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FLETT'S MEAN VALUE THEOREM IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT. We prove some generalizations of Flett's mean value theorem for a class of Gateaux differentiable functions $f : X \rightarrow Y$, where X and Y are topological vector spaces.

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1. Introduction. In 1958, Flett proved that if $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and satisfies $f'(a) = f'(b)$, then there exists η in the open interval (a, b) such that $f(\eta) - f(a) = (\eta - a)f'(\eta)$ [3]. Flett's conclusion implies that the tangent at $(\eta, f(\eta))$ passes through the point $(a, f(a))$. A recent article by Khan [4], which generalizes the Mean Value Theorem to the context of topological vector spaces, stimulated us to see if there was a similar generalization of Flett's theorem. It turns out that such a generalization does exist. However, even more can be done.

This article focuses on three distinct generalizations of Flett's theorem. First, we drop the endpoint condition $f'(a) = f'(b)$. Second, we consider the case where f is not differentiable at a finite number of points. Third, we drop differentiability. As expected, the conclusion at each step is weaker than the previous conclusion. These generalizations are given in [Theorem 2.1](#). We then place these results in a topological vector space setting replacing ordinary differentiability with Gateaux differentiability.

2. Results. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. Following [2] we will say that *the graph of f intersects its chord in the extended sense* if either there is a number $c \in (a, b)$ such that

$$(f(c) - f(a))(b - a) = (c - a)(f(b) - f(a)) \quad (2.1)$$

or

$$\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}. \quad (2.2)$$

We now state some interesting generalizations of Flett's theorem.

THEOREM 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. Let $J = \{x \in [a, b] : f \text{ is not differentiable at } x\}$ and set $j = |J|$. For each $x \in (a, b] \setminus J$, let*

$$F\ell(x) = \frac{1}{(x - a)^2} [(x - a)f'(x) - (f(x) - f(a))]. \quad (2.3)$$

(1) If $j = 0$, then there exists a point $\eta \in (a, b)$ such that

$$F\ell(\eta) = \frac{1}{2} \frac{f'(b) - f'(a)}{b - a}. \quad (2.4)$$

(2) If $j \leq n$ for some nonnegative integer n and $a \notin J$, then there exist $n + 1$ points $\eta_1, \eta_2, \dots, \eta_{n+1} \in (a, b)$ and $n + 1$ positive numbers $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ such that

$$\begin{aligned} \alpha_1 + \alpha_2 + \dots + \alpha_{n+1} &= 1, \\ \sum_{i=1}^{n+1} \alpha_i F\ell(\eta_i) &= \frac{1}{(b-a)^2} [(f(b) - f(a)) - (b-a)f'(a)]. \end{aligned} \quad (2.5)$$

(3) If j is unbounded and the graph of f intersects its chord in the extended case, then there exist c in (a, b) , and two positive numbers δ_1, δ_2 such that either

$$F\ell_1(c, h) \leq 0 \leq F\ell_2(c, k) \quad (2.6)$$

or

$$F\ell_2(c, k) \leq 0 \leq F\ell_1(c, h) \quad (2.7)$$

holds for $0 < h \leq \delta_1$ and $0 < k \leq \delta_2$ where

$$\begin{aligned} F\ell_1(c, h) &= \left[(c-a) \frac{f(c) - f(c-h)}{h} - (f(c) - f(a)) \right], \\ F\ell_2(c, k) &= \left[(c-a) \frac{f(c+k) - f(c)}{k} - (f(c) - f(a)) \right]. \end{aligned} \quad (2.8)$$

Some remarks are in order. Flett's original result is the case where $F\ell(\eta) = 0$. In item (2) we note that if $f'(a) = (f(b) - f(a))/(b - a)$, that is, the second condition for the graph of f intersecting its chord in the extended sense holds, then $\sum_{i=1}^{n+1} \alpha_i F\ell(\eta_i) = 0$. If, in item (3), f is differentiable at c , then

$$\lim_{h \rightarrow 0^+} \frac{F\ell_1(c, h)}{(c-a)^2} = \lim_{k \rightarrow 0^+} \frac{F\ell_2(c, k)}{(c-a)^2} = F\ell(c). \quad (2.9)$$

A proof of item (1) can be found in [1] and a proof of item (3) can be found in [2]. In order to prove item (2) we need the following lemma.

