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Research Article

Damped Algorithms for the Split Fixed Point and Equilibrium Problems

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The main purpose of this paper is to study the split fixed point and equilibrium problems which includes fixed point problems, equilibrium problems, and variational inequality problems as special cases. A damped algorithm is presented for solving this split common problem. Strong convergence analysis is shown.

1. Introduction

Very recently, the split problems (e.g., the split feasibility problem, the split common fixed points problem, and the split variational inequality problem) have been studied extensively, see, for instance, [1–19]. Now we recall the related history. Let H_1 and H_2 be two Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ two nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem is to solve the inclusion:

$$x \in C \cap A^{-1}(Q) \tag{1}$$

which arise in the field of intensity-modulated radiation therapy and was presented in [1]. The iteration $\rho^{n+1} = \text{proj}_C(\rho^n - \varsigma A^*(I - P_Q)A\rho^n)$ is popular with $\varsigma \in (0, 2/\|A\|^2)$. Further, Xu [3] suggested a single step regularized method. Dang and Gao [4] developed a damped projection algorithm. If C and Q are the fixed point sets of mappings U and T, respectively, then (1) becomes a special case of the split common fixed point problem:

$$\text{Find } x \in \text{Fix}(U) \cap A^{-1}(\text{Fix}(T)). \tag{2}$$

Censor and Segal [5] invented a scheme below to solve (2):

$$\rho^{n+1} = U\left(\rho^n - \varsigma A^*(I - T)A\rho^n\right), \quad n \in \mathbb{N}. \tag{3}$$

Cui et al., [6] extended the damped projection algorithm to the split common fixed point problems. Let $\psi : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem is to find $x^\dagger \in C$ such that

$$\psi\left(x^\dagger, x\right) \geq 0, \quad \forall x \in C. \tag{4}$$

We will indicate with $\text{EP}(\psi)$ the set of solutions of (4).

In the present paper, our main purpose is to study the following split fixed point and equilibrium problem.

$$\begin{aligned} &\text{Find a point } u^\S \in \text{Fix}(W) \cap \text{EP}(\psi) \\ &\text{such that } Au^\S \in \text{Fix}(S) \cap \text{EP}(\varphi), \end{aligned} \tag{5}$$

where $\text{Fix}(S)$ and $\text{Fix}(W)$ are the sets of fixed points of two nonlinear mappings S and W, respectively; $\text{EP}(\psi)$ and $\text{EP}(\varphi)$ are the solution sets of two equilibrium problems with bifunctions ψ and φ, respectively, and A is a bounded linear mapping. Denote the solution set of (5) by

$$\Theta = \left\{x \in \text{Fix}(W) \cap \text{EP}(\psi) : Ax \in \text{Fix}(S) \cap \text{EP}(\varphi)\right\}. \tag{6}$$

We develop a damped algorithm to solve this split fixed point and equilibrium problem. Strong convergence of the suggested damped algorithm is demonstrated.

2. Concepts and Lemmas

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. A mapping $W : C \to C$ is called nonexpansive if

$$\left\|Wx^{\dagger} - Wx^{\ddagger}\right\| \leq \left\|x^{\dagger} - x^{\ddagger}\right\|, \tag{7}$$

for all $x^{\dagger}, x^{\ddagger} \in C$. We call $\mathrm{proj}_C : H \to C$ the metric projection if for each $x^{\flat} \in H$

$$\left\|x^{\flat} - \mathrm{proj}_C\left(x^{\flat}\right)\right\| = \inf\left\{\left\|x^{\flat} - x^{\dagger}\right\| : x^{\dagger} \in C\right\}. \tag{8}$$

It is well known that the metric projection $\mathrm{proj}_C : H \to C$ is firmly nonexpansive, that is,

$$\begin{aligned}&\left\|\mathrm{proj}_C\left(x^{\dagger}\right) - \mathrm{proj}_C\left(x^{\ddagger}\right)\right\|^2 \\ &\quad \leq \left\langle x^{\dagger} - x^{\ddagger}, \mathrm{proj}_C\left(x^{\dagger}\right) - \mathrm{proj}_C\left(x^{\ddagger}\right)\right\rangle\end{aligned} \tag{9}$$

for all $x^{\dagger}, x^{\ddagger} \in H$. Hence proj_C is also nonexpansive.

