

Research Article

Graphs with Bounded Maximum Average Degree and Their Neighbor Sum Distinguishing Total-Choice Numbers

Patcharapan Jumnonnit and Kittikorn Nakprasit

Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

Correspondence should be addressed to Kittikorn Nakprasit; kitnak@hotmail.com

Received 31 May 2017; Accepted 4 October 2017; Published 7 November 2017

Academic Editor: Daniel Simson

Copyright © 2017 Patcharapan Jumnonnit and Kittikorn Nakprasit. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let G be a graph and $\phi : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, k\}$ be a k -total coloring. Let $w(v)$ denote the sum of color on a vertex v and colors assigned to edges incident to v . If $w(u) \neq w(v)$ whenever $uv \in E(G)$, then ϕ is called a neighbor sum distinguishing total coloring. The smallest integer k such that G has a neighbor sum distinguishing k -total coloring is denoted by $\text{tndi}_{\Sigma}(G)$. In 2014, Dong and Wang obtained the results about $\text{tndi}_{\Sigma}(G)$ depending on the value of maximum average degree. A k -assignment L of G is a list assignment L of integers to vertices and edges with $|L(v)| = k$ for each vertex v and $|L(e)| = k$ for each edge e . A *total- L -coloring* is a total coloring ϕ of G such that $\phi(v) \in L(v)$ whenever $v \in V(G)$ and $\phi(e) \in L(e)$ whenever $e \in E(G)$. We state that G has a *neighbor sum distinguishing total- L -coloring* if G has a total- L -coloring such that $w(u) \neq w(v)$ for all $uv \in E(G)$. The smallest integer k such that G has a neighbor sum distinguishing total- L -coloring for every k -assignment L is denoted by $\text{Ch}_{\Sigma}''(G)$. In this paper, we strengthen results by Dong and Wang by giving analogous results for $\text{Ch}_{\Sigma}''(G)$.

1. Introduction

Let G be a simple, finite, and undirected graph. We use $V(G)$, $E(G)$, and $\Delta(G)$ to denote the vertex set, edge set, and maximum degree of a graph G , respectively. A vertex v is called a k -vertex if $d(v) = k$. The length of a shortest cycle in G is called the *girth* of a graph G , denoted by $g(G)$. The *maximum average degree* of G is defined by $\text{mad}(G) = \max_{H \subseteq G} (2|E(H)|/|V(H)|)$. The well-known observation for a planar graph G is $\text{mad}(G) < 2g(G)/(g(G) - 2)$. Let $\phi : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, k\}$ be a k -total coloring. We denote the sum (set, resp.) of colors assigned to edges incident to v and the color on the vertex v by $w(v)$ ($C(v)$, resp.); that is, $w(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$ and $C(v) = \{\phi(v)\} \cup \{\phi(uv) \mid uv \in E(G)\}$. The total coloring ϕ of G is a *neighbor sum distinguishing* (neighbor distinguishing, resp.) total coloring if $w(u) \neq w(v)$ ($C(u) \neq C(v)$, resp.) for each edge $uv \in E(G)$. The smallest integer k such that G has a neighbor sum distinguishing (neighbor distinguishing, resp.) total coloring is called the *neighbor sum distinguishing total chromatic number* (neighbor distinguishing total chromatic number, resp.), denoted by $\text{tndi}_{\Sigma}(G)$ ($\text{tndi}(G)$, resp.). In 2005, a neighbor distinguishing

total coloring of graphs was introduced by Zhang et al. [1]. They obtained $\text{tndi}(G)$ for many basic graphs and brought forward the following conjecture.

Conjecture 1 (see [1]). *If G is a graph with order at least two, then $\text{tndi}(G) \leq \Delta(G) + 3$.*

Conjecture 1 has been confirmed for subcubic graphs, K_4 -minor free graphs, and planar graphs with large maximum degree [2–4].

In 2015, Piłśniak and Woźniak [5] obtained $\text{tndi}_{\Sigma}(G)$ for cycles, cubic graphs, bipartite graphs, and complete graphs. Moreover, they posed the following conjecture.

