_1..r_{k-1}}(y_k;y_1,\dots,y_{k-1})$$

$$\overset{2}{H}_{r_1..r_k}(x;y_1,\dots,y_k) = \frac{\partial^{r_{k-1}}}{\partial y_{k-1}^{r_{k-1}}}(G(x,y_{k-1}))\cdot$$

$$\cdot \overset{2(r_{k-1})}{H}_{r_1..r_{k-2}r_k}(y_{k-1};y_1,\dots,y_{k-2},y_k).$$

It is $\overset{i(r)}{H}_{...}(x;\dots) \doteq \frac{\partial^r}{\partial x^r}\overset{i}{H}_{...}(x;\dots)$ and $\overset{i}{H}_{...}(x;\dots) \in C^{(m-1)}(0,1)$.

PROOF. The statement follows for k=1 from (18). After we have obtained the formula for k-1 then it follows for k from (17)

$$u_k(x,\omega) = -\int_0^1 G(x,y)L_1(\omega)u_{k-1}(y,\omega)dy$$

$$= -\sum_{r=0}^{m-1}\int_0^1 \bar{f}_r(y,\omega)u_{k-1}^{(r)}(y)\frac{\partial^r G(x,y)}{\partial y^r}\,dy$$

$$= -\sum_{r=0}^{m-1}\sum_{r_1,\dots,r_{k-1}=0}^{m-1}\int_0^1..\int_0^1\int_0^1 \bar{f}_{r_1}(y_1)\dots\bar{f}_{r_{k-1}}(y_{k-1})\frac{\partial^r G(x,y)}{\partial y^r}\cdot$$

$$\cdot\, \overset{1(r)}{H}_{r_1\dots r_{k-1}}(y;y_1,\dots,y_{k-1})dy_1\dots dy_{k-1}dy$$

$$= \sum_{r=0}^{m-1}\sum_{r_1,\dots,r_{k-2}=0}^{m-1}\int_0^1..\int_0^1\int_0^1 \bar{f}_{r_1}(y_1)\dots\bar{f}_{r_{k-2}}(y_{k-2})\bar{f}_r(y)\bar{g}(y_{k-1})\cdot$$

$$\cdot\,\frac{\partial^r G(x,y)}{\partial y^r}\overset{2(r)}{H}_{r_1\dots r_{k-1}}(y;y_1,\dots,y_{k-1})dy_1..dy_{k-1}dy.$$

The theorem 8 is proved.

Now we deal with questions of the convergence of the development. Since $\frac{\partial^{r+s}G(x,y)}{\partial x^r\partial y^s}$ for $r,s=0,1,\dots,2m$ with $0\le r+s\le 2m$ is continuous in $0\le x<y\le 1$ respectively in $0\le y<x\le 1$, it follows for $(x,y)\in[0,1]\times[0,1]$ and the stated r,s $\left|\frac{\partial^{r+s}G(x,y)}{\partial x^r\partial y^s}\right|\le C$, further on we have

$$|u_0^{(r)}(x)| \le C_0 \qquad \text{for } 0\le r\le 2m \text{ and } x\in[0,1]$$

$$\left|\frac{1}{f_m(x)}\right| \le a \qquad \text{for } x\in[0,1] \text{ and} \tag{19}$$

$$|\bar{f}_p^{(r)}(x,\omega)| \le \eta \quad \text{almost surely for } 0\le p\le m-1, 0\le r\le p, x\in[0,1].$$

C, C_0 and a are constants. By properties of the Green· function

$$u_k^{(2m)}(x,\omega) = \int_0^1 \frac{\partial^{2m}G(x,y)}{\partial x^{2m}} L_1u_{k-1}(y)dy + \frac{(-1)^m}{f_m(x)} L_1u_{k-1}(x)$$

$$u_k^{(p)}(x,\omega) = \int_0^1 \frac{\partial^p G(x,y)}{\partial x^p} L_1u_{k-1}(y)dy, \quad p=0,1,\dots,2m-1$$

and therefore

$$|u_k^{(p)}(x,\omega)| \leq (C+a) \max_{x\in[0,1]} |L_1u_{k-1}(x)| \quad \text{for } p=0,1,\dots,2m.$$

Hence we obtain with

$$|L_1u_o(x)| \leq \sum_{r=o}^{m-1}\left|\sum_{v=o}^{r}\binom{r}{v}\bar{f}_r^{(v)}u_o^{(2r-v)}\right| \leq \eta C_o \sum_{r=o}^{m-1}\sum_{v=o}^{r}\binom{r}{v} = 2^m\eta C_o$$

for almost all ω then

$$|u_1^{(p)}(x,\omega)| \leq 2^m(C+a)\eta C_o \quad \text{for almost all } \omega$$

and through induction for almost all ω

$$|u_k^{(p)}(x,\omega)| \leq (2^m(C+a)\eta)^k C_o$$

for $0\leq p\leq 2m$ and $x \in [0,1]$.

Hence $\sum_{k=o}^{\infty}|u_k^{(p)}(x,\omega)|$ converges almost surely and uniformly in $0\leq x\leq 1$ when $|\eta| \leq (2^m(C+a))^{-1}$ for almost all ω is right. Then $\sum_{k=o}^{\infty}\langle|u_k^{(p)}(x,\omega)|\rangle$ also converges uniformly in $0\leq x\leq 1$ with this condition. Let $u_k(x,\omega)$ be calculated by (17) then $u(x,\omega)=\sum_{k=o}^{\infty}u_k(x,\omega)$ is the solution of the stochastic boundary value problem (15). Now we formulate the theorem 9.