Lemma 1 (see [20]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\psi : C \times C \to \mathbb{R}$ be a bifunction which satisfies the following conditions:*

(H1) $\psi(x^{\ddagger}, x^{\ddagger}) = 0$ *for all* $x^{\ddagger} \in C$;

(H2) ψ *is monotone, that is,* $\psi(x^{\ddagger}, x^{\dagger}) + \psi(x^{\dagger}, x^{\ddagger}) \leq 0$ *for all* $x^{\ddagger}, x^{\dagger} \in C$;

(H3) *for each* $x^{\dagger}, x^{\ddagger}, x^{\natural} \in C$, $\lim_{t\downarrow 0}\psi(tx^{\natural} + (1-t)x^{\dagger}, x^{\ddagger}) \leq \psi(x^{\dagger}, x^{\ddagger})$;

(H4) *for each* $x^{\dagger} \in C$, $x^{\ddagger} \mapsto \psi(x^{\dagger}, x^{\ddagger})$ *is convex and lower semicontinuous.*

Let $\varpi > 0$ and $x^{\dagger} \in C$. Then, there exists $x^{\natural} \in C$ such that

$$\psi\left(x^{\natural}, x^{\ddagger}\right) + \frac{1}{\varpi}\left\langle x^{\ddagger} - x^{\natural}, x^{\natural} - x^{\dagger}\right\rangle \geq 0, \quad \forall x^{\ddagger} \in C. \tag{10}$$

Further, if $U_{\varpi}^{\psi}(x^{\dagger}) = \{x^{\natural} \in C : \psi(x^{\natural}, x^{\ddagger}) + (1/\varpi)\langle x^{\ddagger} - x^{\natural}, x^{\natural} - x^{\dagger}\rangle \geq 0,$ for all $x^{\ddagger} \in C\}$, then the following hold:

(i) U_{ϖ}^{ψ} *is single-valued and* U_{ϖ}^{ψ} *is firmly nonexpansive, that is, for any* $x^{\dagger}, x^{\ddagger} \in H$, $\|U_{\varpi}^{\psi}x^{\dagger} - U_{\varpi}^{\psi}x^{\ddagger}\|^2 \leq \langle U_{\varpi}^{\psi}x^{\dagger} - U_{\varpi}^{\psi}x^{\ddagger}, x^{\dagger} - x^{\ddagger}\rangle$;

(ii) $\mathrm{EP}(\psi)$ *is closed and convex and* $\mathrm{EP}(\psi) = \mathrm{Fix}(U_{\varpi}^{\psi})$.

Lemma 2 (see [21]). *Let H be a Hilbert space and $C \subset H$ a closed convex subset. Let $W : C \to C$ be a nonexpansive mapping. Then, the mapping $I - W$ is demiclosed. That is, if $\{\rho^n\}$ is a sequence in C such that $\rho^n \to \nu$ weakly and $(I-W)\rho^n \to u$ strongly, then $(I-W)\nu = u$.*

Lemma 3 (see [22]). *Assume that $\{\eta_n\}$ is a sequence of nonnegative real numbers such that*

$$\eta_{n+1} \leq (1-\kappa_n)\eta_n + \varsigma_n, \quad n \in \mathbb{N}, \tag{11}$$

where $\{\kappa_n\}$ is a sequence in $(0,1)$ and $\{\varsigma_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty}\kappa_n = \infty$;

(2) $\limsup_{n\to\infty}(\varsigma_n/\kappa_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\varsigma_n| < \infty$.

Then $\lim_{n\to\infty}\eta_n = 0$.

3. Main Results

Let H_1 and H_2 be two Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ two nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $\psi : C \times C \to \mathbb{R}$ and let $\varphi : D \times D \to \mathbb{R}$ be two bifunctions satisfying the conditions (H1)–(H4) in Lemma 1. Let $S : D \to D$ and $W : C \to C$ be two nonexpansive mappings.

Algorithm 4. Let $x_0 \in H_1$. Define a sequence $\{x_n\}$ as follows:

$$\begin{aligned}\rho^{n+1} = {} & WU_{\iota}^{\psi}\left[(1-\zeta_n)\right. \\ & \left.\times\left(\rho^n + \varsigma A^*\left(SU_{\kappa}^{\varphi} - I\right)A\rho^n\right)\right], \quad \forall n \in \mathbb{N},\end{aligned} \tag{12}$$

where ι, κ, and ς are three constants satisfying $\iota \in (0,\infty)$, $\kappa \in (0,\infty)$, $\varsigma \in (0, 1/\|A\|^2)$, and $\{\zeta_n\}$ is a real number sequence in $(0,1)$.

