Hindawi Publishing Corporation
ISRN Applied Mathematics
Volume 2014, Article ID 805798, 14 pages
http://dx.doi.org/10.1155/2014/805798

Research Article

Asymptotic Stability Analysis and Optimality Algorithm for Uncertain Neutral Systems with Saturation

Xinghua Liu

Department of Auto, School of Information Science and Technology, University of Science and Technology of China, Anhui 230027, China

Correspondence should be addressed to Xinghua Liu; salxh@mail.ustc.edu.cn

Received 29 December 2013; Accepted 18 February 2014; Published 27 March 2014

Academic Editors: C.-Y. Lu and X. Meng

The certain and uncertain neutral systems with time-delay and saturating actuator are considered in this paper. In order to analyse and optimize the system, auxiliary functions are presented based on additive decomposition approach and the relationship among them is discussed. As the novel stability criterion, two sufficient conditions are obtained for asymptotic stability of the neutral systems. Furthermore, the paper gives the stability analysis algorithm and optimality algorithm to optimize the result. Finally, from the two-stage dissolution tank of solid caustic soda in a chemical plant, three numerical examples are implemented to show the effectiveness of the proposed method.

1. Introduction

Delay is often inevitable in various practical systems; examples include population ecology [1], steam or water pipes, heat exchanges [2], and many others [3–5]. In the control engineering language, these delays can be categorized as state delay, input or output delay (retarded systems), delay in the state derivative (neutral systems), and so forth. Guaranteeing the stability of systems with delay is one core design objective both in theory and in practice. Particularly, in terms of neutral systems, the focus has mainly been on systems with identical delays in neutral and discrete terms [6–10]. Results also exist that depend only on the size of the discrete delays but not on the size of the neutral delays [11–13].

Besides delays, the saturated controller is apt to cause instability as well. In the presence of actuator saturation, the problem of estimating asymptotic stability regions for linear systems subject to it has been studied by many researches in the past years in [14]. Generally speaking, the existing methods for estimating the stability regions for linear systems with saturating actuators are based on the concept of Lyapunov level set. LMI optimization-based approaches were proposed to estimate the stability regions by using quadratic Lyapunov functions and the Lur's-type Lyapunov functions in [15–19].

For the studies in response to both issues of delay and saturation, the sufficient conditions for systems with delay and saturated actuator are obtained in [18, 20–22]; Lyapunov-Krasovskii functional is employed to investigate the delay-dependent robust stabilization for uncertain neutral systems with saturated actuators in [20]; a controller is constructed in terms of linear matrix inequalities using descriptor model transformation in [18], just to name a few. However, this paper wants to provide a new method to find the system stability region and give the optimality algorithm to obtain the largest region in this method. Besides, it gives the application in the chemical process of the plant.

In this paper, a novel Lyapunov functional is proposed based on the delay-dividing approach, which leads to less conservative stability conditions for linear systems with time-delay and saturated actuator. This is done by introducing auxiliary functions based on the additive decomposition approach [23]. We also propose an algorithm to obtain the optimal auxiliary function. Finally, we design a two-stage dissolution tank of the chemical process by modeling it as a neutral delay system with actuator saturation and demonstrate the effectiveness of the proposed method.

Notations. $\star$ denotes the symmetric part to be a symmetric block matrix, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ is the set of all real $m \times n$ matrices. I is the identity matrix with proper dimensions. C_0 is the set of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ where τ is a constant representing the neutral time-delay. A^T is the transpose of matrix A. $|A| = [|a_{ij}|]$, with $A = [a_{ij}]$. For real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). $\lambda_{\max\,(\min)}(A)$ is the eigenvalue of matrix A with maximum (minimum) real part. $\|v\|$ is the Euclidean norm of vector v, $\|v\| = (v^T v)^{1/2}$, while $\|A\|$ is spectral norm of matrix A, $\|A\| = [\lambda_{\max}(A^T A)]^{1/2}$. ρ represents the domain of attraction. diag$\{\cdots\}$ denotes a block-diagonal matrix decided by the corresponding elements in the brace and finally $B(\rho) = \{y \in R^n : \|y\| \leq \rho\}$.

2. Problem Statement and Preliminaries

The following neutral system with time-delay and actuator saturation is considered:

$$\dot{x}(t) - D\dot{x}(t-\tau) = Ax(t) + Bx(t-h) + C\,\mathrm{Sat}(u(t)), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state and $u(t)$ is the control input. $h > 0$ is the constant discrete time delay and $\tau > 0$ is the constant neutral time-delay. A, B, C, and D are known real constant parameter matrices of appropriate dimensions with $\|D\| < 1$. Sat$(\cdot)$ is used to denote the standard saturation function defined for $u = [u_1, u_2, \ldots, u_m]$:

$$\mathrm{Sat}(u) = [\mathrm{Sat}(u_1), \mathrm{Sat}(u_2), \ldots, \mathrm{Sat}(u_m)]^T,$$
$$\mathrm{Sat}(u_i) = \begin{cases} u_i^+ & \text{if } u_i > u_i^+ \\ u_i & \text{otherwise} \\ -u_i^+ & \text{if } u_i < -u_i^+. \end{cases} \quad (2)$$

The following linear state feedback is to be designed:

$$u(t) = -Kx(t), \quad (3)$$

where the linear state feedback gain $K = [K_1^T, K_2^T, \ldots, K_m^T]^T$; $K_i \in \mathbb{R}^n$ is an n-dimensional row vector.

Here we have slightly abused the notation by using Sat to denote both a scalar valued and a vector valued function. We have also assumed a unity saturation level for the saturation function without loss of generality.

Define $Dz(u(t)) = u(t) - \mathrm{Sat}(u(t))$ where $Dz(u(t)) = [dz(u_1), dz(u_2), \ldots, dz(u_m)]^T$ and

$$dz(u_i) = \begin{cases} u_i - u_i^+ & \text{if } u_i > u_i^+ \\ 0 & \text{otherwise} \\ u_i + u_i^+ & \text{if } u_i < -u_i^+. \end{cases} \quad (4)$$

The saturated system can now be written as follows:

$$\dot{x}(t) - D\dot{x}(t-\tau) = A_c x(t) + Bx(t-h) + CDz(Kx(t)), \quad (5)$$

where $A_c = A - CK$.

The neutral system (5) then leads to the following by model transformation:

$$\frac{d}{dt}[\mathscr{L}(x(t))] = \widehat{A}_c x(t) + CDz(Kx), \quad (6)$$

where $\widehat{A}_c = A_c + B = A + B - CK$ and $\mathscr{L}(x(t)) = x(t) - Dx(t-\tau) + B\int_{t-h}^{t} x(s)ds$.

The following definitions and lemmas are required before proceeding with the main contributions presented in the next section.

Definition 1 (see [24]). The operator $\mathscr{L} : C_0 \to \mathbb{R}^n$ is said to be stable if the zero solution of the homogeneous difference equation

$$\mathscr{L}(x(t)) = 0, \quad t \geq 0,$$
$$x_0 = \psi \in \{\varphi \in C_0 : \mathscr{L}\varphi = 0\} \quad (7)$$

is uniformly asymptotically stable. The stability of operator $\mathscr{L}$ is necessary for the stability of neutral system (1) with (3), which is always satisfied when $\|h|B| + D\| < 1$.

Lemma 2 (see [24]). *For any matrix $B, D \in \mathbb{R}^{m\times n}$, if $\|h|B| + D\| < 1$, then the operator $\mathscr{L} : C_0 \to \mathbb{R}^n$ with*

$$\mathscr{L}(x(t)) = x(t) + B\int_{t-h}^{t} x(s)\,ds - Dx(t-\tau) \quad (8)$$

is stable.

Lemma 3 (see [25]). *Let $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$, and let $x \in \mathbb{R}^n$. Then we have*

$$x^T UVx \leq \|UV\|\, x^T x \leq \|U\|\,\|V\|\, x^T x,$$
$$2x^T UVx \leq x^T\left(\frac{1}{\gamma}UU^T + \gamma V^T V\right)x, \quad \forall \gamma > 0. \quad (9)$$

Definition 4 (see [23]). Auxiliary functions $W_1(\rho_1)$ and $W_2(\rho_2)$ are described in the following

$$W_1(\rho_1) = \begin{cases} \sum_{i=1}^{m}\left(1 - \frac{u_i^+}{\|K_i\|\rho_1}\right)^2 K_i^T K_i & \text{if } \|K_i\|\rho_1 > u_i^+, \\ 0 & \text{otherwise,} \end{cases}$$
$$W_2(\rho_2) = \begin{cases} \sum_{i=1}^{m}\left(\frac{\rho_2}{4u_i^+}\right)^2 \left(K_i^T K_i\right)^2 & \text{if } \|K_i\|\rho_2 > u_i^+, \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

3. Main Results

In this section, we firstly construct the auxiliary functions $W_3(\rho_3), W_4(\rho_4), \ldots, W_n(\rho_n)$ using the geometric method and with them present a new delay-dependent stabilization criterion. Then the relationship between those auxiliary functions is explored which helps to obtain the optimal $W_i(\rho_i)$.

Consider the polynomial function $y = ax^n$ which is tangent with saturated function $y = x - u_i^+$ in positive axis. a can be firstly determined by the geometric relation and auxiliary polynomial functions can be obtained.