Conjecture 2 (see [5]). *If G is a graph with at least two vertices, then $\text{tndi}_{\Sigma}(G) \leq \Delta(G) + 3$.*

Li et al. verified this conjecture for K_4 -minor free graphs [6] and planar graphs with the large maximum degree [7]. Wang et al. [8] confirmed this conjecture by using the famous Combinatorial Nullstellensatz that holds for any triangle free planar graph with maximum degree of at least 7. Several

results about $\text{ndi}_\Sigma(G)$ for planar graphs can be found in [9–11].

In 2014, Dong and Wang [12] proved the following results.

Theorem 3. *If G is a graph with $\text{mad}(G) < 3$, then $\text{ndi}_\Sigma(G) \leq \max\{\Delta(G) + 2, 7\}$.*

Corollary 4. *If G is a graph with $\text{mad}(G) < 3$ and $\Delta(G) \geq 5$, then $\text{ndi}_\Sigma(G) \leq \max\{\Delta(G) + 2, 7\}$.*

Corollary 5. *Let G be a planar graph. If $g(G) \geq 6$ and $\Delta(G) \geq 5$, then $\text{ndi}_\Sigma(G) \leq \Delta(G) + 2$; and $\text{ndi}_\Sigma(G) = \Delta(G) + 2$ if and only if G has two adjacent vertices of maximum degree.*

The concept of list coloring was introduced by Vizing [13] and by Erdős et al. [14]. A k -assignment L of G is a list assignment L of integers to vertices and edges with $|L(v)| = k$ for each vertex v and $|L(e)| = k$ for each edge e . A total- L -coloring is a total coloring ϕ of G such that $\phi(v) \in L(v)$ whenever $v \in V(G)$ and $\phi(e) \in L(e)$ whenever $e \in E(G)$. We state that G has a neighbor sum distinguishing total- L -coloring if G has a total- L -coloring such that $w(u) \neq w(v)$ for all $uv \in E(G)$. The smallest integer k such that G has a neighbor sum distinguishing total- L -coloring for every k -assignment L , denoted by $\text{Ch}_\Sigma''(G)$, is called the neighbor sum distinguishing total-choice number.

Qu et al. [15] proved that $\text{Ch}_\Sigma''(G) \leq \Delta(G) + 3$ for any planar graph G with $\Delta(G) \geq 13$. Yao et al. [16] studied $\text{Ch}_\Sigma''(G)$ of d -degenerate graphs. Later, Wang et al. [17] confirmed Conjecture 2 true for planar graphs without 4-cycles. For $H \subseteq G$, we let L_H denote a list L restricted to any proper subgraph H of G . In this paper, we strengthen Theorem 3 by giving analogous results for $\text{Ch}_\Sigma''(G)$.

2. Main Results

The following lemma is obvious, so we omit the proof.

Lemma 6. *Let $|S_1| = |S_2| = \dots = |S_k| = k + 1$ and $S^* = \{a_1 + a_2 + \dots + a_k \mid a_i \in S_i, a_i \neq a_j, 1 \leq i < j \leq k\}$. Then $|S^*| \geq k + 1$.*

Proof. We proceed by induction on k .

If $k = 1$, then $|S_1| = 2$; then Lemma 6 holds. Assume that $k > 1$. Suppose that Lemma 6 holds for $k - 1$. Let $a = \min(S_1 \cup S_2 \cup \dots \cup S_k)$. Without loss of generality, let $a \in S_1$. Let $T_i \subseteq S_i$ be such that $|T_i| = k$ and $a \notin T_i$ for $i = 1, 2, \dots, k$. By induction hypothesis, we have $|T^*| \geq k$. Thus $\{a + t_2 + t_3 + \dots + t_k\} \subseteq S^*$, where $t_i \in T_i, t_j \in T_j$ for $2 \leq i, j \leq k$ and $t_i \neq t_j$ for $i \neq j$. So $|S^*| \geq k$. Let $t'_2 + \dots + t'_k = \max T^*$ with $t'_i \in T_i, t'_j \in T_j$ for $2 \leq i, j \leq k$ and $t'_i \neq t'_j$ for $i \neq j$ and $b \in S_1 \setminus \{a, t'_2, t'_3, \dots, t'_k\}$. Thus $b + t'_2 + t'_3 + \dots + t'_k > \max\{a + t_2 + t_3 + \dots + t_k\}$ and $b + t'_2 + t'_3 + \dots + t'_k \in S^*$. Therefore, we obtain $|S^*| \geq k + 1$. \square