In the sequel, we assume that

$$\Theta = \left\{x \in \mathrm{Fix}(W) \cap \mathrm{EP}(\psi) : Ax \in \mathrm{Fix}(S) \cap \mathrm{EP}(\varphi)\right\} \neq \emptyset. \tag{13}$$

Theorem 5. *If $\{\zeta_n\}$ satisfies $\lim_{n\to\infty}\zeta_n = 0$, $\sum_{n=1}^{\infty}\zeta_n = \infty$ and $\lim_{n\to\infty}\zeta_{n+1}/\zeta_n = 1$, then $\{\rho^n\}$ generated by algorithm (12) converges strongly to $\mathrm{proj}_{\Theta}(0)$ which is the minimum-norm element in Θ.*

Proof. Let $p = \mathrm{proj}_{\Theta}(0)$. Then, $p \in \mathrm{Fix}(W) \cap \mathrm{EP}(\psi)$ and $Ap \in \mathrm{Fix}(S) \cap \mathrm{EP}(\varphi)$. Set $z^n = U_{\kappa}^{\varphi}A\rho^n$, $y^n = (1-\zeta_n)(\rho^n + \varsigma A^*(SU_{\kappa}^{\varphi} - I)A\rho^n)$ and $u^n = U_{\iota}^{\psi}[(1-\zeta_n)(\rho^n + \varsigma A^*(SU_{\kappa}^{\varphi} - I)A\rho^n)]$ for all $n \in \mathbb{N}$. Then $u^n = U_{\iota}^{\psi}y^n$. From Lemma 1, we know that U_{ι}^{ψ} and U_{κ}^{φ} are firmly nonexpansive. Thus, we have

$$\left\|z^n - Ap\right\| = \left\|U_{\kappa}^{\varphi}A\rho^n - Ap\right\| \leq \left\|A\rho^n - Ap\right\|, \tag{14}$$

$$\left\|u^n - p\right\| = \left\|U_{\iota}^{\psi}y^n - p\right\| \leq \left\|y^n - p\right\|, \tag{15}$$

$$\begin{aligned}\left\|SU_{\kappa}^{\varphi}A\rho^n - Ap\right\|^2 &= \left\|SU_{\kappa}^{\varphi}A\rho^n - SU_{\kappa}^{\varphi}Ap\right\| \\ &\leq \left\|U_{\kappa}^{\varphi}A\rho^n - U_{\kappa}^{\varphi}Ap\right\|^2 \\ &\leq \left\|A\rho^n - Ap\right\|^2 - \left\|U_{\kappa}^{\varphi}A\rho^n - A\rho^n\right\|^2.\end{aligned} \tag{16}$$

Note that

$$\begin{aligned}\left\|u^{n+1}-u^{n}\right\| &= \left\|U_{\iota}^{\psi} y^{n+1}-U_{\iota}^{\psi} y^{n}\right\| \\ &\leq \left\|y^{n+1}-y^{n}\right\|,\end{aligned} \tag{17}$$

$$\begin{aligned}\left\|z^{n+1}-z^{n}\right\| &= \left\|U_{\kappa}^{\varphi} A\rho^{n+1}-U_{\kappa}^{\varphi} A\rho^{n}\right\| \\ &\leq \left\|A\rho^{n+1}-A\rho^{n}\right\|.\end{aligned} \tag{18}$$

From (12) and (15), we have

$$\left\|\rho^{n+1}-p\right\| = \left\|Wu^{n}-p\right\| \leq \left\|u^{n}-p\right\| \leq \left\|y^{n}-p\right\|. \tag{19}$$

Observe that

$$\begin{aligned}\left\|y^{n}-p\right\|^{2} &= \|(1-\zeta_{n}) \\ &\quad \times (\rho^{n}-p+\varsigma A^{*}(Sz^{n}-A\rho^{n}))-\zeta_{n}p\|^{2} \\ &\leq (1-\zeta_{n})\left\|(\rho^{n}-p+\varsigma A^{*}(Sz^{n}-A\rho^{n}))\right\|^{2} \\ &\quad + \zeta_{n}\|p\|^{2} \\ &= (1-\zeta_{n})\left[\left\|\rho^{n}-p\right\|+2\varsigma\right. \\ &\quad \times \left\langle \rho^{n}-p, A^{*}(Sz^{n}-A\rho^{n})\right\rangle \\ &\quad \left. +\varsigma^{2}\left\|A^{*}(Sz^{n}-A\rho^{n})\right\|^{2}\right] \\ &\quad + \zeta_{n}\|p\|^{2}.\end{aligned} \tag{20}$$

Since A^{*} is the adjoint of A, we have

$$\begin{aligned}&\left\langle \rho^{n}-p, A^{*}(Sz^{n}-A\rho^{n})\right\rangle \\ &\quad = \left\langle A(\rho^{n}-p), Sz^{n}-A\rho^{n}\right\rangle \\ &\quad = \left\langle A\rho^{n}-Ap+Sz^{n}-A\rho^{n}\right. \\ &\qquad \left. -(Sz^{n}-A\rho^{n}), Sz^{n}-A\rho^{n}\right\rangle \\ &\quad = \left\langle Sz^{n}-Ap, Sz^{n}-A\rho^{n}\right\rangle - \left\|Sz^{n}-A\rho^{n}\right\|^{2}.\end{aligned} \tag{21}$$