Let

$$\begin{aligned} y &= ax^n, \\ y &= x - u_i^+. \end{aligned} \tag{11}$$

The slope of polynomial function $y = ax^n$ equals to 1 at the tangent point. Thus we have

$$\dot{y} = anx^{n-1} = 1. \tag{12}$$

It can be determined by solving the resulting equations simultaneously and we obtain

$$a = \left(\frac{n-1}{u_i^+}\right)^{n-1} \frac{1}{n^n}. \tag{13}$$

Noting in the polynomial function, we replace x with u_i here so that they have the same independent variable with $dz(u_i)$. Consider

$$y = \left(\frac{n-1}{u_i^+}\right)^{n-1} \frac{1}{n^n} u_i^n. \tag{14}$$

Thus we compare the nonlinear function $dz(u_i)$ with the polynomial function above. For simplicity, we consider the cases $n = 2$ and $n = 3$. As shown in Figure 1, the graphics of quadratic and cubic polynomial functions are above the graphic of $|dz(u_i)|$.

Furthermore, according to the nature of the polynomial function when $n = 4, 5, \ldots$, the graphic of function $y = ((n-1)/u_i^+)^{n-1}(1/n^n)u_i^n$ is also above the graphic of $|dz(u_i)|$. So we have the following inequality:

$$\left|dz(u_i)\right| \le \left(\frac{n-1}{u_i^+}\right)^{n-1} \frac{1}{n^n} u_i^n, \tag{15}$$

where $n = 2, 3, 4, \ldots$.

Lemma 5. *When $n = 2, 3, 4, \ldots$, the following inequality holds:*

$$Dz^T(Kx(t))\,Dz(Kx(t)) \le x^T W_n(\rho_n)\,x, \tag{16}$$

where

$$W_n(\rho_n) = \begin{cases} \displaystyle\sum_{i=1}^{m} \left(\frac{(n-1)\rho_n}{n^{n/(n-1)} u_i^+}\right)^{2n-2} \left(K_i^T K_i\right)^n & \text{if } \|K_i\|\,\rho_n > u_i^+ \\ 0 & \text{otherwise}. \end{cases} \tag{17}$$

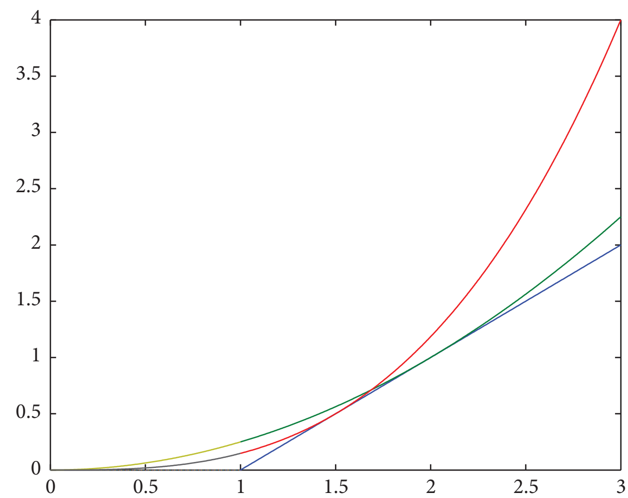

Figure 1: Two polynomial functions are $y = (1/4u_i^+)u_i^2$ and $y = (4/27)(1/u_i^+)^2 u_i^3$ (when $u_i^+ = 2$) and the piecewise linear function is $dz(u_i)$.

Proof. From (15) we obtain that

$$\begin{aligned} Dz^T(Kx)\,Dz(Kx) &= \sum_{i=1}^{m} dz^2(u_i) \\ &\le \sum_{i=1}^{m} \left(\frac{n-1}{u_i^+}\right)^{2n-2} \frac{1}{n^{2n}} u_i^{2n} \\ &\le \sum_{i=1}^{m} \left(\frac{n-1}{u_i^+}\right)^{2n-2} \\ &\quad \times \frac{1}{n^{2n}} x^T K_i^T K_i x \cdots x^T K_i^T K_i x \\ &\le \sum_{i=1}^{m} \left(\frac{n-1}{u_i^+}\right)^{2n-2} \\ &\quad \times \frac{1}{n^{2n}} \left\|xx^T\right\|^{n-1} x^T K_i^T K_i \cdots K_i^T K_i x \\ &\le \sum_{i=1}^{m} \left(\frac{n-1}{u_i^+}\right)^{2n-2} \frac{1}{n^{2n}} \rho_n^{2n-2} x^T \left(K_i^T K_i\right)^n x \\ &\le x^T W_n(\rho_n)\,x. \end{aligned} \tag{18}$$

This completes the proof. □

3.1. Asymptotic Stability for Certain Neutral System

Theorem 6. *The neutral system with time-delay and actuator saturation as described in (1) and (3) is asymptotic stability if $\|h|B| + D\| < 1$ and there exist scalars $\gamma_i > 0$, $i = 1, 2, 3$, $P > 0$, $\widehat{P} > 0$, $Q = [Q_{ij}]_{2\times 2}$, $Q_{ij} > 0$, $R = [R_{ij}]_{2\times 2}$, $R_{ij} > 0$, $T_i > 0$*

$(i = 1, 2, 3, 4, 5, 6)$, *and* $\widetilde{T}_i > 0$ $(i = 3, 4)$ *such that the following symmetric linear matrix inequality holds:*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & \Phi_{14} & \Phi_{15} & \Phi_{16} & 0 \\ \star & \Phi_{22} & \Phi_{23} & 0 & 0 & 0 & 0 \\ \star & \star & \Phi_{33} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \Phi_{44} & \Phi_{45} & 0 & 0 \\ \star & \star & \star & \star & \Phi_{55} & 0 & 0 \\ \star & \star & \star & \star & \star & \Phi_{66} & 0 \\ \star & \star & \star & \star & \star & \star & \Phi_{77} \end{bmatrix} < 0, \quad (19)$$

where

$$W_n(\rho) = \begin{cases} \sum_{i=1}^{m} \left(\dfrac{(n-1)\rho_n}{n^{n/(n-1)} u_i^+} \right)^{2n-2} \left(K_i^T K_i \right)^n & \text{if } \|K_i\| \rho_n > u_i^+, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Phi_{11} = \widehat{A}_c^T P + P \widehat{A}_c + \frac{1}{\gamma_1} \widehat{P} + (\gamma_1 + \gamma_2 + \gamma_3) \left\| C^T C \right\| W_i(\rho_i) + Q_{11} + R_{11} + T_1 + T_2 + \frac{h^2}{2} \left(\widetilde{T}_3 + T_5 \right) + \frac{\tau^2}{2} \left(\widetilde{T}_4 + T_6 \right),$$

$$\Phi_{22} = Q_{22} - Q_{11},$$

$$\Phi_{33} = -Q_{22} - T_1,$$

$$\Phi_{44} = R_{22} - R_{11},$$

$$\Phi_{55} = \frac{1}{\gamma_2} D^T \widehat{P} D - R_{22} - T_2,$$

$$\Phi_{66} = \frac{1}{\gamma_3} B^T \widehat{P} B - T_3,$$

$$\Phi_{77} = -T_4,$$

$$\Phi_{12} = Q_{12},$$

$$\Phi_{14} = R_{12},$$

$$\Phi_{15} = -\widehat{A}_c^T P D,$$

$$\Phi_{16} = \widehat{A}_c^T P B,$$

$$\Phi_{23} = -Q_{12},$$

$$\Phi_{45} = -R_{12}. \quad (20)$$

Proof. Define a legitimate Lyapunov functional candidate as follows:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_4(x(t)) + V_5(x(t)) + V_6(x(t)), \quad (21)$$

where

$$V_1(x(t)) = \mathscr{L}^T(x(t)) P \mathscr{L}(x(t)),$$

$$V_2(x(t)) = \int_{t-(h/2)}^{t} \begin{bmatrix} x(s) \\ x\left(s - \dfrac{h}{2}\right) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22} \end{bmatrix} \times \begin{bmatrix} x(s) \\ x\left(s - \dfrac{h}{2}\right) \end{bmatrix} ds + \int_{t-(\tau/2)}^{t} \begin{bmatrix} x(s) \\ x\left(s - \dfrac{\tau}{2}\right) \end{bmatrix}^T \times \begin{bmatrix} R_{11} & R_{12} \\ \star & R_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ x\left(s - \dfrac{\tau}{2}\right) \end{bmatrix} ds,$$

$$V_3(x(t)) = \int_{t-h}^{t} x^T(s) T_1 x(s) ds + \int_{t-\tau}^{t} x^T(s) T_2 x(s) ds,$$

$$V_4(x(t)) = \int_{t-h}^{t} \left[\int_s^t x^T(s_1) ds_1 \right] T_3 \left[\int_s^t x(s_2) ds_2 \right] ds,$$

$$V_5(x(t)) = \int_{t-\tau}^{t} \left[\int_s^t x^T(s_1) ds_1 \right] T_4 \left[\int_s^t x^T(s_2) ds_2 \right] ds, \quad (22)$$

$$V_6(x(t)) = \frac{1}{2} \int_{t-h}^{t} (s - t + h)^2 x^T(s) T_5 x(s) ds + \frac{1}{2} \int_{t-\tau}^{t} (s - t + \tau)^2 x^T(s) T_6 x(s) ds, \quad (23)$$

where $P > 0$, $Q = [Q_{ij}]_{2\times2}$, $Q_{ij} > 0$, $R = [R_{ij}]_{2\times2}$, $R_{ij} > 0$, and $T_i > 0$, $i = 1, 2, 3, 4, 5, 6$.