THEOREM 9. If $\bar{f}_r(x,\omega)$, $\bar{g}(x,\omega)$ are sequences of independent, weakly correlated processes with the correlation length $\varepsilon\downarrow 0$ and

$$\langle\bar{f}_r(x)\bar{f}_r(y)\rangle = \begin{cases} R_{r\varepsilon}(x,y) & \text{for } y\in K_\varepsilon(x) \\ 0 & \text{for } y\notin K_\varepsilon(x), \end{cases}$$

$$\langle\bar{g}(x)\bar{g}(y)\rangle = \begin{cases} R_{g\varepsilon}(x,y) & \text{for } y\in K_\varepsilon(x) \\ 0 & \text{for } y\notin K_\varepsilon(x), \end{cases}$$

$$\lim_{\varepsilon\downarrow o}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}R_{r\varepsilon}(x,x+y)dy = a_r(y), \quad \lim_{\varepsilon\downarrow o}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}R_{g\varepsilon}(x,x+y)dy = a_g(x),$$

then the random vector

$$\frac{1}{\sqrt{\varepsilon}}(u(x_1,\omega)-u_o(x_1),\dots,u(x_s,\omega)-u_o(x_s))^T \qquad (x_i \in [0,1]),$$

which has been constituted by the solution $u(x,\omega)$ of (15) and the solution of the averaged problem to (15), converges in the distribution to a Gaussian random vector

$$(\eta(x_1,\omega),\dots,\eta(x_s,\omega))^T$$

with $\eta(x,\omega)=\eta_g(x,\omega)+\sum_{r=0}^{m-1}\eta_r(x,\omega)$. Further on the random vectors

$$\tilde{\eta}_g=(\eta_g(x_1,\omega),\dots,\eta_g(x_s,\omega))^T, \quad \tilde{\eta}_r=(\eta_r(x_1,\omega),\dots,\eta_r(x_s,\omega))^T,$$

$r=0,1,\dots,m-1$, are independent Gaussian random vectors with

$$\langle\tilde{\eta}_g\rangle=0, \quad \langle\tilde{\eta}_g\tilde{\eta}_g^T\rangle=\Big(\int_0^1 a_g(z)G(x_i,z)G(x_j,z)dz\Big)_{1\le i,j\le s}$$

$$\langle\tilde{\eta}_r\rangle=0, \quad \langle\tilde{\eta}_r\tilde{\eta}_r^T\rangle=\Big(\int_0^1 a_r(z)\frac{\partial^r G(x_i,z)}{\partial z^r}\,\frac{\partial^r G(x_j,z)}{\partial z^r}\,(u_o^{(r)}(z))^2dz\Big)_{1\le i,j\le s}.$$

REMARK 1. This theorem 9 proves the convergence in the distribution of the processes $\frac{1}{\sqrt{\varepsilon}}(u(x,\omega)-u_o(x))$ which has been constituted by the solution $u(x,\omega)$ of (15) and the solution of the averaged problem to (15) $u_o(x)$ to the Gaussian process $\eta(x,\omega)$ with $\langle\eta(x,\omega)\rangle=0$ and

$$\langle\eta(x)\eta(y)\rangle = \int_0^1 a_g(z)G(x,z)G(y,z)dz + \sum_{r=0}^{m-1}\int_0^1 a_r(z)\frac{\partial^r G(x,z)}{\partial z^r}\,\frac{\partial^r G(y,z)}{\partial z^r}(u_o^{(r)}(z))^2dz.$$

PROOF. We put $\overset{n}{u}(x,\omega)\doteq\sum_{k=0}^{n}u_k(x,\omega)$. Then we see with similar considerations as in the proof of theorem 7 and with theorem 2

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\sqrt{\varepsilon}^p}\langle(\overset{n}{u}(x_{a_1})-u_o(x_{a_1}))\dots(\overset{n}{u}(x_{a_p})-u_o(x_{a_p}))\rangle = \begin{cases}\lim\limits_{\varepsilon\downarrow 0}\frac{1}{\sqrt{\varepsilon}^p}\langle u_1(x_{a_1})\dots u_1(x_{a_p})\rangle & \text{for } p \text{ even}\\ 0 & \text{for } p \text{ odd}\end{cases}$$

where $a_q\in\{1,\dots,s\}$. If we set $u_1(x,\omega)=c(x,\omega)+\sum_{r=0}^{m-1}b_r(x,\omega)$ (see (18)) with

$$c(x,\omega) = \int_0^1 G(x,y)\bar{g}(y,\omega)dy$$

$$b_r(x,\omega) = -\int_0^1 \bar{f}_r(y,\omega)u_o^{(r)}(y)\frac{\partial^r G(x,y)}{\partial y^r}dy$$

it follows as in the proof of the theorem 7

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\sqrt{\varepsilon}^p}\langle u_1(x_{a_1})\dots u_1(x_{a_p})\rangle = \begin{cases}\lim\limits_{\varepsilon\downarrow 0}A_\varepsilon & \text{for } p \text{ even}\\ 0 & \text{for } p \text{ odd}\end{cases}$$

where

$$\lim_{\varepsilon\downarrow 0} A_\varepsilon = \sum_{\substack{v_g=0\\ v_0^g=0\\ \vdots\\ v_{m-1}=0\\ v_g+\sum_{j=1}^{m-1} v_j=\frac{p}{2}}}^{p/2} \sum_{\substack{\text{all splittings}\\ \text{of } (1,\dots,n) \text{ in}\\ (i_1^g,\dots,i_{2v_g}^g)\\ (i_1^0,\dots,i_{2v_0}^0)\\ \vdots\\ (i_1^{m-1},\dots,i_{2v_{m-1}}^{m-1})}} A^g_{i_1^g\dots i_{2v_g}^g} A^0_{i_1^0\dots i_{2v_0}^0}\dots A^{m-1}_{i_1^{m-1}\dots i_{2v_{m-1}}^{m-1}}$$

and

$$A^g_{i_1^g\dots i_{2v_g}^g} = \sum_{(i_{j_1}^g, i_{j_2}^g),\dots,(i_{j_{2v_g-1}}^g, i_{2v_g}^g)} \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\langle c(x_{a_{i_{j_1}^g}})c(x_{a_{i_{j_2}^g}})\rangle\cdot\ldots\cdot$$

$$\cdot \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\langle c(x_{a_{i_{j_{2v_g-1}}^g}})c(x_{a_{i_{j_{2v_g}}^g}})\rangle$$

and appropriate for $A^r_{\dots 1}$ with b_r instead of c. Then the formulas

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\langle c(x)c(y)\rangle = \int_0^1 a_g(z)G(x,z)G(y,z)dz$$