Using parallelogram law, we obtain

$$\begin{aligned}&\left\langle Sz^{n}-Ap, Sz^{n}-A\rho^{n}\right\rangle \\ &\quad = \frac{1}{2}\left(\left\|Sz^{n}-Ap\right\|^{2}+\left\|Sz^{n}-A\rho^{n}\right\|^{2}\right. \\ &\qquad \left.-\left\|A\rho^{n}-Ap\right\|^{2}\right).\end{aligned} \tag{22}$$

From (16), (21) and (22), we have

$$\begin{aligned}&\left\langle \rho^{n}-p, A^{*}(Sz^{n}-A\rho^{n})\right\rangle \\ &\quad = \frac{1}{2}\left(\left\|Sz^{n}-Ap\right\|^{2}+\left\|Sz^{n}-A\rho^{n}\right\|^{2}\right. \\ &\qquad \left.-\left\|A\rho^{n}-Ap\right\|^{2}\right) \\ &\qquad - \left\|Sz^{n}-A\rho^{n}\right\|^{2} \\ &\quad \leq \frac{1}{2}\left(\left\|A\rho^{n}-Ap\right\|^{2}-\left\|z^{n}-A\rho^{n}\right\|^{2}\right. \\ &\qquad \left.+\left\|Sz^{n}-A\rho^{n}\right\|^{2}-\left\|A\rho^{n}-Ap\right\|^{2}\right) \\ &\qquad - \left\|Sz^{n}-A\rho^{n}\right\|^{2} \\ &\quad = -\frac{1}{2}\left\|z^{n}-A\rho^{n}\right\|^{2} \\ &\qquad - \frac{1}{2}\left\|Sz^{n}-A\rho^{n}\right\|^{2}.\end{aligned} \tag{23}$$

By (20) and (23), we deduce

$$\begin{aligned}&\left\|y^{n}-p\right\|^{2} \\ &\quad \leq (1-\zeta_{n})\left[\left\|\rho^{n}-p\right\|^{2}+\varsigma^{2}\|A\|^{2}\left\|Sz^{n}-A\rho^{n}\right\|^{2}\right. \\ &\qquad + 2\varsigma\left(-\frac{1}{2}\left\|z^{n}-A\rho^{n}\right\|^{2}\right. \\ &\qquad \left.\left. -\frac{1}{2}\left\|Sz^{n}-A\rho^{n}\right\|^{2}\right)\right] + \zeta_{n}\|p\|^{2} \\ &\quad = (1-\zeta_{n}) \\ &\qquad \times \left[\left\|\rho^{n}-p\right\|^{2}+\left(\varsigma^{2}\|A\|^{2}-\varsigma\right)\left\|Sz^{n}-A\rho^{n}\right\|^{2}\right. \\ &\qquad \left. -\varsigma\left\|z^{n}-A\rho^{n}\right\|^{2}\right] + \zeta_{n}\|p\|^{2} \\ &\quad \leq (1-\zeta_{n})\left\|\rho^{n}-p\right\|^{2}+\zeta_{n}\|p\|^{2}.\end{aligned} \tag{24}$$

It follows from (19), we get

$$\begin{aligned}\left\|\rho^{n+1}-p\right\|^{2} &\leq \left\|y^{n}-p\right\|^{2} \\ &\leq (1-\zeta_{n})\left\|\rho^{n}-p\right\|^{2}+\zeta_{n}\|p\|^{2} \\ &\leq \max\left\{\left\|\rho^{n}-p\right\|^{2}, \|p\|^{2}\right\}.\end{aligned} \tag{25}$$

The boundedness of the sequence $\{\rho^{n}\}$ yields.