Then

$$\dot{V}(x(t)) = \dot{V}_1(x(t)) + \dot{V}_2(x(t)) + \dot{V}_3(x(t)) + \dot{V}_4(x(t)) + \dot{V}_5(x(t)) + \dot{V}_6(x(t)). \quad (24)$$

By (19)-(22), we obtain that

$$\begin{aligned} \dot{V}_1(x(t)) &= 2 \left\{ x^T(t) - x^T(t-\tau) D^T + \left[\int_{t-h}^{t} x^T(s) ds \right] B^T \right\} \times P \left[\widehat{A}_c x(t) + CDz(Kx) \right] \\ &= 2x^T P \widehat{A}_c x + 2x^T PCDz(Kx) - 2x^T \widehat{A}_c^T PDx(t-\tau) \\ &\quad - 2x^T(t-\tau) D^T PCDz(Kx) \\ &\quad + 2x^T(t) \widehat{A}_c^T PB \left[\int_{t-h}^{t} x(s) ds \right] \\ &\quad + 2 \left[\int_{t-h}^{t} x^T(s) ds \right] B^T PCDz(Kx) \end{aligned}$$

$$\leq x^T\left[\widehat{A}_c^T P + P\widehat{A}_c + \frac{1}{\gamma_1}\widehat{P} + (\gamma_1+\gamma_2+\gamma_3)\left\|C^T C\right\| W_i(\rho_i)\right]x + \frac{1}{\gamma_2}x^T(t-\tau)D^T\widehat{P}Dx(t-\tau) + \frac{1}{\gamma_3}\left[\int_{t-h}^{t} x^T(s)\,ds\right]B^T\widehat{P}B\left[\int_{t-h}^{t} x(s)\,ds\right] - 2x^T(t)\widehat{A}_c^T PDx(t-\tau) + 2x^T(t)\widehat{A}_c^T PB\left[\int_{t-h}^{t} x(s)\,ds\right], \quad (25)$$

where $\widehat{P} > 0$, since $\widehat{P} = P^2$. Consider

$$\dot{V}_2(x(t)) = \begin{bmatrix} x(t) \\ x\left(t-\frac{h}{2}\right)\end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22}\end{bmatrix} \begin{bmatrix} x(t) \\ x\left(t-\frac{h}{2}\right)\end{bmatrix} - \begin{bmatrix} x\left(t-\frac{h}{2}\right) \\ x(t-h)\end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22}\end{bmatrix} \begin{bmatrix} x\left(t-\frac{h}{2}\right) \\ x(t-h)\end{bmatrix} + \begin{bmatrix} x(t) \\ x\left(t-\frac{\tau}{2}\right)\end{bmatrix}^T \begin{bmatrix} R_{11} & R_{12} \\ \star & R_{22}\end{bmatrix} \begin{bmatrix} x(t) \\ x\left(t-\frac{\tau}{2}\right)\end{bmatrix} - \begin{bmatrix} x\left(t-\frac{\tau}{2}\right) \\ x(t-\tau)\end{bmatrix}^T \begin{bmatrix} R_{11} & R_{12} \\ \star & R_{22}\end{bmatrix} \begin{bmatrix} x\left(t-\frac{\tau}{2}\right) \\ x(t-\tau)\end{bmatrix},$$

$$\dot{V}_3(x(t)) = x^T(t)T_1x(t) - x^T(t-h)T_1x(t-h) + x^T(t)T_2x(t) - x^T(t-\tau)T_2x(t-\tau),$$

$$\begin{aligned}\dot{V}_4(x(t)) &= -\left[\int_{t-h}^{t} x^T(s_1)\,ds_1\right]T_3\left[\int_{t-h}^{t} x(s_2)\,ds_2\right] + \int_{t-h}^{t} x^T(t)T_3\left[\int_{s}^{t} x(s_2)\,ds_2\right]ds + \int_{t-h}^{t}\left[\int_{s}^{t} x(s_2)\,ds_2\right]T_3x(t)\,ds \\ &= -\left[\int_{t-h}^{t} x^T(s_1)\,ds_1\right]T_3\left[\int_{t-h}^{t} x(s_2)\,ds_2\right] + \int_{t-h}^{t}(s-t+h)x^T(t)T_3x(s)\,ds + \int_{t-h}^{t}(s-t+h)x^T(s)T_3x(t)\,ds \\ &= -\left[\int_{t-h}^{t} x^T(s_1)\,ds_1\right]T_3\left[\int_{t-h}^{t} x(s_2)\,ds_2\right] - \int_{t-h}^{t}(s-t+h)\left(T_3x(t)-T_5x(s)\right)^T \times T_5^{-1}\left(T_3x(t)-T_5x(s)\right)ds + \frac{h^2}{2}x^T(t)T_3T_5^{-1}T_3x(t) + \int_{t-h}^{t}(s-t+h)x^T(s)T_5x(s)\,ds \\ &\leq -\left[\int_{t-h}^{t} x^T(s_1)\,ds_1\right]T_3\left[\int_{t-h}^{t} x(s_2)\,ds_2\right] + \frac{h^2}{2}x^T(t)\widetilde{T}_3x(t) + \int_{t-h}^{t}(s-t+h)x^T(s)T_5x(s)\,ds,\end{aligned} \quad (26)$$

where $\widetilde{T}_3 > 0$, since $\widetilde{T}_3 = T_3T_5^{-1}T_3$.

Similarly we have

$$\dot{V}_5(x(t)) \leq -\left[\int_{t-\tau}^{t} x^T(s_1)\,ds_1\right]T_4\left[\int_{t-\tau}^{t} x(s_2)\,ds_2\right] + \frac{\tau^2}{2}x^T(t)\widetilde{T}_4x(t) + \int_{t-\tau}^{t}(s-t+\tau)x^T(s)T_6x(s)\,ds, \quad (27)$$

where $\widetilde{T}_4 > 0$, since $\widetilde{T}_4 = T_4T_6^{-1}T_4$. Consider

$$\dot{V}_6(x(t)) = \frac{1}{2}h^2x^T(t)T_5x(t) - \int_{t-h}^{t}(s-t+h)x^T(s)T_5x(s)\,ds + \frac{1}{2}\tau^2x^T(t)T_6x(t) - \int_{t-\tau}^{t}(s-t+\tau)x^T(s)T_6x(s)\,ds. \quad (28)$$

Substituting these into (24), the time-derivative of V has new upper bound as follows:

$$\dot{V}(x(t)) \leq \xi^T(t)\Phi\xi(t), \quad (29)$$

where

$$\xi^T(t) = \left[x^T(t)\ \ x^T\left(t-\frac{h}{2}\right)\ \ x^T(t-h)\ \ x^T\left(t-\frac{\tau}{2}\right)\ \ x^T(t-\tau)\ \ \int_{t-h}^{t} x^T(s)\,ds\ \ \int_{t-\tau}^{t} x^T(s)\,ds\right]; \tag{30}$$

Φ is defined as stated in (19).

If linear matrix inequality (19) is feasible, then we can get $\dot{V}(x(t)) < 0$, for all $x \in B(\rho)$. Therefore, if constant scalar h, constant parameter matrices B, D such that $\|h|B| + D\| < 1$ and there exist $P > 0$, $\widehat{P} > 0$, $Q = [Q_{ij}]_{2\times 2}$, $Q_{ij} > 0$, $R = [R_{ij}]_{2\times 2}$, $R_{ij} > 0$, $T_i > 0$ $(i = 1,2,3,4,5,6)$, and $\widetilde{T}_i > 0$ $(i = 3,4)$ satisfying (19) for real scalars $\gamma_i > 0$ $(i = 1,2,3)$, from Hale and Verduyn Lunel [24], we can draw the neutral system which can be described by (1) and (3) is asymptotic stability. This completes the proof. □

Remark 7. $W_i(\rho_i)(i = 1,2,\ldots,n)$ are created by the parameter ρ_i which is a measure tool for domain of attraction. With these functions, we obtain the novel stability criterion. However, in Theorem 6 we need to look for the largest value of ρ_i with the optimal $W_i(\rho_i)$. These can be seen in Section 3.3 below.

Remark 8. Theorem 6 gives a delay-dependent stability criterion for neutral system with (1) and (3) using a delay-dividing approach. The delay differential conditions in other works, such as in [26], are usually more strict. These facts mean that our result is less conservative than some previous approaches.

The delay-dependent stability criterion for system (1) with $\tau \equiv h$ is presented in the following corollary.