Lemma 7 (see [12]). *Let S_1, S_2 be two sets and let $S_3 = \{a + b \mid a \in S_1, b \in S_2, a \neq b\}$. If $|S_1| \geq 2$ and $|S_2| \geq 3$, then $|S_3| \geq 3$.*

Theorem 8. *If G is a graph with $\text{mad}(G) < 3$, then $\text{Ch}_\Sigma''(G) \leq k$, where $k = \max\{\Delta(G) + 2, 7\}$.*

Proof. The proof is proceeded by contradiction. Assume that G is a minimum counterexample. Let $|L(v)| \geq k$ for each vertex v and $|L(e)| \geq k$ for each edge e in G . For any proper subgraph G' of G , we always assume that there is a neighbor sum distinguishing total- $L_{G'}$ -coloring ϕ of G' by minimality of G . For convenience, we use a total- $L_{G'}$ -coloring ϕ of G' to denote a neighbor sum distinguishing total- $L_{G'}$ -coloring ϕ of G' and we use $F(v) = \{\phi(u), \phi(uv) \mid uv \in E(G')\}$ for $v \in V(G')$ and $F(uv) = \{\phi(u), \phi(v), \phi(ur), \phi(vs) \mid ur \in E(G'), vs \in E(G')\}$ for $uv \in E(G)$.

Let H be the graph obtained by removing all leaves of G . Then H is a connected graph with $\text{mad}(H) \leq \text{mad}(G) < 3$. The properties of the graph H are collected in the following claims.

Claim 1. Each vertex in H has degree of at least 2.

Proof. Suppose to the contrary that H contains a vertex v with $d_H(v) \leq 1$. If $d_H(v) = 0$, then G is the star $K_{1, \Delta(G)-1}$ and $\text{Ch}_\Sigma''(G) = \Delta(G)$; then we obtain a total- L_G -coloring ϕ of G , a contradiction to the choice of G . Assume that $d_H(v) = 1$. Let u and v_i be the neighbors of v where $i = 1, 2, \dots, l = \Delta(G) - 1$ and $d_G(v_i) = 1$. Let $G' = G - vv_1$. First, we uncolor v_i where $i = 1, 2, \dots, \Delta(G) - 1$. Then we color vv_1 with a color in $L(vv_1) \setminus (F(vv_1) \cup \{w(u) - w(v)\})$. Next, we color v_i with a color in $L(v_i) \setminus (F(v_i) \cup \{w(v) - w(v_i)\})$ for $i = 1, 2, \dots, \Delta(G) - 1$; then we obtain a total- L_G -coloring ϕ of G , a contradiction to the choice of G . \square

Claim 2. If $d_H(u) = 2$, then $d_G(u) = 2$.

Proof. Suppose to the contrary that $d_G(u) = k \geq 3$. Let u_1, u_2 be the neighbors of u and v_i be all neighbors of u which are leaves in G for $i = 1, 2, \dots, l = d_G(u) - 2$.

Case 1 ($d_G(u) = 3$). Let $G' = G - v_1$ and $L'(uv_1) = L(uv_1) \setminus (F(uv_1) \cup \{w(u_1) - w(u), w(u_2) - w(u)\})$. We color uv_1 with a color in $L'(uv_1)$ and color v_1 with a color in $L(v_1) \setminus (F(v_1) \cup \{w(u) - w(v_1)\})$. Thus we obtain a total- L_G -coloring ϕ of G , which is a contradiction to the choice of G .