LEMMA 2.2. *Let $g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on the open interval (a, b) except possibly at a finite number, n , of points. Then there exist $n + 1$ points $\eta_1, \eta_2, \dots, \eta_{n+1} \in (a, b)$ and $n + 1$ positive numbers $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ such that*

$$\begin{aligned} \alpha_1 + \alpha_2 + \dots + \alpha_{n+1} &= 1, \\ g(b) - g(a) &= (b-a) \sum_{i=1}^{n+1} \alpha_i g'(\eta_i). \end{aligned} \quad (2.10)$$

Notice that [Lemma 2.2](#) is a generalization of the mean value theorem. A proof of [Lemma 2.2](#) can be found in [5].

PROOF OF (2). Consider the function $g : [a, b] \rightarrow \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{f(x) - f(a)}{x - a} & \text{if } x \in (a, b], \\ f'(a) & \text{if } x = a. \end{cases} \tag{2.11}$$

Note that g is continuous on $[a, b]$ and satisfies the hypotheses of [Lemma 2.2](#). Further

$$g'(x) = -\frac{f(x) - f(a)}{(x - a)^2} + \frac{f'(x)}{x - a}, \tag{2.12}$$

where f' exists; this implies

$$g'(x) = -\frac{g(x)}{x - a} + \frac{f'(x)}{x - a}. \tag{2.13}$$

Applying [Lemma 2.2](#), we get

$$g(b) - g(a) = (b - a) \sum_{i=1}^{n+1} \alpha_i g'(\eta_i). \tag{2.14}$$

After simplifying the previous expression the result follows. □

Our main goal is to place [Theorem 2.1](#) in the context of topological vector spaces. However, an exact analogue of [Theorem 2.1](#) for vector-valued functions is not true. For example, to see why item (1) fails consider the function $f : [0, 2\pi] \rightarrow \mathbb{R}^2$ defined by

$$f(x) = (\cos(x), \sin(x) - x) \tag{2.15}$$

for all $x \in [0, 2\pi]$. Then

$$f'(x) = (-\sin(x), \cos(x) - 1). \tag{2.16}$$

Therefore, we have $f'(0) = (0, 0) = f'(2\pi)$, that is, the derivatives of f at the endpoints of the closed interval $[0, 2\pi]$ are equal. Nevertheless, it is not hard to show that the equation

$$f(\eta) - f(0) = \eta f'(\eta) \tag{2.17}$$

has no solution in $(0, 2\pi)$.

In the sequel, we will let X and Y be Hausdorff topological vector spaces over the field \mathbb{R} of real numbers, and $A \subset X$ be an open set. Furthermore, we assume that Y has a continuous dual Y^* . A function $f : A \rightarrow Y$ is said to be *Gateaux differentiable* at $x_0 \in A$ if there exists a mapping from X into Y , denoted by $f'(x_0)$, such that, given any $z \in X$ and a balanced neighborhood V of 0 in Y , there exists a $\delta > 0$ satisfying

$$\frac{f(x_0 + tz) - f(x_0)}{t} - f'(x_0)(z) \in V \tag{2.18}$$

whenever $0 < |t| < \delta$; $f'(x_0)$ is called the Gateaux derivative of f at x_0 and we write

$$f'(x_0)(z) = \lim_{t \rightarrow 0} \frac{f(x_0 + tz) - f(x_0)}{t}, \quad z \in X. \tag{2.19}$$

Note that a Gateaux differentiable function need not be continuous. For example, the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, given by

$$f(z) = \begin{cases} \frac{uv^2}{u^2 + v^4} & \text{if } z = (u, v) \neq (0, 0), \\ 0 & \text{if } z = (u, v) = (0, 0), \end{cases} \tag{2.20}$$

is not continuous at $(0, 0)$ although $f'((0, 0))(z)$ exists.

Let $[a, b] \subset A$. A function $f : [a, b] \rightarrow Y$ is said to intersect its chord in the extended sense if either there is an l in $(0, 1)$ and a $u \in Y^*$ such that

$$\langle f(a + l(b - a)) - f(a), u \rangle = \langle l(f(b) - f(a)), u \rangle. \tag{2.21}$$

or there is a $u \in Y^*$ such that

$$\left\langle \lim_{t \rightarrow 0^+} \frac{f(a + t(b - a)) - f(a)}{t}, u \right\rangle = \langle f(b) - f(a), u \rangle. \tag{2.22}$$

We note that if $Y = \mathbb{R}$, then we may choose $u = 1$ and $\langle \cdot, \cdot \rangle$ is merely multiplication. In this case, the previous condition reduces to the condition where the graph of f intersects its chord in the extended sense.