Corollary 9. *The neutral systems* (1) *and* (3) *with* $\tau \equiv h$ *are asymptotic stability if* $\|h|B| + D\| < 1$ *and there exist* $P > 0$, $\widehat{P} > 0$, $Q = [Q_{ij}]_{2\times 2}$, $Q_{ij} > 0$, $T_i > 0$ $(i = 1,2,3)$, *and* $\widetilde{T}_2 > 0$ *such that the following symmetric linear matrix inequality holds for real constant scalars* $\gamma_i > 0$, $i = 1,2,3$:

$$\overline{\Phi} = \begin{bmatrix} \overline{\Phi}_{11} & \overline{\Phi}_{12} & \overline{\Phi}_{13} & \overline{\Phi}_{14} \\ \star & \overline{\Phi}_{22} & \overline{\Phi}_{23} & 0 \\ \star & \star & \overline{\Phi}_{33} & 0 \\ \star & \star & \star & \overline{\Phi}_{44} \end{bmatrix} < 0, \tag{31}$$

where $W_i(\rho_i)$, $i = 1,2,3,\ldots,n$, *are defined as before. Consider*

$$\begin{aligned}
\overline{\Phi}_{11} &= \widehat{A}_c^T P + P\widehat{A}_c + \frac{1}{\gamma_1}\widehat{P} + (\gamma_1+\gamma_2+\gamma_3)\left\|C^T C\right\| W_i(\rho_i) \\
&\quad + Q_{11} + T_1 + \frac{h^2}{2}\left(\widetilde{T}_2 + T_3\right), \\
\overline{\Phi}_{22} &= Q_{22} - Q_{11}, \\
\overline{\Phi}_{33} &= \frac{1}{\gamma_2} D^T \widehat{P} D - Q_{22} - T_1, \\
\overline{\Phi}_{44} &= \frac{1}{\gamma_3} B^T \widehat{P} B - T_2, \\
\overline{\Phi}_{12} &= Q_{12}, \\
\overline{\Phi}_{13} &= -\widehat{A}_c^T P D, \\
\overline{\Phi}_{14} &= \widehat{A}_c^T P B, \\
\overline{\Phi}_{23} &= -Q_{12}.
\end{aligned} \tag{32}$$

Proof. Define a legitimate Lyapunov functional candidate as

$$\begin{aligned} V(x(t)) &= V_1(x(t)) + V_2(x(t)) \\ &\quad + V_3(x(t)) + V_4(x(t)) + V_5(x(t)), \end{aligned} \tag{33}$$

where

$$\begin{aligned}
V_1(x(t)) &= \mathscr{L}^T(x(t))\,P\mathscr{L}(x(t)), \\
V_2(x(t)) &= \int_{t-(h/2)}^{t} \begin{bmatrix} x(s) \\ x\left(s-\frac{h}{2}\right) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ x\left(s-\frac{h}{2}\right) \end{bmatrix} ds, \\
V_3(x(t)) &= \int_{t-h}^{t} x^T(s)\,T_1 x(s)\,ds, \\
V_4(x(t)) &= \int_{t-h}^{t} \left[\int_s^t x^T(s_1)\,ds_1\right] T_2 \left[\int_s^t x(s_2)\,ds_2\right] ds, \\
V_5(x(t)) &= \frac{1}{2}\int_{t-h}^{t} (s-t+h)^2 x^T(s)\,T_3 x(s)\,ds,
\end{aligned} \tag{34}$$

where $P > 0$, $Q = [Q_{ij}]_{2\times 2}$, $Q_{ij} > 0$, and $T_i > 0$, $i = 1,2,3$.

According to (34) we obtain

$$\begin{aligned}
\dot{V}_1(x(t)) \le{}& x^T\left[\widehat{A}_c^T P + P\widehat{A}_c + \frac{1}{\gamma_1}\widehat{P}\right. \\
&\left. + (\gamma_1+\gamma_2+\gamma_3)\left\|C^T C\right\| W_i(\rho_i)\right] x \\
&+ \frac{1}{\gamma_2} x^T(t-h)\,D^T\widehat{P}Dx(t-h) \\
&+ \frac{1}{\gamma_3}\left[\int_{t-h}^{t} x^T(s)\,ds\right] B^T\widehat{P}B\left[\int_{t-h}^{t} x(s)\,ds\right]
\end{aligned}$$

$$- 2x^T(t)\widehat{A}_c^T PDx(t-h) + 2x^T(t)\widehat{A}_c^T PB\left[\int_{t-h}^{t} x(s)\,ds\right], \quad (35)$$

where $\widehat{P} > 0$, since $\widehat{P} = P^2$. Consider

$$\begin{aligned}
\dot{V}_2(x(t)) &= \begin{bmatrix} x(t) \\ x\left(t-\frac{h}{2}\right)\end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22}\end{bmatrix} \begin{bmatrix} x(t) \\ x\left(t-\frac{h}{2}\right)\end{bmatrix} \\
&\quad - \begin{bmatrix} x\left(t-\frac{h}{2}\right) \\ x(t-h)\end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22}\end{bmatrix} \begin{bmatrix} x\left(t-\frac{h}{2}\right) \\ x(t-h)\end{bmatrix}, \\
\dot{V}_3(x(t)) &= x^T(t)T_1x(t) - x^T(t-h)T_1x(t-h), \\
\dot{V}_4(x(t)) &\le -\left[\int_{t-h}^{t} x^T(s_1)\,ds_1\right] T_2 \left[\int_{t-h}^{t} x(s_2)\,ds_2\right] \\
&\quad + \frac{h^2}{2}x^T(t)\widetilde{T}_2 x(t) \\
&\quad + \int_{t-h}^{t}(s-t+h)x^T(s)T_3x(s)\,ds,
\end{aligned} \quad (36)$$

where $\widetilde{T}_2 > 0$, since $\widetilde{T}_2 = T_2T_3^{-1}T_2$. Consider

$$\begin{aligned}
\dot{V}_5(x(t)) &= \frac{1}{2}h^2x^T(t)T_3x(t) \\
&\quad - \int_{t-h}^{t}(s-t+h)x^T(s)T_3x(s)\,ds.
\end{aligned} \quad (37)$$

Then, the time-derivative of V has new upper bound as follows:

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{V}_1(x(t)) + \dot{V}_2(x(t)) + \dot{V}_3(x(t)) \\
&\quad + \dot{V}_4(x(t)) + \dot{V}_5(x(t)) \le \overline{\xi}^T(t)\overline{\Phi}\overline{\xi}(t),
\end{aligned} \quad (38)$$

where $\overline{\Phi}$ is defined as stated in (31) and

$$\overline{\xi}^T(t) = \left[x^T(t) \;\; x^T\left(t-\frac{h}{2}\right) \;\; x^T(t-h) \;\; \int_{t-h}^{t} x^T(s)\,ds\right]. \quad (39)$$

The corollary can then be proved following [24]. □

3.2. Asymptotic Stability for Uncertain Neutral System. Consider the following uncertain neutral system with time-delay and actuator saturation:

$$\begin{aligned}
\dot{x}(t) - D\dot{x}(t-\tau) &= (A + \Delta A(t))x(t) \\
&\quad + Bx(t-h) + (C + \Delta C(t))\,\mathrm{Sat}(u(t)),
\end{aligned} \quad (40)$$

where $\Delta A(t)$ and $\Delta C(t)$ stand for the uncertainties. For simplicity, the constant parameter matrices A, B, C, and D are square matrices. The spectral norm bound of the unknown uncertainties is

$$\|\Delta A(t)\| \le \alpha, \quad \|\Delta C(t)\| \le \beta, \quad \forall t \ge 0. \quad (41)$$

Using the nonlinear function $Dz(\cdot)$, rewrite the uncertain neutral system as follows:

$$\begin{aligned}
\dot{x}(t) - D\dot{x}(t-\tau) &= \left(A_c + \Delta A(t) - \Delta C(t)K\right)x(t) \\
&\quad + Bx(t-h) + (C + \Delta C(t))Dz(Kx(t)),
\end{aligned} \quad (42)$$

where A_c is defined as the same with certain neutral system with time-delay and actuator saturation. In particular, when $\|\Delta A(t)\| = 0$ and $\|\Delta C(t)\| = 0$, the uncertain neutral system becomes the certain case.

Similarly, we employ the operator $\mathscr{L} : C_0 \to \mathbb{R}^n$ with

$$\mathscr{L}(x(t)) = x(t) + B\int_{t-h}^{t} x(s)\,ds - Dx(t-\tau). \quad (43)$$

The following transformed system is then obtained:

$$\frac{d}{dt}\mathscr{L}(x(t)) = \overline{A}_c x(t) + (C + \Delta C(t))Dz(Kx(t)), \quad (44)$$

where $\overline{A}_c = \widehat{A}_c + \Delta A(t) - \Delta C(t)K = A + B - CK + \Delta A(t) - \Delta C(t)K$.

Theorem 10. *The uncertain neutral system in* (40) *with feedback control* (3) *is asymptotic stability if* $\|h|B| + D\| < 1$ *and there exist scalars* $\gamma_i > 0$, $i = 1, 2, 3, 4, 5, 6$, $P > 0$, $\widehat{P} > 0$, $Q = [Q_{ij}]_{2\times 2}$, $Q_{ij} > 0$, $R = [R_{ij}]_{2\times 2}$, $R_{ij} > 0$, $T_i > 0$ $(i = 1, 2, 3, 4, 5, 6)$, *and* $\widetilde{T}_i > 0$ $(i = 3, 4)$ *such that the following symmetric linear matrix inequality holds:*

$$\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & 0 & \Xi_{14} & \Xi_{15} & \Xi_{16} & 0 \\
\star & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 & 0 \\
\star & \star & \Xi_{33} & 0 & 0 & 0 & 0 \\
\star & \star & \star & \Xi_{44} & \Xi_{45} & 0 & 0 \\
\star & \star & \star & \star & \Xi_{55} & 0 & 0 \\
\star & \star & \star & \star & \star & \Xi_{66} & 0 \\
\star & \star & \star & \star & \star & \star & \Xi_{77}
\end{bmatrix} < 0, \quad (45)$$