Set $v^n = \rho^n + \varsigma A^*(SU_\kappa^\varphi - I)A\rho^n$. Then, we have

$$\begin{aligned}
\left\|v^{n+1} - v^n\right\|^2 &= \Big\|\rho^{n+1} - \rho^n \\
&\quad + \varsigma\left[A^*\left(Sz^{n+1} - A\rho^{n+1}\right) - A^*\left(Sz^n - A\rho^n\right)\right]\Big\|^2 \\
&= \left\|\rho^{n+1} - \rho^n\right\|^2 \\
&\quad + 2\varsigma\left\langle \rho^{n+1} - \rho^n, \right. \\
&\qquad \left. A^*\left[\left(Sz^{n+1} - A\rho^{n+1}\right) - \left(Sz^n - A\rho^n\right)\right]\right\rangle \\
&\quad + \varsigma^2\left\|A^*\left(Sz^{n+1} - A\rho^{n+1}\right) - A^*\left(Sz^n - A\rho^n\right)\right\|^2 \\
&\le \left\|\rho^{n+1} - \rho^n\right\|^2 \\
&\quad + 2\varsigma\left\langle A\rho^{n+1} - A\rho^n, \right. \\
&\qquad \left. Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\rangle \\
&\quad + \varsigma^2\|A\|^2\left\|Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\|^2 \\
&= \left\|\rho^{n+1} - \rho^n\right\|^2 \\
&\quad + \varsigma^2\|A\|^2\left\|Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\|^2 \\
&\quad + 2\varsigma\left\langle Sz^{n+1} - Sz^n, \right. \\
&\qquad \left. Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\rangle \\
&\quad - 2\varsigma\left\|Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\|^2 \\
&= \left\|\rho^{n+1} - \rho^n\right\|^2 \\
&\quad + \varsigma^2\|A\|^2\left\|Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\|^2 \\
&\quad + \varsigma\Big(\left\|Sz^{n+1} - Sz^n\right\|^2 \\
&\qquad + \left\|Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\|^2 \\
&\qquad - \left\|A\rho^{n+1} - A\rho^n\right\|^2\Big) \\
&\quad - 2\varsigma\left\|Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\|^2 \\
&= \left\|\rho^{n+1} - \rho^n\right\|^2 \\
&\quad + \left(\varsigma^2\|A\|^2 - \varsigma\right) \\
&\qquad \times \left\|Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\|^2 \\
&\quad + \varsigma\left(\left\|Sz^{n+1} - Sz^n\right\|^2 - \left\|\left(A\rho^{n+1} - A\rho^n\right)\right\|^2\right) \\
&\le \left\|\rho^{n+1} - \rho^n\right\|^2 \\
&\quad + \left(\varsigma^2\|A\|^2 - \varsigma\right) \\
&\qquad \times \left\|Sz^{n+1} - Sz^n - \left(A\rho^{n+1} - A\rho^n\right)\right\|^2 \\
&\quad + \varsigma\left(\left\|z^{n+1} - z^n\right\|^2 - \left\|A\rho^{n+1} - A\rho^n\right\|^2\right).
\end{aligned} \tag{26}$$

Since $\varsigma \in (0, 1/\|A\|^2)$, we derive by virtue of (18) and (26) that

$$\left\|v^{n+1} - v^n\right\| \le \left\|\rho^{n+1} - \rho^n\right\|. \tag{27}$$

According to (17) and (27), we have

$$\begin{aligned}
\left\|\rho^{n+1} - \rho^n\right\| &= \left\|Wu^{n+1} - Wu^n\right\| \\
&\le \left\|u^{n+1} - u^n\right\| \\
&\le \left\|y^{n+1} - y^n\right\| \\
&= \left\|\left(1 - \zeta_{n+1}\right)v^{n+1} - \left(1 - \zeta_n\right)v^n\right\| \\
&= \left\|\left(1 - \zeta_{n+1}\right)\left(v^{n+1} - v^n\right) + \left(\zeta_n - \zeta_{n+1}\right)v^n\right\| \\
&\le \left(1 - \zeta_{n+1}\right)\left\|v^{n+1} - v^n\right\| + \left|\zeta_{n+1} - \zeta_n\right|\left\|v^n\right\| \\
&\le \left(1 - \zeta_{n+1}\right)\left\|\rho^{n+1} - \rho^n\right\| + \left|\zeta_{n+1} - \zeta_n\right|\left\|v^n\right\|.
\end{aligned} \tag{28}$$

It follows that

$$\left\|\rho^{n+1} - \rho^n\right\| \le \frac{\left|\zeta_{n+1} - \zeta_n\right|}{\zeta_{n+1}}\left\|v^n\right\|. \tag{29}$$

Since $\{\rho^n\}$ is bounded, we can deduce $\{v^n\}$ is also bounded. From (29), we have

$$\lim_{n\to\infty}\left\|\rho^{n+1} - \rho^n\right\| = 0. \tag{30}$$

Hence,

$$\lim_{n\to\infty}\left\|\rho^n - Wu^n\right\| = 0. \tag{31}$$

Using the firmly-nonexpansivenessity of U_ι^ψ, we have

$$\begin{aligned}
\left\|u^n - p\right\|^2 &= \left\|U_\iota^\psi y^n - p\right\|^2 \\
&\le \left\|y^n - p\right\|^2 - \left\|U_\iota^\psi y^n - y^n\right\|^2 \\
&= \left\|y^n - p\right\|^2 - \left\|u^n - y^n\right\|^2.
\end{aligned} \tag{32}$$

Thus, we get

$$\begin{aligned}
\left\|\rho^{n+1} - p\right\|^2 &\le \left\|u^n - p\right\|^2 \\
&\le \left\|y^n - p\right\|^2 - \left\|u^n - y^n\right\|^2 \\
&\le \left(1 - \zeta_n\right)\left\|\rho^n - p\right\|^2 + \zeta_n\|p\|^2 - \left\|u^n - y^n\right\|^2.
\end{aligned} \tag{33}$$