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\langle b_r(x)b_r(y)\rangle = \int_0^1 a_r(z)\frac{\partial^r G(x,z)}{\partial z^r}\frac{\partial^r G(y,z)}{\partial z^r}(u_0^{(r)}(z))^2 dz$$

imply the statement of the theorem for $\frac{1}{\sqrt{\varepsilon}}(\overset{n}{u}(x_1)-u_0(x_1),\dots, \overset{n}{u}(x_s)-u_0(x_s))^T$.

The complete proof of this theorem follows as in theorem 7 from the uniform convergence relativ to ε

$$\lim_{n\to\infty} \frac{1}{\varepsilon}\langle(\overset{n}{u}(x,\omega) - u(x,\omega))^2\rangle = 0. \tag{20}$$

We consider $\frac{1}{\sqrt{\varepsilon}}\langle u_p(x)u_q(x)\rangle$ with $p,q \ge N_0$ to prove formula (20). Then we obtain with theorem 8 and the following estimations

$$\begin{aligned}\left|\frac{1}{\varepsilon}\langle u_p(x)u_q(x)\rangle\right| \le \frac{1}{\varepsilon}\sum_{r_1,\dots,r_{p+q}=0}^{m-1} c_0^2 c^{p+q}\int_0^1\cdots\int_0^1 \left|\left\langle\prod_{i=1}^{p+q} f_{r_i}(y_i)\right\rangle\right| \prod_{i=1}^{p+q} dy_i \\ + \frac{1}{\varepsilon}\sum_{r_3,\dots,r_{p+q}=0}^{m-1} c^{p+q}\int_0^1\cdots\int_0^1 |\langle\bar{g}(y_1)\bar{g}(y_2)\rangle\langle\prod_{i=3}^{p+q}\bar{f}_{r_i}(y_i)\rangle| \prod_{i=1}^{p+q} dy_i .\end{aligned} \tag{21}$$

Let $h(y,\omega)$ be a weakly correlated process with the correlation length ε, $|h(y,\omega)| < \eta$ almost everywhere and $l \ge 2$. Then we estimate an integral of the form

$$\int_0^1 \cdots \int_0^1 |\langle h(y_1)\ldots h(y_l)\rangle|\, dy_1 \ldots dy_l .$$

For $l^2 \geq 1/(2\varepsilon)$ we have

$$\int_0^1 \cdots \int_0^1 |\langle h(y_1)\ldots h(y_l)\rangle|\, dy_1 \ldots dy_l \leq \eta^l \leq 2\varepsilon l^2 \eta^l$$

and for $l^2 < 1/(2\varepsilon)$

$$\int_0^1 \cdots \int_0^1 |\langle h(y_1)\ldots h(y_l)\rangle|\, dy_1 \ldots dy_l \leq \eta^l (1-V_l).$$

V_l designes the volume of the points $(y_1,\ldots,y_l)$ of $[0,1]^l$ for which the ε-maximum adjoining splitting is $\{(y_1),(y_2),\ldots,(y_l)\}$. Since

$$V_l = \int_0^1 dy_1 \int_{|y_1-y_2|>\varepsilon} dy_2 \ldots \int_{\substack{|y_1-y_l|>\varepsilon \\ \vdots \\ |y_{l-1}-y_l|>\varepsilon}} dy_l \geq \prod_{t=1}^{l-1}(1-2\varepsilon t) \geq 1-2\varepsilon l^2$$

and we obtain for all $l \geq 2$

$$\int_0^1 \cdots \int_0^1 |\langle h(y_1)\ldots h(y_l)\rangle|\, dy_1 \ldots dy_l \leq 2\varepsilon l^2 \eta^l . \tag{22}$$

Under the condition $N_0 \geq \frac{m}{2}+3$ a factor of the form as in (22) with $h(x,\omega) \in \{\bar f_0(x,\omega),\ldots,\bar f_{m-1}(x,\omega)\}$ is included in every item of (21) because the processes $\bar f_r(x,\omega)$, $\bar g(x,\omega)$ are independent. Hence it follows form (21)

$$\left|\frac{1}{\varepsilon}\langle u_p(x)u_q(x)\rangle\right| \leq 2(m\eta C)^{p+q-2}C^2(p+q)^2\left[m^2C_0^2\eta^2+\int_0^1\int_0^1 |\langle \bar g(y_1)\bar g(y_2)\rangle|\, dy_1 dy_2\right. .$$

The uniform convergence from (20) we obtain through the convergence of the series $\sum_{p,q=1}^{\infty}(p+q)^2(m\eta C)^{p+q}$ for $|m\eta C| < 1$. The theorem 9 is proved.