where

$$
\begin{aligned}
\Xi_{11} = {} & \widehat{A}_c^T P + P\widehat{A}_c + \frac{1}{\gamma_1}\widehat{P} + \frac{1}{\gamma_2}\widehat{P} \\
& + (\gamma_1 + \gamma_3 + \gamma_5)\left(\alpha^2 + 2\alpha\beta\|K\| + \beta^2\|K\|^2\right) I \\
& + (\gamma_2 + \gamma_4 + \gamma_6)\left(\beta^2 + 2\beta\|C\| + \|C\|^2\right) W_i(\rho_i) \\
& + Q_{11} + R_{11} + T_1 + T_2 + \frac{h^2}{2}\left(\widetilde{T}_3 + T_5\right) + \frac{\tau^2}{2}\left(\widetilde{T}_4 + T_6\right), \\
\Xi_{22} = {} & Q_{22} - Q_{11}, \\
\Xi_{33} = {} & -Q_{22} - T_1, \\
\Xi_{44} = {} & R_{22} - R_{11}, \\
\Xi_{55} = {} & \left(\frac{1}{\gamma_3} + \frac{1}{\gamma_4}\right) D^T \widehat{P} D - R_{22} - T_2, \\
\Xi_{66} = {} & \left(\frac{1}{\gamma_5} + \frac{1}{\gamma_6}\right) B^T \widehat{P} B - T_3, \\
\Xi_{77} = {} & -T_4, \\
\Xi_{12} = {} & Q_{12}, \\
\Xi_{14} = {} & R_{12}, \\
\Xi_{15} = {} & -\widehat{A}_c^T P D, \\
\Xi_{16} = {} & \widehat{A}_c^T P B, \\
\Xi_{23} = {} & -Q_{12}, \\
\Xi_{45} = {} & -R_{12}.
\end{aligned} \tag{46}
$$

Proof. Define the legitimate Lyapunov functional candidate as

$$
\begin{aligned}
V(x(t)) = {} & V_1(x(t)) + V_2(x(t)) + V_3(x(t)) \\
& + V_4(x(t)) + V_5(x(t)) + V_6(x(t)),
\end{aligned} \tag{47}
$$

where $V_1(x(t))$, $V_2(x(t))$, $V_3(x(t))$, $V_4(x(t))$, $V_5(x(t))$, and $V_6(x(t))$ are the same as in Theorem 6.

The time-derivative of $V(x(t))$ along the trajectories of closed system (44) is given by the following:

$$
\begin{aligned}
\dot{V}(x(t)) = {} & \dot{V}_1(x(t)) + \dot{V}_2(x(t)) + \dot{V}_3(x(t)) \\
& + \dot{V}_4(x(t)) + \dot{V}_5(x(t)) + \dot{V}_6(x(t)).
\end{aligned} \tag{48}
$$

Then

$$
\begin{aligned}
\dot{V}_1(x(t)) = {} & 2\left\{x^T(t) - x^T(t-\tau) D^T + \left[\int_{t-h}^{t} x^T(s)\,ds\right] B^T\right\} \\
& \times P\left[\overline{A}_c x(t) + (C + \Delta C) Dz(Kx)\right] \\
= {} & 2x^T P\overline{A}_c x + 2x^T P(C + \Delta C) Dz(Kx) \\
& - 2x^T \overline{A}_c{}^T P D x(t-\tau) - 2x^T(t-\tau) D^T \\
& \times P(C + \Delta C) Dz(Kx) \\
& + 2x^T(t) \overline{A}_c{}^T P B\left[\int_{t-h}^{t} x(s)\,ds\right] \\
& + 2\left[\int_{t-h}^{t} x^T(s)\,ds\right] B^T P(C + \Delta C) Dz(Kx) \\
= {} & 2x^T P\widehat{A}_c x + 2x^T P(\Delta A - \Delta C K) x \\
& + 2x^T P(C + \Delta C) Dz(Kx) \\
& - 2x^T(t-\tau) D^T P\widehat{A}_c x(t) \\
& - 2x^T(t-\tau) D^T P(\Delta A - \Delta C K) x \\
& - 2x^T(t-\tau) D^T P(C + \Delta C) Dz(Kx) \\
& + 2\left[\int_{t-h}^{t} x^T(s)\,ds\right] B^T P\widehat{A}_c x \\
& + 2\left[\int_{t-h}^{t} x^T(s)\,ds\right] B^T P(\Delta A - \Delta C K) x \\
& + 2\left[\int_{t-h}^{t} x^T(s)\,ds\right] B^T P(C + \Delta C) Dz(Kx) \\
\le {} & x^T\left[\widehat{A}_c^T P + P\widehat{A}_c\right] x - 2x^T(t)\widehat{A}_c^T P D x(t-\tau) \\
& + 2x^T(t)\widehat{A}_c^T P B\left[\int_{t-h}^{t} x(s)\,ds\right] \\
& + \frac{1}{\gamma_1} x^T \widehat{P} x + \gamma_1 x^T (\Delta A - \Delta C K)^T (\Delta A - \Delta C K) x \\
& + \frac{1}{\gamma_2} x^T \widehat{P} x + \gamma_2 D z^T(Kx)(C + \Delta C)^T \\
& \times (C + \Delta C) Dz(Kx) \\
& + \frac{1}{\gamma_3} x^T(t-\tau) D^T \widehat{P} D x(t-\tau) \\
& + \gamma_3 x^T (\Delta A - \Delta C K)^T (\Delta A - \Delta C K) x \\
& + \frac{1}{\gamma_4} x^T(t-\tau) D^T \widehat{P} D x(t-\tau) \\
& + \gamma_4 D z^T(Kx)(C + \Delta C)^T (C + \Delta C) Dz(Kx) \\
& + \frac{1}{\gamma_5}\left[\int_{t-h}^{t} x^T(s)\,ds\right] B^T \widehat{P} B\left[\int_{t-h}^{t} x(s)\,ds\right] \\
& + \gamma_5 x^T (\Delta A - \Delta C K)^T (\Delta A - \Delta C K) x \\
& + \frac{1}{\gamma_6}\left[\int_{t-h}^{t} x^T(s)\,ds\right] B^T \widehat{P} B\left[\int_{t-h}^{t} x(s)\,ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma_6 Dz^T(Kx)(C+\Delta C)^T(C+\Delta C)Dz(Kx)\\
&\leq x^T\Big\{\left(\widehat{A}_c^T P + P\widehat{A}_c\right)\\
&\quad+\left[\frac{1}{\gamma_1}\widehat{P}+\gamma_1\left(\alpha^2+2\alpha\beta\|K\|+\beta^2\|K\|^2\right)I\right]\\
&\quad+\left[\frac{1}{\gamma_2}\widehat{P}+\gamma_2\left(\beta^2+2\beta\|C\|+\|C\|^2\right)W_i(\rho)\right]\\
&\quad+\gamma_3\left(\alpha^2+2\alpha\beta\|K\|+\beta^2\|K\|^2\right)I\\
&\quad+\gamma_4\left(\beta^2+2\beta\|C\|+\|C\|^2\right)W_i(\rho_i)\\
&\quad+\gamma_5\left(\alpha^2+2\alpha\beta\|K\|+\beta^2\|K\|^2\right)I\\
&\quad+\gamma_6\left(\beta^2+2\beta\|C\|+\|C\|^2\right)W_i(\rho_i)\Big\}x\\
&+x^T(t-\tau)\left(\frac{1}{\gamma_3}D^T\widehat{P}D+\frac{1}{\gamma_4}D^T\widehat{P}D\right)x(t-\tau)\\
&+\left[\int_{t-h}^{t}x^T(s)\,ds\right]\left[\frac{1}{\gamma_5}B^T\widehat{P}B+\frac{1}{\gamma_6}B^T\widehat{P}B\right]\\
&\times\left[\int_{t-h}^{t}x(s)\,ds\right]-2x^T(t)\widehat{A}_c^T PDx(t-\tau)\\
&+2x^T(t)\widehat{A}_c^T PB\left[\int_{t-h}^{t}x(s)\,ds\right],
\end{aligned}
\tag{49}
$$

where $\widehat{P}>0$, since $\widehat{P}=P^2$.

$\dot{V}_2(x(t))$, $\dot{V}_3(x(t))$, $\dot{V}_4(x(t))$, $\dot{V}_5(x(t))$, and $\dot{V}_6(x(t))$ are obtained similarly as in Theorem 6. Substituting these into (48), the time-derivative of V has new upper bound as follows:

$$
\dot{V}(x(t)) \leq \xi^T(t)\,\Xi\,\xi(t), \tag{50}
$$

where

$$
\xi^T(t)=\left[x^T(t)\;\; x^T\left(t-\frac{h}{2}\right)\;\; x^T(t-h)\;\; x^T\left(t-\frac{\tau}{2}\right)\;\; x^T(t-\tau)\;\; \int_{t-h}^{t}x^T(s)\,ds\;\; \int_{t-\tau}^{t}x^T(s)\,ds\right]; \tag{51}
$$

Ξ is defined as stated in (45).

If linear matrix inequality (45) is feasible, then $\dot{V}(x(t))<0$, for all $x\in B(\rho)$. The theorem can then be proved following [24]. □

Theorem 10 provides new asymptotic stability conditions for the uncertain neutral systems in (40) and (3). The following corollary is presented as a special case of the theorem.