Case 2 ($d_G(u) \geq 4$). Let $G' = G - \{v_1, \dots, v_l\}$, where $l = d_G(u) - 2$. Let $A_i = L(uv_i) - \{\phi(u), \phi(uu_1), \phi(uu_2)\}$, where $i = 1, 2, \dots, l$. Then $|A_i| \geq \Delta(G) - 1 \geq l + 1 \geq 3$, where $i = 1, 2, \dots, l$. By Lemma 6, we have at least $l + 1 \geq 3$ color sets available for the edge set $\{uv_i \mid i = 1, 2, \dots, l\}$ to guarantee $w(u) = w(u_i)$ for $i = 1, 2$. Since at most two color sets may cause $w(u) = w(u_1)$ or $w(u) = w(u_2)$, we have at least one color set available for the edge set $\{uv_i \mid i = 1, 2, \dots, l\}$. Finally, we color v_i with the color in $L(v_i) \setminus (F(v_i) \cup \{w(u) - w(v_i)\})$ for $i = 1, 2, \dots, l = d_G(u) - 2$; then we obtain a total- L_G -coloring ϕ of G , which is a contradiction to the choice of G . \square

Claim 3. A 2-vertex u is not adjacent to a 3-vertex.

Proof. Suppose to the contrary that u is adjacent to a 3-vertex v in H . Let v_1, v_2 be the neighbors of v and s be the other neighbor of u .

Case 1 ($d_G(v) = 3$). Let $G' = G - uv$. First, we uncolor u . Next, we color uv with a color in $L(uv) \setminus (F(uv) \cup \{w(v_1) - w(v), w(v_2) - w(v)\})$. Later, we color u with a color in $L(u) \setminus (F(u) \cup \{w(v) - w(u), w(s) - w(u)\})$; then we obtain a total- L_G -coloring ϕ of G , which is a contradiction to the choice of G .

Case 2 ($d_G(v) \geq 4$). Let x_1, x_2, \dots, x_t be the other neighbors of v such that $d_G(x_i) = 1$ for all $i = 1, 2, \dots, t = d_G(u) - 3$. Let $G' = G - \{uv, vx_1\}$. First, we uncolor all vertices u and $x_i, i = 1, 2, \dots, t$. Consider $L'(vx_1) = L(vx_1) \setminus F(vx_1)$ and $L'(uv) = L(uv) \setminus F(uv)$. We can see that $|L'(vx_1)| \geq 3$ and $|L'(uv)| \geq 2$. By Lemma 7, we can choose $\phi(vx_1) \in L'(vx_1)$ and $\phi(uv) \in L'(uv)$ such that $w(v) \neq w(v_1)$ and $w(v) \neq w(v_2)$. Next, we color u with a color in $L(u) \setminus (F(u) \cup \{w(v) - w(u), w(s) - w(u)\})$ and color x_i with a color in $L(x_i) \setminus (F(x_i) \cup \{w(v) - w(x_i)\})$ for $i = 1, 2, \dots, t$; then we obtain a total- L_G -coloring ϕ of G , which is a contradiction to the choice of G . \square

Claim 4. A 4-vertex u is adjacent to at most two 2-vertices.

Proof. Suppose to the contrary that u is adjacent to three 2-vertices v_1, v_2, v_3 and the other vertex v . Let v'_i be the neighbor of v_i for $i = 1, 2, 3$.

Case 1 ($d_G(u) = 4$). Let $G' = G - uv_1$ and $L'(uv_1) = L(uv_1) \setminus (F(uv_1) \cup \{w(v) - w(u)\})$. First, we uncolor all vertices v_1, v_2, v_3 . Next, we color uv_1 with a color in $L'(uv_1)$ and color v_i with a color in $L(v_i) \setminus (F(v_i) \cup \{w(u) - w(v_i), w(v'_i) - w(v_i)\})$ for $i = 1, 2, 3$. Thus we obtain a total- L_G -coloring ϕ of G , which is a contradiction to the choice of G .

Case 2 ($d_G(u) \geq 5$). Let x_1, x_2, \dots, x_t be the neighbors of u such that $d_G(x_i) = 1$ for all $i = 1, 2, \dots, t = d_G(u) - 4$. Let $G' = G - ux_1$. First, we uncolor vertices v_i and x_j where $1 \leq i \leq 3, 1 \leq j \leq t$. Next, we choose $\phi(ux_1) \in L(ux_1) \setminus (F(ux_1) \cup \{w(v) - w(u)\})$. After that, we color v_i with a color in $L(v_i) \setminus (F(v_i) \cup \{w(u) - w(v_i), w(v'_i) - w(v_i)\})$ for $i = 1, 2, 3$ and color x_j with a color in $L(x_j) \setminus (F(x_j) \cup \{w(u) - w(x_j)\})$ for $j = 1, 2, \dots, t$. Thus we obtain a total- L_G -coloring ϕ of G , which is a contradiction to the choice of G . \square

Claim 5. A 5-vertex u is adjacent to at most four 2-vertices.