REMARK 2. Let $\bar f_r(x,\omega)\equiv 0$ for $r=0,1,\ldots,m-1$, i.e. let L a deterministic operator, then the statement of theorem 9 is right without someone ristriction of the weakly correlated process $\bar g(x,\omega)$ related to the smallness. This follows from the proof of theorem 9 or through a direct use of the theorems 1 and 3 on the solution of this boundary value problem

$$u(x,\omega) = u_0(x) + \int_0^1 G(x,y)\bar g(y,\omega)dy \quad \text{with } u_0(x) = \int_0^1 G(x,y)\langle g(y)\rangle dy.$$

Hence we obtain in this case that $\frac{1}{\sqrt{\varepsilon}}(u(x,\omega)-u_o(x))$ converges in the distribution to a Gaussian process $\eta(x,\omega)$ with

$$\langle\eta(x)\rangle = 0, \quad \langle\eta(x)\eta(y)\rangle = \int_0^1 a_g(z)G(x,z)G(y,z)dz.$$

W.E. Boyce deals with this case of a stochastic boundary value problem in [1] and he showed that $\frac{1}{\sqrt{\varepsilon}}(u(x,\omega)-u_o(x))$ is in the limit a Gaussian random variable $\eta(x)$ for any $x \in [0,1]$.

REMARK 3. Let $w_i(x)$ be the eigenfunctions of the averaged operator $\langle L\rangle$ with $\int_0^1 w_i(x)w_j(x)dx=\delta_{ij}$ and μ_i the eigenvalues of $\langle L\rangle$. Then we can calculate the correlation function of the limit process $\eta(x,\omega)$ of $\frac{1}{\sqrt{\varepsilon}}(u(x,\omega)-u_o(x))$ through

$$\langle\eta(x)\eta(y)\rangle = \sum_{p,q=1}^{\infty}\{b_{pq} + \sum_{r=o}^{m-1} c^r_{pq}\}\frac{w_p(x)w_q(y)}{\mu_p\mu_q}$$

with

$$b_{pq} \doteq \int_0^1 a_g(z)w_p(z)w_q(z)dz \qquad \text{and}$$

$$c^r_{pq} \doteq \int_0^1 a_r(z)(u_o^{(r)}(z))^2 w_p^{(r)}(z)w_q^{(r)}(z)dz.$$

Specially we have for a wide-sense stationary, weakly correlated process

$$\langle\eta(x)\eta(y)\rangle = a_g\sum_{p=1}^{\infty}\frac{1}{\mu_p^2}w_p(x)w_p(y) + \sum_{r=o}^{m-1}a_r\sum_{p,q=1}^{\infty}\frac{\bar{c}^r_{pq}}{\mu_p\mu_q}w_p(x)w_q(y)$$

with $\bar{c}^r_{pq} \doteq \int_0^1 (u_o^{(r)}(z))^2 w_p^{(r)}(z)w_q^{(r)}(z)dz.$

These statements can be proved easly with $G(x,y)=\sum_{p=1}^{\infty}\frac{w_p(x)w_p(y)}{\mu_p}$ (see also [7]).

Now we regard the boundary value problem (15) with the conditions given above. It denotes $\{\varphi_i\}_{1\leq i<\infty}$ a system of functions of the energetic space $H_{\langle L\rangle}$. This system be complete in $H_{\langle L\rangle}$. The solution $u(x,\omega)$ of the equation $L(\omega)u=g(\omega)$ is the minimum of the energetic functional $(Lu,u)-2(g,u)$ in the energetic space. The Ritz-method with the co-ordinate functions $\{\varphi_i\}$ conducts to the following system

$$\sum_{k=1}^{n}(L(\omega)\varphi_k,\varphi_j)x_k^{(n)} = (g,\varphi_j), \quad j=1,2,\dots,n. \tag{23}$$

It is $u_n(x,\omega)=\sum_{k=1}^{n} x_k^{(n)}(\omega)\varphi_k(x)$ the n-th Ritz-approximation of the solution $u(x,\omega)$ of $L(\omega)u=g(\omega)$. We can write the formula (23) in the following form:

$$(\underline{A}_o + \underline{B}(\omega))\underline{x}^{(n)} = \underline{b}_o + \underline{c}(\omega)$$

with $\underline{A}_o = ((\langle L\rangle\varphi_k,\varphi_j))_{1\le k,j\le n}$, $\quad \underline{b}_o = ((\langle g\rangle,\varphi_j))^T_{1\le j\le n}$,

$\underline{B}(\omega) = ((L_1(\omega)\varphi_k,\varphi_j))_{1\le k,j\le n}$, $\quad \underline{c}(\omega) = ((\bar{g},\varphi_j))^T_{1\le j\le n}$.

The matrix $\underline{A}_o$ is regular through the conditions for $\{\varphi_j\}$: $\underline{A}_o^{-1}=(\alpha_{ij})_{1\le i,j\le n}$. Let $\underline{x}_o^{(n)}=\underline{A}_o^{-1}\underline{b}_o=(x_{o1}^{(n)},\dots,x_{on}^{(n)})^T$ be then we obtain for the n-th Ritz-approximation $u_n(x,\omega)$

$$u_n(x,\omega) = \sum_{k=1}^{n} x_{ok}^{(n)}\varphi_k(x) + \sum_{k=1}^{n}(\underline{A}_o^{-1}(-\underline{B}\underline{x}_o^{(n)}+\underline{c}))_k\varphi_k(x)$$

$$+ \text{ terms of higher order than first in } b_{ij},\ c_j$$

respectively with $u_{on}(x) \doteq \sum_{k=1}^{n} x_{ok}^{(n)}\varphi_k(x)$

$$u_n(x,\omega) - u_{on}(x) = (\bar{g}(z),\sum_{k,j=1}^{n}\alpha_{kj}\varphi_j(z)\varphi_k(x))_z$$

$$-\sum_{r=o}^{m-1}(\bar{f}_r(z),\sum_{k,j=1}^{n}\alpha_{kj}\varphi_j^{(r)}(z)\varphi_k(x)\sum_{l=1}^{n}x_{ol}^{(n)}\varphi_l^{(r)}(z))_z$$