Corollary 11. *The uncertain neutral system in* (40) *and* (3) *with $\tau \equiv h$ is asymptotic stability if $\|h|B| + D\| < 1$ and there exist $P>0$, $\widehat{P}>0$, $Q=[Q_{ij}]_{2\times 2}$, $Q_{ij}>0$, $T_i>0$ $(i=1,3)$, and $\widetilde{T}_2>0$ such that the following symmetric linear matrix inequality holds for real constant scalars $\gamma_i>0$, $i=1,2,3,4,5,6$:*

$$
\Xi=\begin{bmatrix}\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14}\\ \star & \Xi_{22} & \Xi_{23} & 0\\ \star & \star & \Xi_{33} & 0\\ \star & \star & \star & \Xi_{44}\end{bmatrix}<0, \tag{52}
$$

where $W_i(\rho)$, $i=1,2,3,\ldots,n$ are defined as before and

$$
\begin{aligned}
\Xi_{11}&=\widehat{A}_c^T P + P\widehat{A}_c+\frac{1}{\gamma_1}\widehat{P}+\frac{1}{\gamma_2}\widehat{P}\\
&\quad+(\gamma_1+\gamma_3+\gamma_5)\left(\alpha^2+2\alpha\beta\|K\|+\beta^2\|K\|^2\right)I\\
&\quad+(\gamma_2+\gamma_4+\gamma_6)\left(\beta^2+2\beta\|C\|+\|C\|^2\right)W_i(\rho_i)\\
&\quad+Q_{11}+T_1+\frac{h^2}{2}\left(\widetilde{T}_2+T_3\right),\\
\Xi_{22}&=Q_{22}-Q_{11},\\
\Xi_{33}&=\left(\frac{1}{\gamma_3}+\frac{1}{\gamma_4}\right)D^T\widehat{P}D-Q_{22}-T_1,\\
\Xi_{44}&=\left(\frac{1}{\gamma_5}+\frac{1}{\gamma_6}\right)B^T\widehat{P}B-T_2,\\
\Xi_{12}&=Q_{12},\\
\Xi_{13}&=-\widehat{A}_c^T PD,\\
\Xi_{14}&=\widehat{A}_c^T PB,\\
\Xi_{23}&=-Q_{12}.
\end{aligned}
\tag{53}
$$

Proof. Choose a legitimate Lyapunov functional candidate as

$$
\begin{aligned}
V(x(t))&=V_1(x(t))+V_2(x(t))\\
&\quad+V_3(x(t))+V_4(x(t))+V_5(x(t))
\end{aligned}
\tag{54}
$$

which are the same as (34).

$\dot{V}_1(x(t))$ can be evaluated similarly as in Theorem 10 and Corollary 9. The proof can be readily obtained. □

3.3. The Algorithm with $W_2(\rho_2)$ and the Algorithm to Solve the Optimal $W_i(\rho_i)$. From Definition 4, it is seen that $W_1(\rho_1)$ are different from $W_2(\rho_2)$, $W_3(\rho_3)$, and $W_4(\rho_4),\ldots,W_n(\rho_n)$. We compare between $W_2(\rho_2)$, $W_3(\rho_3)$, and $W_4(\rho_4),\ldots,W_n(\rho_n)$ and intend to reduce the conservativeness of the result. To that end, we should obtain ρ_2 in the first place. In what follows we present the stability analysis algorithm with $W_2(\rho_2)$ to solve ρ_2.

Step 1. Give α, β, K.

Step 2. Set positive values γ_i $(i = 1, 2, 3, 4, 5, 6)$.

Step 3. Initialize ρ_2.

Step 4. With $W_2(\rho_2)$, solve linear matrix inequality (45) by Matlab LMI Toolbox.

Step 5. If the solution satisfies the stability condition, go to Step 6; otherwise, reduce ρ_2 and go to Step 3.

Step 6. Increase ρ_2 and go to Step 3.

Step 7. End.

Remark 12. The above algorithm is stated with respect to uncertain systems (40) and (3). Other cases can be dealt with similarly.

After defining ρ_2, we replace $W_2(\rho_2)$ by $W_i(\rho_i)$ to reduce the conservativeness. We analyse these functions $W_2(\rho_2), W_3(\rho_3), W_4(\rho_4), \ldots, W_n(\rho_n)$ and find the optimal $W_i(\rho_i)$ to obtain the maximum ρ_i among $\rho_2, \rho_3, \ldots, \rho_n$.

Theorem 13. *Given $W_2(\rho_2), W_3(\rho_3), W_4(\rho_4), \ldots, W_n(\rho_n)$, if there exists $j \le n$, such that $\rho_{j-1} \ge \rho_k$ and $\|W_j(\rho_{j-1})\| \le \|W_k(\rho_k)\|$, for all $k = 2, 3, \ldots, j-1$, then we have*

$$\frac{\|K_i\| \rho_{j-1}}{u_i^+} \le f_k(j), \quad \forall k = 2, 3, \ldots, j-1, \tag{55}$$

where ρ_j is the domain attraction obtained by $W_j(\rho_j)$ and

$$f_k(j) = \frac{(k-1)^{(k-1)/(j-k)}}{k^{k/(j-k)}} \times \frac{j^{j/(j-k)}}{(j-1)^{(j-1)/(j-k)}}. \tag{56}$$

Proof. Recall the definition of $W_j(\rho_j)$, we know it can be expressed in the following equality:

$$W_j(\rho_j) = \sum_{i=1}^{m} \left(\frac{(j-1)\rho_j}{j^{j/(j-1)} u_i^+} \right)^{2j-2} \left(K_i^T K_i \right)^j. \tag{57}$$

If we have $\rho_{j-1} \ge \rho_k$, for all $k = 2, 3, \ldots, j-1$, then from the above equality we obtain

$$\|W_k(\rho_k)\| \le \left\| W_k(\rho_{j-1}) \right\|, \quad \forall k = 2, 3, \ldots, j-1. \tag{58}$$

With regard to $W_j(\rho_{j-1})$, we have

$$\begin{aligned} W_j(\rho_{j-1}) &= \sum_{i=1}^{m} \left(\frac{(j-1)\rho_{j-1}}{j^{j/(j-1)} u_i^+} \right)^{2j-2} \left(K_i^T K_i \right)^k \left(K_i^T K_i \right)^{j-k} \\ &= \sum_{i=1}^{m} \left[\frac{(k-1)\rho_{j-1}}{k^{k/(k-1)} u_i^+} \right]^{2k-2} \left(K_i^T K_i \right)^k \left(K_i^T K_i \right)^{j-k} \\ &\quad \times \left(\frac{\rho_{j-1}}{u_i^+} \right)^{2(j-k)} \frac{k^{2k}}{(k-1)^{2k-2}} \frac{(j-1)^{2j-2}}{j^{2j}} \\ &= \|K_i\|^{2(j-k)} \left(\frac{\rho_{j-1}}{u_i^+} \right)^{2(j-k)} \\ &\quad \times \frac{k^{2k}}{(k-1)^{2k-2}} \frac{(j-1)^{2j-2}}{j^{2j}} W_k(\rho_{j-1}). \end{aligned} \tag{59}$$

If we have $\|W_j(\rho_{j-1})\| \le \|W_k(\rho_k)\|$, then we can obtain

$$\begin{aligned} \left\| W_j(\rho_{j-1}) \right\| &= \|K_i\|^{2(j-k)} \left(\frac{\rho_{j-1}}{u_i^+} \right)^{2(j-k)} \frac{k^{2k}}{(k-1)^{2k-2}} \\ &\quad \times \frac{(j-1)^{2j-2}}{j^{2j}} \left\| W_k(\rho_{j-1}) \right\| \le \left\| W_k(\rho_k) \right\|. \end{aligned} \tag{60}$$

By (58), we have

$$\left(\frac{\|K_i\| \rho_{j-1}}{u_i^+} \right)^{2(j-k)} \frac{k^{2k}}{(k-1)^{2k-2}} \frac{(j-1)^{2j-2}}{j^{2j}} \le 1, \tag{61}$$

where $j = 3, 4, 5, \ldots$, and $k = 2, 3, \ldots, j-1$.

Then we easily obtain

$$\frac{\|K_i\| \rho_{j-1}}{u_i^+} \le f_k(j) = \frac{(k-1)^{(k-1)/(j-k)}}{k^{k/(j-k)}} \times \frac{j^{j/(j-k)}}{(j-1)^{(j-1)/(j-k)}} \tag{62}$$

which completes the proof. □

Remark 14. Theorem 13 provides us a criterion to find the optimal $W_j(\rho_j)$. That is, if condition (55) cannot be satisfied, the procedure to find the optimal one should be ended. For example, when ρ_4 is obtained, we compute $\|K_i\|\rho_4/u_i^+$ and the following inequalities need to be satisfied:

$$\begin{gathered} \frac{\|K_i\| \rho_4}{u_i^+} \le f_2(5), \qquad \frac{\|K_i\| \rho_4}{u_i^+} \le f_3(5), \\ \frac{\|K_i\| \rho_4}{u_i^+} \le f_4(5). \end{gathered} \tag{63}$$

Otherwise, the algorithm terminates with ρ_4.

The following optimality algorithm is used to obtain the optimal $W_n(\rho_n)$.

Step 1. Set positive value δ (small value, for example, 10^{-5}).

Step 2. Solve ρ_2 based on the above stability analysis algorithm with $W_2(\rho_2)$.

Step 3. Compute $\|K_i\|\rho_2/u_i^+$, if $(\|K_i\|\rho_2/u_i^+) - f_k(j) \le f_2(3)$, then go to Step 4; otherwise, go to Step 7.

Step 4. Set $\rho_{j+1}=\rho_j$, repeat the stability analysis algorithm with $W_{j+1}(\rho_j)$ then obtain ρ_{j+1} iteratively.