It follows that

$$\begin{aligned}
\left\|u^n - y^n\right\|^2 &\le \left\|\rho^n - p\right\|^2 - \left\|\rho^{n+1} - p\right\|^2 + \zeta_n\|p\|^2 \\
&\le \left(\left\|\rho^n - p\right\| + \left\|\rho^{n+1} - p\right\|^2\right)\left\|\rho^{n+1} - p\right\|^2 + \zeta_n\|p\|^2.
\end{aligned} \tag{34}$$

This together with (30) and (C1) implies that

$$\lim_{n\to\infty} \|u^n - y^n\| = 0. \tag{35}$$

Note that

$$\begin{aligned}\|\rho^{n+1} - p\|^2 &\le \|y^n - p\|^2 \\ &\le (1-\zeta_n)\|\rho^n - p\|^2 \\ &\quad + (1-\zeta_n)\left(\varsigma^2\|A\|^2 - \varsigma\right)\|Sz^n - A\rho^n\|^2 \\ &\quad - (1-\zeta_n)\varsigma\|z^n - A\rho^n\|^2 + \zeta_n\|p\|^2.\end{aligned} \tag{36}$$

Hence,

$$\begin{aligned}&(1-\zeta_n)\left(\varsigma - \varsigma^2\|A\|^2\right)\|Sz^n - A\rho^n\|^2 \\ &\quad + (1-\zeta_n)\varsigma\|z^n - A\rho^n\|^2 \\ &\le \|\rho^n - p\|^2 - \|\rho^{n+1} - p\|^2 + \zeta_n\|p\|^2 \\ &\le \left(\|\rho^n - p\| + \|\rho^{n+1} - p\|\right)\|\rho^{n+1} - \rho^n\| \\ &\quad + \zeta_n\|p\|^2,\end{aligned} \tag{37}$$

which implies that

$$\lim_{n\to\infty}\|Sz^n - A\rho^n\| = \lim_{n\to\infty}\|z^n - A\rho^n\| = 0. \tag{38}$$

So, we get

$$\lim_{n\to\infty}\|Sz^n - z^n\| = 0. \tag{39}$$

Since

$$\begin{aligned}\|y^n - p^n\| &= \|\varsigma A^*\left(SU_\kappa^\varphi - I\right)A\rho^n + \zeta_n v^n\| \\ &\le \varsigma\|A\|\|Sz^n - A\rho^n\| + \zeta_n\|v^n\|,\end{aligned} \tag{40}$$

we get

$$\lim_{n\to\infty}\|\rho^n - y^n\| = 0. \tag{41}$$

From (31), (35), and (41), we get

$$\lim_{n\to\infty}\|\rho^n - W\rho^n\| = 0. \tag{42}$$

Now, we show that

$$\limsup_{n\to\infty}\langle p, y^n - p\rangle \ge 0. \tag{43}$$

Choose a subsequence $\{y^{n_i}\}$ of $\{y^n\}$ such that

$$\limsup_{n\to\infty}\langle p, y^n - p\rangle = \lim_{i\to\infty}\langle p, y^{n_i} - p\rangle. \tag{44}$$

Notice that $\{y^{n_i}\}$ is bounded, we can choose $\{y^{n_{i_j}}\}$ of $\{y^{n_i}\}$ such that $y^{n_{i_j}} \rightharpoonup z$. Without loss of generality, we assume that $y^{n_i} \rightharpoonup z$. From the above conclusions, we derive that

$$\begin{aligned}\rho^{n_i} &\rightharpoonup z, & u_{n_i} &\rightharpoonup z, \\ A\rho^{n_i} &\rightharpoonup Az, & z^{n_i} &\rightharpoonup Az.\end{aligned} \tag{45}$$

By Lemma 2, (39), and (41), we deduce $z \in \mathrm{Fix}(W)$ and $Az \in \mathrm{Fix}(S)$.

Next, we show that $z \in \mathrm{EP}(\psi)$. Since $u^n = U_\iota^\psi y^n$, we have

$$\psi\left(u^n, x^\dagger\right) + \frac{1}{\iota}\left\langle x^\dagger - u^n, u^n - y^n\right\rangle \ge 0, \quad \forall x^\dagger \in C. \tag{46}$$

By the monotonicity of ψ, we have

$$\frac{1}{\iota}\left\langle x^\dagger - u^n, u^n - y^n\right\rangle \ge \psi\left(x^\dagger, u^n\right), \tag{47}$$

and so

$$\left\langle x^\dagger - u_{n_i}, \frac{u_{n_i} - y^{n_i}}{\iota}\right\rangle \ge \psi\left(x^\dagger, u_{n_i}\right). \tag{48}$$