Proof. Suppose to the contrary that u is adjacent to five 2-vertices v_1, v_2, v_3, v_4, v_5 . Let x_1, x_2, \dots, x_t be the other neighbors of u (if they exist) such that $d_G(x_i) = 1$ for all $i = 1, 2, \dots, t = d_G(u) - 5$ and v'_i be the neighbor of v_i for $i = 1, 2, 3, 4, 5$. Let $i = 1, 2, 3, 4, 5$ and $j = 1, 2, \dots, t = d_G(u) - 5$ and $G' = G - uv_1$. First, we uncolor vertices v_i and x_j . Next, we color uv_1 with a color in $L(uv_1) \setminus F(uv_1)$. After that, we color v_i with a color in $L(v_i) \setminus (F(v_i) \cup \{w(u) - w(v_i), w(v'_i) - w(v_i)\})$. Finally, we color x_j with a color in $L(x_j) \setminus (F(x_j) \cup \{w(u) - w(x_j)\})$. Thus we obtain a total- L_G -coloring ϕ of G , which is a contradiction to the choice of G . \square

By Claim 1, we have $\Delta(H) \geq 2$.

Suppose that $\Delta(H) = 2$. By Claims 1 and 2, G is a cycle. One can obtain that $Ch''_2(G) \leq 7$, a contradiction to the choice of G .

Suppose that $\Delta(H) = 3$. By Claim 3, H is a 3-regular graph. Thus we have $mad(H) = 3$, which is a contradiction.

Suppose that $\Delta(H) \geq 4$. We complete the proof by using the discharging method. Define an initial charge function $ch(v) = d_H(v)$ for every $v \in V(H)$. Next, rearrange the weights according to the designed rule. When the discharging is finished, we have a new charge $ch'(v)$. However, the sum of all charges is kept fixed. Finally, we want to show that $ch'(v) \geq 3$ for all $v \in V(H)$. This leads to the following contradiction:

$$\begin{aligned} 3 &= \frac{3|V(H)|}{|V(H)|} \leq \frac{\sum_{v \in V(H)} w'(v)}{|V(H)|} = \frac{\sum_{v \in V(H)} w(v)}{|V(H)|} \\ &= \frac{2|E(H)|}{|V(H)|} \leq mad(H) < 3. \end{aligned} \tag{1}$$

Let $v \in V(H)$. Assume that $d_H(v) = 2$ and $uv \in E(H)$. Then vertex u gives charge $1/2$ to v .

Consider a vertex $v \in V(H)$. By Claim 1, we have $d_H(v) \geq 2$.

If $d_H(v) = 2$, then v is adjacent to at least two 4-vertices by Claim 3. Hence $ch'(v) \geq ch(v) + (2 \times (1/2)) = 3$.

If $d_H(v) = 3$, then $ch'(v) = ch(v) = 3$.

If $d_H(v) = 4$, then v is adjacent to at most two 2-vertices by Claim 4. Hence $ch'(v) \geq ch(v) - (2 \times (1/2)) = 3$.

If $d_H(v) = 5$, then v is adjacent to at most four 2-vertices by Claim 5. Hence $ch'(v) \geq ch(v) - (4 \times (1/2)) = 3$.

If $d_H(v) \geq 6$, then $ch'(v) \geq ch(v) - ((1/2)d_H(v)) = (1/2)d_H(v) \geq 3$.

From the above discussion, we have $\sum_{v \in V(H)} ch'(v) \geq 3$, which is a contradiction. This completes the proof of Theorem 8.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

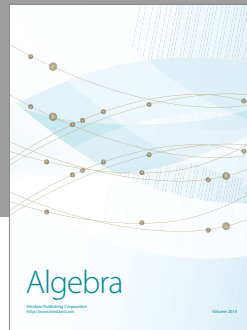
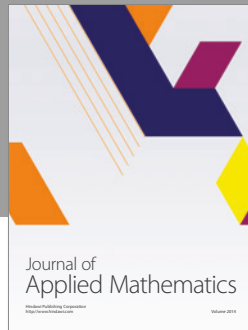
Acknowledgments

The first author is supported by University of Phayao, Thailand. In addition, the authors would like to thank Dr. Keaitsuda Nakprasit for her helpful comments.