$$+ \text{ terms of higher order than first in } b_{ij},\ c_j$$

$((h_1(z,x),h_2(z,x))_z \doteq \int_0^1 h_1(z,x)h_2(z,x)dz)$. From theorem 5 follows that $\frac{1}{\sqrt{\varepsilon}}(u_n(x,\omega)-u_{on}(x))$ converges in the distribution as $\varepsilon\downarrow 0$ to the Gaussian process $\eta_n(x,\omega)$ with $\langle\eta_n(x)\rangle = 0$ and

$$\langle\eta_n(x)\eta_n(y)\rangle = \sum_{k,j,p,q=1}^{n}\alpha_{kj}\alpha_{pq}\varphi_k(x)\varphi_p(y)(a_g\varphi_j,\varphi_q)$$
$$+\sum_{r=o}^{m-1}\sum_{k,j,p,q=1}^{n}\sum_{l,i=1}^{n}\alpha_{kj}\alpha_{pq}x_{ol}^{(n)}x_{oi}^{(n)}\varphi_k(x)\varphi_p(y)(a_r\varphi_j^{(r)}\varphi_q^{(r)},\varphi_l^{(r)}\varphi_i^{(r)}). \tag{24}$$

Particularly we obtain for $\varphi_i(x)=w_i(x)$ (see remark 3) and wide-sense stationary, weakly correlated processes

$$\langle\eta_n(x)\eta_n(y)\rangle = a_g\sum_{k=1}^{n}\frac{1}{\mu_k^2}w_k(x)w_k(y) +$$
$$+\sum_{r=o}^{m-1}a_r\sum_{k,l=1}^{n}\frac{1}{\mu_k\mu_l}((u_{on}^{(r)})^2,w_k^{(r)}w_l^{(r)})w_k(x)w_l(y)$$

an approximation of the correlation function how it has been represented in the remark 3. The formula is used by an approximation of the correlation function $\langle\eta(x)\eta(y)\rangle$ and (24) is suitable for a explicit calculation of $\langle\eta(x)\eta(y)\rangle$ if the Green function is calculable difficult.

4.2. EXAMPLE. Let $g(x,\omega)$ be a wide-sense stationary, weakly correlated process then the simple boundary value problem

$$-u'' + bu = g(x,\omega), \quad u(0) = u(1) = 0$$

with b=const. possesses the averaged problem

$$-w'' + bw = 0, \quad w(0) = w(1) = 0.$$

This problem has only the trivial solution when $b \neq -(k\pi)^2$ is and the Green function $G_b(x,y)$ is

$$G_b(x,y) = \begin{cases} \dfrac{sh(\beta x)sh(\beta(1-y))}{sh(\beta)} & \text{for } 0 \leq x < y \leq 1 \\ \dfrac{sh(\beta y)sh(\beta(1-x))}{sh(\beta)} & \text{for } 0 \leq y < x \leq 1 \end{cases} \qquad (\beta \doteq \sqrt{b}).$$

Hence we obtain the correlation function for the limit process $\eta_b(x,\omega)$ from theorem 9

$$\langle\eta_b(x)\eta_b(y)\rangle = a \int_0^1 G_b(x,z)G_b(y,z)dz$$

and the variance

$$\langle\eta_b^2(x)\rangle = a \int_0^1 G_b^2(x,z)dz = \frac{a}{\beta^2 sh^2(\beta)}(f_\beta(x) + f_\beta(1-x))$$

with $f_\beta(x) = sh^2(\beta(1-x))(\frac{1}{4\beta}sh(2\beta x) - \frac{1}{2}x)$. a is the parameter which belongs to the wide-sense stationary, weakly correlated process $g(x,\omega)$. It is $\beta = i\sqrt{-b}$ for $b < 0$ and

$$\langle\eta_b^2(x)\rangle = \frac{a}{\bar\beta^2 \sin^2(\bar\beta)}(\bar f_{\bar\beta}(x) + \bar f_{\bar\beta}(1-x)) \qquad (\bar\beta \doteq \sqrt{-b})$$

with $\bar f_{\bar\beta}(x) = -\sin(\bar\beta(1-x))(\frac{1}{4\bar\beta}\sin(2\bar\beta x) - \frac{1}{2}x)$. Expecially we have

$$\lim_{b\to 0} \langle\eta_b^2(x)\rangle = \frac{1}{3}x^2(1-x)^2 \quad \text{and} \quad \lim_{b\to-(k\pi)^2} \langle\eta_b^2(x)\rangle = \infty .$$

In the following Fig.4 the variance of the limit process $\eta_b(x,\omega)$ is plotted for some values of the parameter b. Since $\langle\eta_b^2(x)\rangle = \langle\eta_b^2(1-x)\rangle$, we have plotted this function for $0 \le x \le \frac{1}{2}$ only. The following values submit for $b=\pi^2-\frac{1}{2}$ by contrast to it:

x	0	0,1	0,2	0,3	0,4	0,5
$\frac{1}{a}\langle\eta_b^2(x)\rangle$	0	0,7650	2,7663	5,2382	7,2570	8,0004

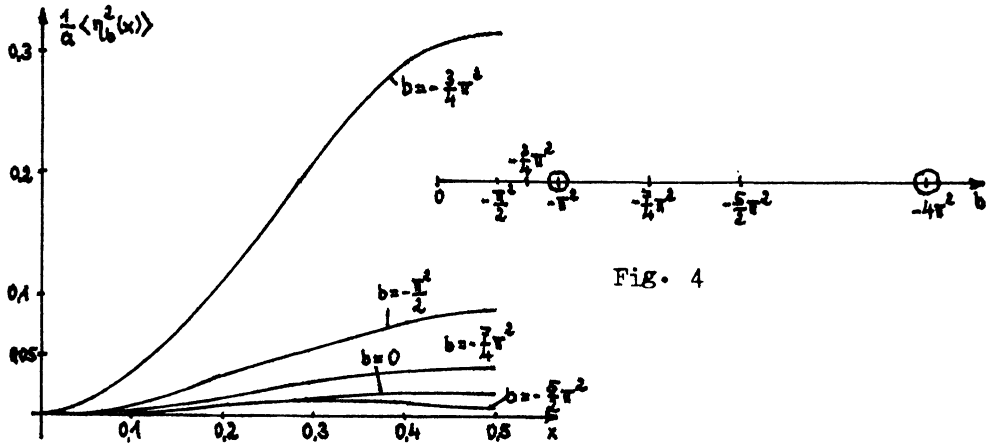


Fig. 4

5. STOCHASTIC INITIAL VALUE PROBLEMS

We consider a system of ordinary differential equations of the first order

$$\frac{d\underline{x}}{dt} = \underline{A}(t)\underline{x} + \underline{f}(t,\omega) \tag{26}$$