Since $\|u^n - y^n\| \to 0$, $u_{n_i} \rightharpoonup z$, we obtain $(u_{n_i} - y^{n_i})/\iota \to 0$. Thus, $0 \ge \psi(x^\dagger, z)$. For t with $0 < t \le 1$ and $x^\dagger \in C$, let $y^t = tx^\dagger + (1-t)z \in C$. We obtain $\psi(y^t, z) \le 0$. Hence,

$$0 = \psi\left(y^t, y^t\right) \le t\psi\left(y^t, x^\dagger\right) + (1-t)\psi\left(y^t, z\right) \le t\psi\left(y^t, x^\dagger\right). \tag{49}$$

So, $0 \le \psi(y^t, x^\dagger)$. And, thus, $0 \le \psi(z, x^\dagger)$. This implies that $z \in \mathrm{EP}(\psi)$. Similarity, we can prove that $Az \in \mathrm{EP}(\varphi)$. To this end, we deduce $z \in \mathrm{Fix}(W) \cap \mathrm{EP}(\psi)$ and $Az \in \mathrm{Fix}(S) \cap \mathrm{EP}(\varphi)$. That is to say, $z \in \Theta$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\langle p, y^n - p\rangle &= \lim_{i\to\infty}\langle p, y^{n_i} - p\rangle \\ &= \lim_{i\to\infty}\langle p, z - p\rangle \\ &\ge 0.\end{aligned} \tag{50}$$

Finally, we prove $\rho^n \to p$. From (12), we have

$$\begin{aligned}\|\rho^{n+1} - p\|^2 &\le \|y^n - p\|^2 \\ &= \|(1-\zeta_n)(v^n - p) - \zeta_n p\|^2 \\ &\le (1-\zeta_n)\|v^n - p\|^2 - 2\zeta_n\langle p, y^n - p\rangle \\ &\le (1-\zeta_n)\|\rho^n - p\|^2 - 2\zeta_n\langle p, y^n - p\rangle.\end{aligned} \tag{51}$$

Applying Lemma 3 and (50) to (51), we deduce $\rho^n \to p$. The proof is completed. □

Algorithm 6. Let $\rho^0 \in H_1$ arbitrarily define a sequence $\{\rho^n\}$ by the following:

$$\rho^{n+1} = W\left((1-\zeta_n)\left(\rho^n + \varsigma A^*(S - I)A\rho^n\right)\right), \tag{52}$$

for all $n \in \mathbb{N}$, where $\varsigma \in (0, 1/\|A\|^2)$ and $\{\zeta_n\}$ is a real number sequence in $(0, 1)$.

Corollary 7. *Suppose $\Theta_1 = \{x \in \mathrm{Fix}(W) : Ax \in \mathrm{Fix}(S)\} \ne \emptyset$. If $\{\zeta_n\}$ satisfies $\lim_{n\to\infty}\zeta_n = 0$, $\sum_{n=1}^{\infty}\zeta_n = \infty$, and $\lim_{n\to\infty}\zeta_{n+1}/\zeta_n = 1$, then the sequence $\{\rho^n\}$ generated by algorithm (52) converges strongly to $p = \mathrm{proj}_{\Theta_1}(0)$ which is the mum-norm element in Θ_1.*

Algorithm 8. Let $\rho^0 \in H_1$ arbitrarily define a sequence $\{\rho^n\}$ by the following:

$$\rho^{n+1} = U_\iota^\psi \left((1 - \zeta_n) \left(\rho^n + \varsigma A^* \left(U_\kappa^\varphi - I \right) A \rho^n \right) \right), \quad (53)$$

for all $n \in \mathbb{N}$, where ι, κ, and ς are three constants satisfying $\iota \in (0, \infty), \kappa \in (0, \infty)$, $\varsigma \in (0, 1/\|A\|^2)$, and $\{\zeta_n\}$ is a real number sequence in $(0, 1)$.

Corollary 9. *Suppose* $\Theta_2 = \{x \in \mathrm{EP}(\psi) : Ax \in \mathrm{EP}(\varphi)\} \neq \emptyset$. *If* $\{\zeta_n\}$ *satisfies* $\lim_{n\to\infty} \zeta_n = 0$, $\sum_{n=1}^{\infty} \zeta_n = \infty$, *and* $\lim_{n\to\infty} \zeta_{n+1}/\zeta_n = 1$, *then the sequence* $\{\rho^n\}$ *generated by algorithm* (53) *converges strongly to* $p = \mathrm{proj}_{\Theta_2}(0)$ *which is the mum-norm element in* Θ_2.