Step 5. Compute $\|K_i\|\rho_{j+1}/u_i^+$, if

$$\frac{\|K_i\|\rho_{j+1}}{u_i^+} \le f_k(j+2), \qquad \left|\frac{\|K_i\|\rho_{j+1}}{u_i^+} - f_k(j+2)\right| > \delta, \qquad \forall k = 2, 3, \ldots, j+1 \tag{64}$$

then go to Step 6; otherwise go to Step 7.

Step 6. Set $j = j + 1$ and go to Step 4.

Step 7. End.

4. Application and Numerical Examples

The two-stage system has been applied to many factories worldwide. For the interest of efficiency, a lot of equipment such as converter, cylinder, and ejector is designed in the two-stage form. The natural, social, and economic systems also exhibit the two-stage mode. A typical example is the two-stage ditch design which is a conservation tool supported by the conservancy in Indiana. The advantages of a two-stage ditch against the typical agricultural ditch include both improved drainage function and ecological function. The two-stage design improves ditch stability by reducing water flow and the need for maintenance, saving both labor and money.

Consider Figure 2 which shows two-stage dissolution tank in the chemical process. The solute in the hopper is transported by the conveyor belt and falls into DT1 within a certain time. The solute is part of the solution in DT1; then within a certain time the solution in DT1 and the undissolved solute flow into DT2 to continue to dissolve and dilute.

Under normal circumstances, the dilute fluid flow of DT1 and the velocity of the conveyor belt are constant. The dilute fluid flow of DT2 is controllable. In practical system design, the pipeline of the DT2 dilute fluid is thin and the regulating action of DT2 is tiny, which can thus be represented by the saturating form $\mathrm{Sat}(u(t))$. Let $\lambda_1(t)$ and $\lambda_2(t)$ be the solution concentration of DT1 and DT2 and λ_{10} and λ_{20} the solution concentration of DT1 and DT2 when the system reaches an equilibrium, respectively. $x_1(t) = \lambda_1(t) - \lambda_{10}$ and $x_2(t) = \lambda_2(t) - \lambda_{20}$ are the considered system states and the dilute fluid flow into DT2 $u(t)$ is the control input. The latter is of the saturating form $\mathrm{Sat}(u(t))$ because the dilute fluid flow into DT2 $u(t)$ is a fine adjustment role. The chemical plant puts the solute into DT1 with constant speed; that is, the velocity of the conveyor belt is constant in a period of time interval. In addition, the velocity of the dilute fluid flow into DT1 is also constant in a period of time interval.

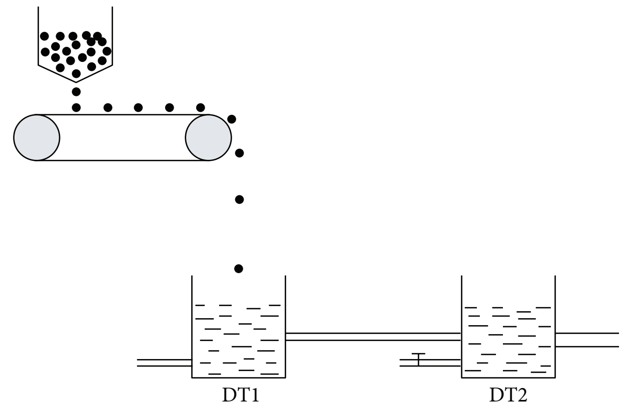

Figure 2: Two-stage dissolution tank.

4.1. No Uncertainty Case

Example 1. The model for the above described system is

$$\dot{x}(t) - \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \dot{x}(t-\tau) = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} x(t-h) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{Sat}(u(t)), \tag{65}$$

where the saturation limit is $u^+ = 15$. The discrete time-delay $h = 0.2$ and neutral time-delay $\tau = 0.4$.

It is seen that

$$\|h|B| + D\| = \left\| 0.2 \left| \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right| + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \right\| = 0.1047 < 1 \tag{66}$$

and the operator $\mathscr{L} : C_0 \to \mathbb{R}^n$ with

$$\mathscr{L}(x(t)) = x(t) + B\int_{t-h}^{t} x(s)\,ds - Dx(t-\tau) \tag{67}$$

is stable.

The following linear state feedback control is used and the same as [18]:

$$u = -Kx = \begin{bmatrix} -0.1325 & 0.0153 \end{bmatrix} x \tag{68}$$

To begin with the stability analysis algorithm, we have $\rho_2 = 161.73$, and compute $\|K\|\rho_2/u^+ = 1.4383$. It is seen that

$$1 < \frac{\|K\|\rho_2}{u^+} \le f_2(3) = \frac{27}{16} \doteq 1.6875. \tag{69}$$

According to the optimality algorithm, let initial data $\rho_3 = \rho_2$. Repeating the stability algorithm with $W_3(\rho_3)$, we obtain $\rho_3 = 163.46$. Similarly, we compute $\|K\|\rho_3/u^+$ and $(\|K\|\rho_3/u^+) = 1.4537$.

It is seen that

$$\frac{\|K\|\rho_3}{u^+} < f_2(4) \doteq 1.5396, \qquad \text{but } \frac{\|K\|\rho_3}{u^+} > f_3(4) \doteq 1.4047 \tag{70}$$

which terminates the algorithm. In this example, we obtain that the optimal auxiliary function is $W_3(\rho_3)$. By Theorem 13, we must verify $\|W_3(\rho_3)\| \le \|W_2(\rho_2)\|$, which is true as $\|W_3(\rho_3)\| = 0.0017$ and $\|W_2(\rho_2)\| = 0.0023$. So we infer that $W_3(\rho_3)$ is the optimal auxiliary function among $W_2(\rho_2), W_3(\rho_3), W_4(\rho_4), \ldots, W_n(\rho_n)$ and ρ_3 is the largest one among $\rho_2, \rho_3, \rho_4, \ldots, \rho_n$.

The comparison of these results can be seen in Figure 3. Notice that ρ is not the domain of attraction.

We obtain the largest parameter ρ_i with optimal $W_i(\rho_i)$ to enlarge the stability region. Thus the chemical plant can make the conveyor belt and dilute fluid flow of DT1 speed up accordingly. In this way, this chemical plant can improve the production efficiency and reduce the cost of production.

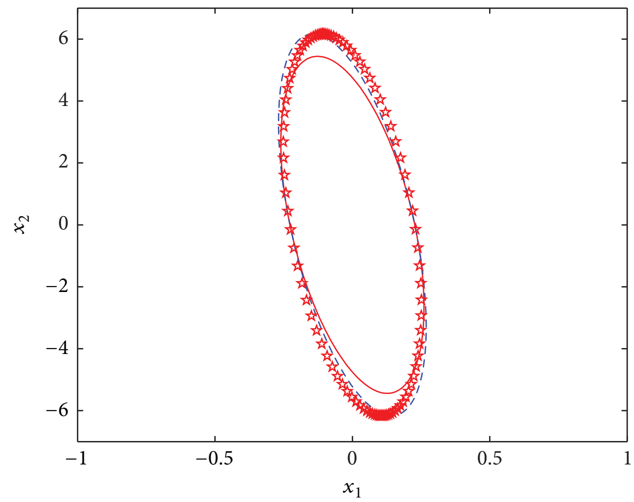

FIGURE 3: Attraction domain expanding schematic diagram: the largest parameter ρ_3 with optimal $W_3(\rho_3)$ is in red pentagram, with $W_2(\rho_2)$ in blue, and another one with $W_4(\rho_4)$ in red.

However, in some circumstances, we cannot enlarge the domain of attraction by replacing $W_2(\rho_2)$ with $W_i(\rho_i)$, $i = 3, 4, \ldots, n$. For example, let us see the following example.

Example 2. Consider the following linear neutral system with time-delay and actuator saturation:

$$\begin{aligned} \dot{x}(t) - \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \dot{x}(t-\tau) &= \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3.0 \end{bmatrix} x(t) \\ &\quad + \begin{bmatrix} 0.5 & -1 \\ 0 & 0.5 \end{bmatrix} x(t-h) + \begin{bmatrix} 0.7 & 0 \\ 0 & 0.1 \end{bmatrix} \operatorname{Sat}(u(t)), \end{aligned} \tag{71}$$

where the saturation limit values are $u_1^+ = 10$ and $u_2^+ = 4$. The discrete time-delay $h = 0.2$ and neutral time-delay $\tau = 0.1$.