with the initial condition $\underline{x}(t_o)=\underline{x}_o$. $\underline{A}(t)$ is a nxn deterministic matrix, $\underline{A}(t)=(a_{ij}(t))_{1\le i,j\le n}$, $\underline{x}$ is the vector $\underline{x}(t)=(x_i(t))^T_{1\le i\le n}$, $\underline{x}_o=(\overset{o}{x}_i)^T_{1\le i\le n}$ and $\underline{f}(t,\omega)=(f_i(t,\omega))^T_{1\le i\le n}$ is a stochastic vector process. Let $f_i(t,\omega)$ be processes of which a.s. the trajectories are continuous and $\underline{Q}(t,t_o)$ be the principal matrix associated with $\underline{A}(t)$ (i.e. $\underline{Q}(t_o,t_o)=\underline{I}$ ($\underline{I}$ is the identity matrix) and $\frac{d}{dt}\underline{Q}(t,t_o)=\underline{A}(t)\underline{Q}(t,t_o)$), then the unique solution a.s. of the initial value problem (26) may be written in the form

$$\underline{x}(t,\omega) = \underline{Q}(t,t_o)(\underline{x}_o + \int_{t_o}^{t} \underline{Q}^{-1}(s,t_o)\underline{f}(s,\omega)ds).$$

The integral is defined by the integral of sample functions. In generally we cannot calculate the distribution of the solution $\underline{x}(t,\omega)$ if we know the distribution of $\underline{f}(t,\omega)$. Let $\underline{f}(t,\omega)$ be a Gaussian vector process then is $\int_{t_o}^{t} \underline{Q}^{-1}(s,t_o)\underline{f}(s,\omega)ds$ also a Gaussian vector process and in the same way the solution $\underline{x}(t,\omega)$ of this system of linear differential equations (see /2/). The first moments of the solution are

$$\langle\underline{x}(t)\rangle = \underline{Q}(t,t_o)(\underline{x}_o + \int_{t_o}^{t} \underline{Q}^{-1}(s,t_o)\langle\underline{f}(s)\rangle ds) \quad \text{and}$$

$$\underline{R}(t_1,t_2) \doteq \langle[\underline{x}(t_1)-\langle\underline{x}(t_1)\rangle][\underline{x}(t_2)-\langle\underline{x}(t_2)\rangle]^T\rangle$$
$$= \underline{Q}(t_1,t_o)\int_{t_o}^{t} \underline{Q}^{-1}(s_1,t_o)\underline{K}(s_1,s_2)\underline{Q}^{-T}(s_2,t_o)ds_1 ds_2 \underline{Q}^T(t_2,t_o)$$

with

$$\underline{K}(t_1,t_2) \doteq \langle[\underline{f}(t_1)-\langle\underline{f}(t_1)\rangle][\underline{f}(t_2)-\langle\underline{f}(t_2)\rangle]^T\rangle,$$

provided that

$$\int_{t_o}^{t_1}\langle\|\underline{f}(s)\|\rangle ds < \infty \quad , \qquad \int_{t_o}^{t_1}\int_{t_o}^{t_2}\langle\|\underline{f}(s_1)\underline{f}^T(s_2)\|\rangle ds_1 ds_2 < \infty$$

for all $t_1,t_2 \geq t_o$ ($\|.\|$ denotes the Euklidian norm of R^n).

In the following we consider the solution (26) if $\underline{f}(t,\omega) = \frac{1}{\sqrt{\varepsilon}}\underline{g}_\varepsilon(t,\omega)$ denotes a vector process with independent, weakly correlated processes $g_{i\varepsilon}(t,\omega)$ as components. This leads to the following theorem.

THEOREM 10. Let $\underline{g}_\varepsilon(t,\omega)$ a sequence of weakly correlated vector processes for $\varepsilon \downarrow 0$ with independent components $g_{i\varepsilon}(t,\omega)$ and

$$\langle g_{i\varepsilon}(t)g_{i\varepsilon}(s)\rangle = \begin{cases} R_{i\varepsilon}(t,s) & \text{for } s \in K_\varepsilon(t) \\ 0 & \text{for } s \notin K_\varepsilon(t) \end{cases}, \quad \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} R_{i\varepsilon}(t,t+s)ds = a_i(t),$$

then the solution

$$\underline{x}_\varepsilon(t,\omega) = \underline{y}(t) + \frac{1}{\sqrt{\varepsilon}}\underline{Q}(t,t_o) \int_{t_o}^{t} \underline{Q}^{-1}(s,t_o)\underline{g}_\varepsilon(s,\omega)ds$$

with

$$\underline{y}(t) = \underline{Q}(t,t_o)(\underline{x}_o + \int_{t_o}^{t} \underline{Q}^{-1}(s,t_o)\underline{h}(s)ds)$$

of the initial value problem

$$\frac{dx}{dt} = \underline{A}(t)\underline{x} + \underline{h}(t) + \frac{1}{\sqrt{\varepsilon}}g_\varepsilon(t,\omega), \qquad \underline{x}(t_o,\omega) = \underline{x}_o \tag{27}$$

converges in the distribution to a Gaussian vector process

$$\eta(t,\omega) = \underline{y}(t) + \xi(t,\omega)$$

with $\langle \xi(t)\rangle = 0$ and

$$\langle \xi(t_1)\xi(t_2)^T\rangle = \sum_{k=1}^{n} \int_{t_o}^{\min(t_1,t_2)} (a_k(s)q_{ik}(t_1,t_o;s)q_{jk}(t_2,t_o;s))_{1\le i,j\le n}ds.$$

It is $(q_{ij}(t,t_o;s))_{1\le i,j\le n} \doteq \underline{Q}(t,t_o)\underline{Q}^{-1}(s,t_o)$.