We can now state a generalization of [Theorem 2.1](#) for functions defined on topological vector spaces.

THEOREM 2.3. *Let X, Y be Hausdorff topological vector spaces over the field \mathbb{R} of real numbers, let $A \subset X$ be an open set and let Y^* denote the continuous dual of Y . Let $f : A \rightarrow Y$ be a function continuous on the line segment $[x_0, x_0 + h] \subset A$. Let $J = \{x \in [x_0, x_0 + h] : f \text{ is not Gateaux differentiable at } x\}$ and set $j = |J|$. Let*

$$F\ell(v) = \frac{1}{v^2} [vf'(x_0 + vh)(h) - (f(x_0 + vh) - f(x_0))]. \tag{2.23}$$

(1) *If $j = 0$, then for each $u \in Y^*$ there exists $v \in (0, 1)$ such that*

$$\langle F\ell(v), u \rangle = \frac{1}{2} \langle f'(x_0 + vh)(h) - f'(x_0)(h), u \rangle. \tag{2.24}$$

(2) *If $j \leq n$ for some nonnegative integer n and $x_0 \notin J$, then for each $u \in Y^*$ there exist $n + 1$ points $v_1, v_2, \dots, v_{n+1} \in (0, 1)$ and $n + 1$ positive numbers $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ such that*

$$\alpha_1 + \alpha_2 + \dots + \alpha_{n+1} = 1, \tag{2.25}$$

$$\left\langle \sum_{i=1}^{n+1} \alpha_i F\ell(v_i), u \right\rangle = \langle (f(x_0 + h) - f(x_0)) - f'(x_0)(h), u \rangle.$$

(3) If j is unbounded and f intersects its chord in the extended sense for some $u \in Y^*$, then there is a $t_0 \in (0, 1)$ and $\delta_1, \delta_2 > 0$ such that either

$$\langle Fl_1(t_0, s), u \rangle \leq 0 \leq \langle Fl_2(t_0, t), u \rangle \tag{2.26}$$

or

$$\langle Fl_2(t_0, t), u \rangle \leq 0 \leq \langle Fl_1(t_0, s), u \rangle \tag{2.27}$$

holds for $0 < s \leq \delta_1$ and $0 < t \leq \delta_2$ where

$$\begin{aligned} Fl_1(t_0, s) &= \left[t_0 \frac{f(x_0 + t_0h) - f(x_0 + (t_0 - s)h)}{s} - (f(x_0 + t_0h) - f(x_0)) \right], \\ Fl_2(t_0, t) &= \left[t_0 \frac{f(x_0 + (t_0 + t)h) - f(x_0 + t_0h)}{t} - (f(x_0 + t_0h) - f(x_0)) \right]. \end{aligned} \tag{2.28}$$

PROOF. Let $u \in Y^*$ and define the function $\phi : [0, 1] \rightarrow \mathbb{R}$ by

$$\phi(t) = \langle f(x_0 + th), u \rangle. \tag{2.29}$$

For each t in $[0, 1]$ where ϕ is differentiable, we have

$$\phi'(t) = \langle f'(x_0 + th)(h), u \rangle. \tag{2.30}$$

If $j = 0$, then ϕ is differentiable on the entire interval $[0, 1]$. It follows from [Theorem 2.1\(1\)](#) that there exists a $v \in (0, 1)$ such that

$$\frac{1}{2} \frac{\phi'(1) - \phi'(0)}{1 - 0} v^2 = \frac{1}{v^2} [(v - 0)\phi'(v) - (\phi(v) - \phi(0))]. \tag{2.31}$$

Using (2.29) and (2.30) and simplifying yields the first item in [Theorem 2.3](#).

If $j \leq n$ for some nonnegative integer n and $x_0 \notin J$, then ϕ is differentiable on $[0, 1]$ except possibly at n points in $(0, 1]$. Thus, the second item in [Theorem 2.3](#) follows directly from the second item in [Theorem 2.1](#).

Finally, if j is unbounded and f intersects its chord in the extended sense for some $u \in Y^*$, then the graph of ϕ intersects its chord in the extended sense and so the third item in [Theorem 2.3](#) follows directly from the third item in [Theorem 2.1](#). This completes the proof. □

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