Compute $\|h|B| + D\|$ and we also know that the operator $\mathscr{L} : C_0 \to \mathbb{R}^n$ with

$$\mathscr{L}(x(t)) = x(t) + B\int_{t-h}^{t} x(s)\,ds - Dx(t-\tau) \tag{72}$$

is stable. Then we obtain the linear state feedback control without considering the saturation using optimal control law in [27]. Consider

$$u = -Kx = -\begin{bmatrix} 0.8456 & 0.0627 \\ 0.0142 & 1.3754 \end{bmatrix} x. \tag{73}$$

To begin with the stability analysis algorithm, we have $\rho_2 = 17.84$, and compute

$$\frac{\|K_1\|\rho_2}{u_1^+} = 1.5127, \qquad \frac{\|K_2\|\rho_2}{u_2^+} = 6.1780. \tag{74}$$

It is seen that

$$\begin{aligned} 1 < \frac{\|K_1\|\rho_2}{u_1^+} \le f_2(3) = \frac{27}{16} \doteq 1.6875, \\ \text{but } \frac{\|K_2\|\rho_2}{u_2^+} > f_2(3) \end{aligned} \tag{75}$$

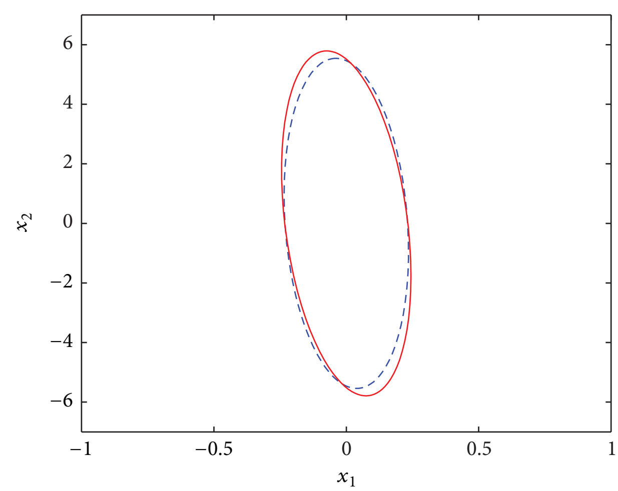

FIGURE 4: Attraction domain expanding schematic diagram: the largest parameter ρ_2 with optimal $W_2(\rho_2)$ is in red and another one with $W_3(\rho_3)$ in blue.

which terminates the algorithm. So we infer that $W_2(\rho_2)$ is the optimal auxiliary function among $W_2(\rho_2), W_3(\rho_3), W_4(\rho_4), \ldots, W_n(\rho_n)$. ρ_2 has been the largest one among $\rho_2, \rho_3, \rho_4, \ldots, \rho_n$ and we cannot maximize the value of ρ_i.

The comparison of these results can be seen in Figure 4.

4.2. Uncertainty Case. In this subsection, we consider the existence of uncertainty due to outside interference. The numerical example demonstrates the effectiveness of the proposed algorithm.

Example 3. Consider system (65) with norm bounded uncertainties:

$$\|\Delta A(t)\| \le 0.1, \qquad \|\Delta B(t)\| \le 0.1. \tag{76}$$

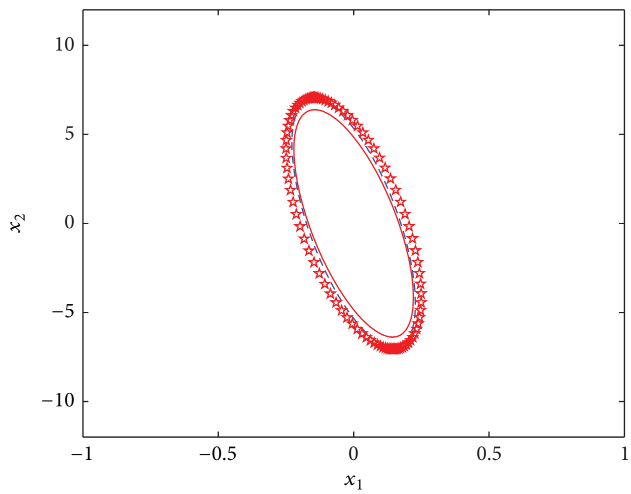

FIGURE 5: Attraction domain expanding schematic diagram: the largest parameter ρ_3 with optimal $W_3(\rho_3)$ is in red pentagram, with $W_2(\rho_2)$ in blue, and another one with $W_4(\rho_4)$ in red.

It is known that this uncertainty decreases the initial state bound that guarantees the asymptotic stability of the system (65). To compare with the above case, the feedback control (68) is applied and we use (45) in Theorem 10.

Using the stability analysis algorithm and the optimal algorithm, we see that $W_3(\rho_3)$ is the optimal auxiliary function among $W_2(\rho_2), W_3(\rho_3), W_4(\rho_4), \ldots, W_n(\rho_n)$. $\rho_3 = 43.57$ is the largest one among $\rho_2, \rho_3, \rho_4, \ldots, \rho_n$. This is shown in Figure 5.

5. Conclusions

By model transformation and Lyapunov method, delay-dependent criteria for a class of uncertain neutral delay systems with saturation are derived in terms of the spectral radius and LMIs. Both conditions for asymptotic stability and algorithms are presented. These theoretical findings are successfully verified via a two-stage dissolution tank in the chemical plant. Potentially, the result is of great significance to the chemical production process.

Conflict of Interests

It has no any conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to thank the Associate Editor and the anonymous reviewers for their constructive comments and suggestions to improve the quality of the paper. This work was supported in part by the Fundamental Research Funds for the Central Universities.

References

[1] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, San Diego, Calif, USA, 1993.

[2] V. B. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[3] J.-E. Feng and S. Xu, "Output feedback stabilization for a class of stochastic non-linear systems with delays in input," *Asian Journal of Control*, vol. 12, no. 1, pp. 110–115, 2010.

[4] L. Dugard and E. I. Verriest, *Stability and Control of Time-Delay Systems*, Springer, London, UK, 1998.

[5] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, vol. 269 of *Lecture Notes in Control and Information Sciences*, Springer, London, UK, 2001.

[6] C.-H. Lien, "New stability criterion for a class of uncertain nonlinear neutral time-delay systems," *International Journal of Systems Science*, vol. 32, no. 2, pp. 215–219, 2001.

[7] C.-H. Lien, K.-W. Yu, and J.-G. Hsieh, "Stability conditions for a class of neutral systems with multiple time delays," *Journal of Mathematical Analysis and Applications*, vol. 245, no. 1, pp. 20–27, 2000.

[8] E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems," *Systems & Control Letters*, vol. 43, no. 4, pp. 309–319, 2001.

[9] J. D. Chen, C. H. Lien, K. K. Fan, and J. H. Chou, "Criteria for asymptotic stability of a class of neutral systems via a LMI approach," *IEE Proceedings: Control Theory and Applications*, vol. 148, no. 6, pp. 442–447, 2001.

[10] Q.-L. Han, "Robust stability of uncertain delay-differential systems of neutral type," *Automatica*, vol. 38, no. 4, pp. 719–723, 2002.

[11] D. Ivănescu, S.-I. Niculescu, L. Dugard, J.-M. Dion, and E. I. Verriest, "On delay-dependent stability for linear neutral systems," *Automatica*, vol. 39, no. 2, pp. 255–261, 2003.

[12] Q.-L. Han, "On delay-dependent stability for neutral delay-differential systems," *International Journal of Applied Mathematics and Computer Science*, vol. 11, no. 4, pp. 965–976, 2001.

[13] S.-I. Niculescu, "On delay-dependent stability under model transformations of some neutral linear systems," *International Journal of Control*, vol. 74, no. 6, pp. 609–617, 2001.

[14] T. S. Hu and Z. L. Lin, *Control Systems with Actuator Saturation: Analysis and Design*, Birkhauser, Boston, Mass, USA, 2001.

[15] C. Pittet, S. Tarbouriech, and C. Burgat, "Stability regions for linear systems with saturating controls via circle and Popov criteria," in *Proceedings of the 36th IEEE Conference on Decision and Control*, pp. 4518–4523, San Diego, Calif, USA, December 1997.

[16] G. Scorletti, J. P. Folcher, and L. El Ghaoui, "Output feedback control with input saturations: LMI design approaches," *European Journal of Control*, vol. 7, no. 6, pp. 567–579, 2001.

[17] H. Hindi and S. Boyd, "Analysis of linear systems with saturation using convex optimization," in *Proceedings of the 37th IEEE Conference on Decision and Control (CDC '98)*, pp. 903–908, Tampa, Fla, USA, December 1998.

[18] J. M. Gomes da Silva, Jr., A. Seuret, E. Fridman, and J. P. Richard, "Stabilisation of neutral systems with saturating control inputs," *International Journal of Systems Science*, vol. 42, no. 7, pp. 1093–1103, 2011.

[19] S. Srivastava and F. Jabbari, "Scheduled controllers for disturbance attenuation of systems with bounded inputs," in *Proceedings of the American Control Conference*, pp. 735–739, Chicago, Ill, USA, June 2000.

[20] A. Haurani, H. H. Michalska, and B. Boulet, "Delay-dependent robust stabilization of uncertain neutral systems with saturating actuators," in *Proceedings of the American Control Conference*, pp. 509–514, Denver, Colo, USA, June 2003.

[21] J.-J. Yan, J.-S. Lin, and T.-L. Liao, "Robust dynamic compensator for a class of time delay systems containing saturating control input," *Chaos, Solitons & Fractals*, vol. 31, no. 5, pp. 1223–1231, 2007.

[22] P.-L. Liu, "Stabilization of input delay constrained systems with delay dependence," *International Journal of Systems Science*, vol. 26, no. 2, pp. 245–255, 1995.

[23] J.-H. Kim and Z. Bien, "Robust stability of uncertain linear systems with saturating actuators," *IEEE Transactions on Automatic Control*, vol. 39, no. 1, pp. 202–207, 1994.

[24] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, Applied Mathematical Sciences, Springer, New York, NY, USA, 1993.

[25] H. B. Ji, *Algebra Foundation of Control Theory*, University of Science and Technology of China Press, Hefei, China, 2008.

[26] W.-A. Zhang and L. Yu, "Delay-dependent robust stability of neutral systems with mixed delays and nonlinear perturbations," *Acta Automatica Sinica*, vol. 33, no. 8, pp. 863–866, 2007.

[27] D. E. Kirk, *Optimal Control Theory-An Introducion*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1970.