PROOF. The proof is following from theorem 5 then we obtain with the definition of the q_{ij}

$$\underline{x}_\varepsilon(t,\omega) = \underline{y}(t) + \frac{1}{\sqrt{\varepsilon}}\sum_{k=1}^{n} \int_{t_o}^{t} (q_{ik}(t,t_o;s)g_{k\varepsilon}(s,\omega))_{1\le i\le n}ds$$

and from this for the limit process $\xi(t,\omega)=(\xi_i(t,\omega))_{1\le i\le n}$ with the notations of theorem 5

$$\langle \xi_i(t_1)\xi_j(t_2)\rangle = \sum_{k=1}^{n}\langle \xi_{ik}(t_1)\xi_{jk}(t_2)\rangle$$
$$= \sum_{k=1}^{n} \int_{t_o}^{\min(t_1,t_2)} a_k(s)q_{ik}(t_1,t_o;s)q_{jk}(t_2,t_o;s)ds.$$

In generally $\xi(t,\omega)$ does not denote a wide-sense stationary vector process while $\langle \xi(t_1)\xi^T(t_2)\rangle$ is not a function which only depends on t_2-t_1. We assume that the processes $g_{i\varepsilon}(t,\omega)$ are wide-sense stationary and weakly correlated and $\underline{A}(t)=\underline{A}$ is a constant matrix, the eigenvalues λ_i of which have negative real parts. Then we obtain for $t \ge T > t_o$ the formula $Q(t,t_o)\underline{x}_o = e^{\underline{A}(t-t_o)}\underline{x}_o \approx 0$ and in the same way the solutions $\int_{t_o}^{t} e^{\underline{A}(t-s)}(\underline{h}(s)+\frac{1}{\sqrt{\varepsilon}}g_\varepsilon(s))ds$ and $\int_{-\infty}^{t} e^{\underline{A}(t-s)}(\underline{h}(s)+\frac{1}{\sqrt{\varepsilon}}g_\varepsilon(s))ds$ of the system of differential equations differ by a solution of the homogeneous system of (27) which is near by zero for $t \ge T > t_o$. Hence the solution of (27) is described by

$$\bar{\underline{x}}_\varepsilon(t,\omega) = \int_{-\infty}^{t} e^{\underline{A}(t-s)}\underline{h}(s)ds + \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{t} e^{\underline{A}(t-s)}g_\varepsilon(s,\omega)ds.$$

Then the term $\frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{t} e^{\underline{A}(t-s)} g_\varepsilon(s,\omega)ds$ is wide-sense stationary as we obtain from

$$\langle \int_{-\infty}^{t} e^{\underline{A}(t-s)} g_\varepsilon(s,\omega)ds\rangle = 0$$

and
$$\langle \int_{-\infty}^{t_1} e^{\underline{A}(t_1-s)} g_\varepsilon(s,\omega)ds(\int_{-\infty}^{t_2} e^{\underline{A}(t_2-s)} g_\varepsilon(s,\omega)ds)^T\rangle$$
$$= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} e^{\underline{A}(t_1-s_1)}\langle g_\varepsilon(s_1)g_\varepsilon(s_2)^T\rangle e^{\underline{A}^T(t_2-s_2)}ds_1 ds_2$$
$$= \int_0^{\infty}\int_0^{\infty} e^{\underline{A}s_1}\langle g_\varepsilon(t_1-s_1)g_\varepsilon(t_2-s_2)^T\rangle e^{\underline{A}^T s_2}\,ds_1 ds_2.$$

The theorem 11 can be proved like the theorem 10.

THEOREM 11. Let $g_\varepsilon(t,\omega)$ be a sequence of wide-sense stationary, weakly correlated vector processes for $\varepsilon\downarrow 0$ with independent components and

$$\langle g_{i\varepsilon}(t)g_{i\varepsilon}(s)\rangle = \begin{cases}\sigma_{2i}\varrho_{i\varepsilon}(t-s) & \text{for } |t-s|\leq\varepsilon\\ 0 & \text{for } |t-s|>\varepsilon\end{cases},\quad \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\varrho_{i\varepsilon}(s)ds = a_i < \infty$$

$(a_i \neq 0)$, then the solution

$$\underline{x}_\varepsilon(t,\omega) = \int_{-\infty}^{t} e^{\underline{A}(t-s)}\underline{h}(s)ds + \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{t} e^{\underline{A}(t-s)} g_\varepsilon(s,\omega)ds$$

of the system of differential equations

$$\frac{dx}{dt} = \underline{Ax} + \underline{h}(t) + \frac{1}{\sqrt{\varepsilon}}g_\varepsilon(t,\omega)$$

converges in the distribution to a Gaussian vector process

$$\underline{\eta}(t,\omega) = \int_{-\infty}^{t} e^{\underline{A}(t-s)}\underline{h}(s)ds + \underline{\zeta}(t,\omega).$$

$\underline{\zeta}(t,\omega)$ is a wide-sense stationary, weakly correlated vector process with $\langle\underline{\zeta}(t)\rangle = 0$ and

$$\langle\underline{\zeta}(t_1)\underline{\zeta}(t_2)^T\rangle = \left(\sum_{k=1}^{n} a_k\sigma_{2k}\int_{-\infty}^{\min(t_1,t_2)}(e^{\underline{A}(t_1-s)})_{ik}(e^{\underline{A}(t_2-s)})_{jk}ds\right)_{1\leq i,j\leq n}.$$