Algorithm 10. Let $\rho^0 \in H_1$ arbitrarily define a sequence $\{\rho^n\}$ by the following:

$$\rho^{n+1} = \mathrm{proj}_C \left((1 - \zeta_n) \left(\rho^n + \varsigma A^* \left(\mathrm{proj}_Q - I \right) A \rho^n \right) \right), \quad (54)$$

for all $n \in \mathbb{N}$, where $\varsigma \in (0, 1/\|A\|^2)$ and $\{\zeta_n\}$ is a real number sequence in $(0, 1)$.

Corollary 11. *Suppose* $\Theta_3 = \{x \in C : Ax \in Q\} \neq \emptyset$. *If* $\{\zeta_n\}$ *satisfies* $\lim_{n\to\infty} \zeta_n = 0$, $\sum_{n=1}^{\infty} \zeta_n = \infty$, *and* $\lim_{n\to\infty} \zeta_{n+1}/\zeta_n = 1$, *then the sequence* $\{\rho^n\}$ *generated by algorithm* (54) *converges strongly to* $p = \mathrm{proj}_{\Theta_3}(0)$ *which is the mum-norm element in* Θ_3.

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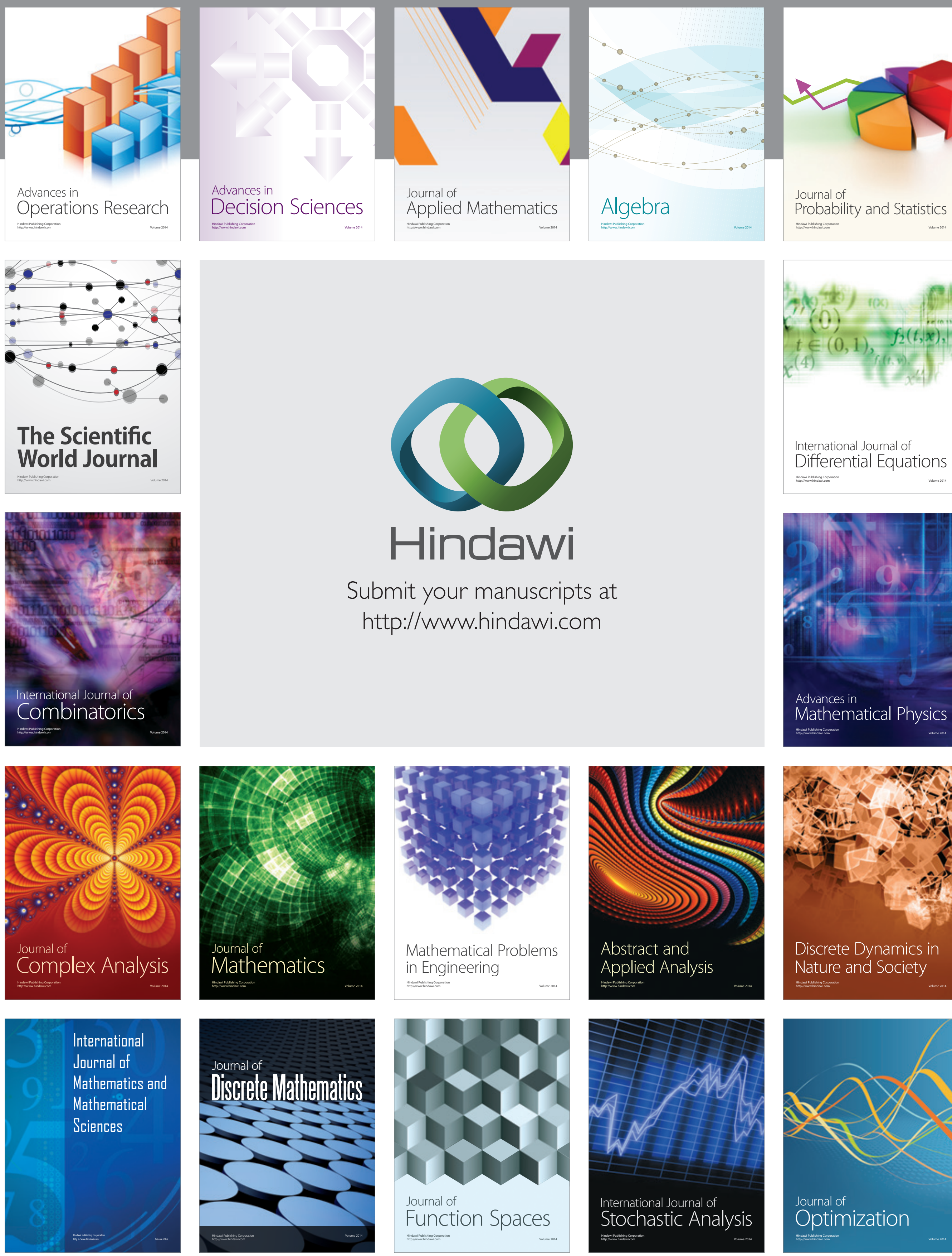
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