References

- [1] Z. Zhang, X. E. Chen, J. Li, B. Yao, X. Lu, and J. Wang, "On adjacent-vertex-distinguishing total coloring of graphs," *Science China Mathematics*, vol. 48, no. 3, pp. 289–299, 2005.
- [2] X. Chen, "On the adjacent vertex distinguishing total coloring numbers of graphs with $\Delta = 3$," *Discrete Mathematics*, vol. 308, no. 17, pp. 4003–4007, 2008.
- [3] W. Wang and D. Huang, "The adjacent vertex distinguishing total coloring of planar graphs," *Journal of Combinatorial Optimization*, vol. 27, no. 2, pp. 379–396, 2014.
- [4] W. Wang and P. Wang, "On adjacent-vertex-distinguishing total coloring of K_4 -minor free graphs," *Sci. China Ser. A*, vol. 39, no. 12, pp. 1462–1472, 2009.

- [5] M. Piłśniak and M. Woźniak, "On the total-neighbor-distinguishing index by sums," *Graphs and Combinatorics*, vol. 31, no. 3, pp. 771–782, 2015.
- [6] H. Li, B. Liu, and G. Wang, "Neighbor sum distinguishing total colorings of K_4 -minor free graphs," *Frontiers of Mathematics in China*, vol. 8, no. 6, pp. 1351–1366, 2013.
- [7] H. Li, L. Ding, B. Liu, and G. Wang, "Neighbor sum distinguishing total colorings of planar graphs," *Journal of Combinatorial Optimization*, vol. 30, no. 3, pp. 675–688, 2015.
- [8] J. H. Wang, Q. L. Ma, and X. Han, "Neighbor sum distinguishing total colorings of triangle free planar graphs," *Acta Mathematica Sinica*, vol. 31, no. 2, pp. 216–224, 2015.
- [9] X. Cheng, D. Huang, G. Wang, and J. Wu, "Neighbor sum distinguishing total colorings of planar graphs with maximum degree Δ ," *Discrete Applied Mathematics: The Journal of Combinatorial Algorithms, Informatics and Computational Sciences*, vol. 190–191, pp. 34–41, 2015.
- [10] C. Qu, G. Wang, J. Wu, and X. Yu, "On the neighbor sum distinguishing total coloring of planar graphs," *Theoretical Computer Science*, vol. 609, no. part 1, pp. 162–170, 2016.
- [11] H. J. Song, W. H. Pan, X. N. Gong, and C. Q. Xu, "A note on the neighbor sum distinguishing total coloring of planar graphs," *Theoretical Computer Science*, vol. 640, pp. 125–129, 2016.
- [12] A. J. Dong and G. H. Wang, "Neighbor sum distinguishing total colorings of graphs with bounded maximum average degree," *Acta Mathematica Sinica*, vol. 30, no. 4, pp. 703–709, 2014.
- [13] V. G. Vizing, "Vertex colorings with given colors" (Russian), *Metody Diskret. Analiz.*, 29, 3–10.
- [14] P. Erdős, A. L. Rubin, and H. Taylor, "Choosability in graphs," in *In Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, Congr. Num.*, vol. 26, pp. 125–157, 1979.
- [15] C. Qu, G. Wang, G. Yan, and X. Yu, "Neighbor sum distinguishing total choosability of planar graphs," *Journal of Combinatorial Optimization*, vol. 32, no. 3, pp. 906–916, 2016.
- [16] J. Yao, X. Yu, G. Wang, and C. Xu, "Neighbor sum (set) distinguishing total choosability of d -degenerate graphs," *Graphs and Combinatorics*, vol. 32, no. 4, pp. 1611–1620, 2016.
- [17] J. Wang, J. Cai, and Q. Ma, "Neighbor sum distinguishing total choosability of planar graphs without 4-cycles," *Discrete Applied Mathematics: The Journal of Combinatorial Algorithms, Informatics and Computational Sciences*, vol. 206, pp. 215–219, 2016.



Hindawi

Submit your manuscripts at
<https://www.hindawi.com>