Indeed $\underline{\zeta}(t,\omega)$ is a wide-sense stationary vector process because we obtain for $t_1 \leq t_2$

$$\int_{-\infty}^{\min(t_1,t_2)}(e^{\underline{A}(t_1-s)})_{ik}(e^{\underline{A}(t_2-s)})_{jk}ds = \int_0^{\infty}(e^{\underline{A}s})_{ik}(e^{\underline{A}(t_2-t_1+s)})_{jk}ds.$$

In the following we describe the connection with the Itô differential equations. The Itô differential equation

$$d\underline{X}_t = (\underline{A}(t)\underline{X}_t + \underline{h}(t))dt + \underline{G}(t)d\underline{W}_t, \qquad \underline{X}_{t_o}(\omega)=\underline{x}_o \tag{28}$$

corresponds to the initial value problem (27) as $\varepsilon\downarrow 0$ where $\underline{G}(t)=(\delta_{ij}\sqrt{a_i(t)})_{1\le i,j\le n}$ and $\underline{W}_t=(W_t^i)_{1\le i\le n}$ is the Wiener process with independent components. This statement is following from the correspondence of the integral equation

$$\underline{x}(t) = \underline{x}_o + \int_{t_o}^{t} (\underline{A}(s)\underline{x}(s) + \underline{h}(s))ds + \frac{1}{\sqrt{\varepsilon}}\int_{t_o}^{t} g_\varepsilon(s,\omega)ds$$

with (27) and of the Itô integral equation

$$\underline{X}_t(\omega) = \underline{x}_o + \int_{t_o}^{t} (\underline{A}(s)\underline{X}_s(\omega) + \underline{h}(s))ds + \int_{t_o}^{t} \underline{G}(s)d\underline{W}_s \tag{29}$$

with (28) if we take into consideration the convergence in the distribution from theorem 5

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\sqrt{\varepsilon}}\int_{t_o}^{t} g_\varepsilon(s,\omega)ds = \int_{t_o}^{t} \underline{G}(s)d\underline{W}_s.$$

We note the relation

$$\left\langle \int_{t_o}^{t_1} \underline{G}(s)d\underline{W}_s\left(\int_{t_o}^{t_2}\underline{G}(s)d\underline{W}_s\right)^T\right\rangle = \int_{t_o}^{\min(t_1,t_2)} \underline{G}(s)\underline{G}(s)^T ds$$

for the Itô integral $\int_{t_o}^{t} \underline{G}(s)d\underline{W}_s$. It is following from the theory of the Itô differential equations that the solution of (28) (the solution of the integral equation (28)) is a Gaussian vector process with $\langle\underline{X}_t\rangle = \underline{Q}(t,t_o)(\underline{x}_o + \int_{t_o}^{t} \underline{Q}^{-1}(s,t_o)\underline{h}(s)ds)$

and $\langle(\underline{X}_{t_1}-\langle\underline{X}_{t_1}\rangle)(\underline{X}_{t_2}-\langle\underline{X}_{t_2}\rangle)^T\rangle =$

$$= \underline{Q}(t_1,t_o)\int_{t_o}^{\min(t_1,t_2)} \underline{Q}^{-1}(s,t_o)\underline{G}(s)\underline{G}(s)^T\underline{Q}^{-T}(s,t_o)ds\,\underline{Q}^T(t_2,t_o).$$

A comparison of the limit solution $\eta(t,\omega)$ from theorem 10 with this solution from (28) shows the correspondence of the solution processes: $\underline{X}_t(\omega)=\eta(t,\omega)$. Hence the same solutions of the initial value problem (27) is obtained if one takes up the limit value $\varepsilon\downarrow 0$ in the equation and the Itô differential equation solves which one obtains and if one solves (27) in the sample functions and takes up the limit value $\varepsilon\downarrow 0$ in the solution.

We can make many applications (e.g. the number of threshold crossings and other) because the limit process $\underline{\eta}(t,\omega)$ or $\bar{\underline{\eta}}(t,\omega)$, respectively, is a Gaussian process. Hence we obtain approximately a good general view of the behaviour of the solution $\underline{x}_\varepsilon(t,\omega)$ or $\bar{\underline{x}}_\varepsilon(t,\omega)$, respectively, if weakly correlated processes come in the differential equation.

REFERENCES

1. Boyce, W. E. Stochastic nonhomogeneous Sturm-Liouville problem, J.Franklin Inst. 282 (1966) 206-215.

2. Bunke, H. Gewöhnliche Differentialgleichungen mit zufälligen Parametern, Akademie-Verlag, Berlin, 1972.

3. Purkert, W. and vom Scheidt, J. Zum Mittelungsproblem bei stochastischen Eigenwertaufgaben, Beiträge zur Analysis 10 (1977) 47-61.

4. Purkert, W. and vom Scheidt, J. Zur approximativen Lösung des Mittelungsproblems für die Eigenwerte stochastischer Differentialoperatoren, ZAMM 57 (1977) 515-526.

5. Purkert, W. and vom Scheidt, J. Stochastic eigenvalue problems for differential equations, Reports on mathem. physics, Torun (in print).

6. Purkert, W. and vom Scheidt, J. Ein Grenzverteilungssatz für stochastische Eigenwertprobleme, ZAMM (in print).

7. Purkert, W. and vom Scheidt, J. Randwertprobleme mit schwach korrelierten Prozessen als Koeffizienten, Transactions of the 8th Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, Prag (1978) 107-118.

8. Uhlenbeck, G. E. and Ornstein, L. S. On the theory of Brownian motion, Physical review 36 (1930) 823-841.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