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Research Article

Fuzzy Approximating Spaces

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Relationships between fuzzy relations and fuzzy topologies are deeply researched. The concept of fuzzy approximating spaces is introduced and decision conditions that a fuzzy topological space is a fuzzy approximating space are obtained.

1. Introduction

Rough set theory, proposed by Pawlak [1], is a new mathematical tool for data reasoning. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems, and many other fields [2–5].

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules. A key notion in Pawlak rough set model is equivalence relations. The equivalence classes are the building blocks for the construction of these approximations. In the real world, the equivalence relation is, however, too restrictive for many practical applications. To address this issue, many interesting and meaningful extensions of Pawlak rough sets have been presented. Equivalence relations can be replaced by tolerance relations [6], similarity relations [7], binary relations [8, 9], and so on.

Various fuzzy generalizations of rough approximations have been proposed [10–14]. The most common fuzzy rough set is obtained by replacing the crisp relations with fuzzy relations on the universe and crisp subsets with fuzzy sets. Dubois and Prade [10] first proposed the concept of rough fuzzy sets and fuzzy rough sets and pointed out that a rough fuzzy set is a special case of a fuzzy rough set. Now, fuzzy rough sets have been used to solve practical problems such as data mining [15], approximate reasoning [5], and medical time series.

An interesting and natural research topic in rough set theory is to study the relationship between rough sets and topologies. Many authors studied topological properties of rough sets [16–21]. It is known that the pair of lower and upper approximation operators induced by a reflexive and transitive relation is exactly the pair of interior and closure operators of a topology [16, 22].

The purpose of this paper is to investigate further topological properties of fuzzy rough sets.

The remaining part of this paper is organized as follows. In Section 2, we recall some basic concepts about fuzzy sets and fuzzy topologies. In Section 3, fuzzy rough approximation operators are further investigated. In Section 4, relationships between fuzzy approximation spaces and fuzzy topologies are established. In Section 5, the concept of fuzzy approximating spaces is introduced and decision conditions that a fuzzy topological space is a fuzzy approximating space are obtained. Conclusions are in Section 6.

2. Preliminaries

Throughout this paper, U denotes a nonempty finite set, I denotes $[0,1]$, and $F(U)$ denotes the set of all fuzzy sets in U. For $a \in I$, $\overline{a}$ denotes the constant fuzzy set in U.

For all $A \in F(U)$, we denote

$$R_A = \{(x, y) \in U \times U : A(x) > A(y)\}. \tag{1}$$

Obviously, $R_A = \emptyset \Leftrightarrow A = \overline{a}$ for some $\lambda \in I$.

A fuzzy set is called a fuzzy point in U, if it takes the value 0 for each $y \in U$ except one, say, $x \in U$. If its value at x is λ $(0 < \lambda \leq 1)$, we denote this fuzzy point by x_λ, where the point x is called its support and λ is called its height (see [23, 24]).

Specially,

$$x_1(y) = \begin{cases} 1, & y = x, \\ 0, & y \neq x. \end{cases} \tag{2}$$

Remark 1. Consider

$$A = \bigcup_{x \in U} (A(x) x_1) \quad (A \in F(U)). \tag{3}$$

Definition 2 (see [25]). $\tau \subseteq F(U)$ is called a fuzzy topology on U, if

(i) $\forall a \in I$, $\overline{a} \in \tau$,

(ii) $A, B \in \tau \Rightarrow A \cap B \in \tau$,

(iii) $\{A_j : j \in J\} \subseteq \tau \Rightarrow \bigcup_{j \in J} A_j \in \tau$.

The pair (U, τ) is called a fuzzy topological space. Every member of τ is called a fuzzy open set in U. Its complement is called a fuzzy closed set in U.

We denote $\tau^c = \{A \in F(U) : A^c \in \tau\}$.

It should be pointed out that if (i) in Definition 2 is replaced [26] by

(i)′ $\overline{0}, \overline{1} \in \tau$,

then τ is a fuzzy topology in the sense of Chang [26]. We can see that a fuzzy topology in the sense of Lowen must be a fuzzy topology in the sense of Chang. In this paper, we always consider the fuzzy topology in the sense of Lowen.

A fuzzy topology τ is called Alexandrov [27] if (ii) in Definition 2 is replaced by

(ii)′ $\{A_j : j \in J\} \subseteq \tau \Longrightarrow \bigcap_{j \in J} A_j \in \tau$.

Definition 3 (see [28]). Let R be a relation on U. $\forall x \in U$, denote

$$\begin{aligned} R_p(x) &= \{y \in U : (y, x) \in R\}, \\ R_s(x) &= \{y \in U : (x, y) \in R\}. \end{aligned} \tag{4}$$

Then $R_p(x)$ and $R_s(x)$ are called the predecessor and successor neighborhood of x, respectively.

3. Fuzzy Approximation Spaces and Fuzzy Rough Approximation Operators

Recall that R is called a fuzzy relation on U if $R \in F(U \times U)$.

Definition 4 (see [14, 29]). Let R be a fuzzy relation on U. Then R is called

(1) reflexive, if $R(x, x) = 1$ for any $x \in U$,

(2) symmetric, if $R(x, y) = R(y, x)$ for any $x, y \in U$,

(3) transitive, if $R(x, z) \geq R(x, y) \wedge R(y, z)$ for any $x, y, z \in U$.

Let R be a fuzzy relation on U. R is called preorder if R is reflexive and transitive. R is called equivalence if R is reflexive, symmetric, and transitive.

Definition 5 (see [14, 29]). Let R be a fuzzy relation on U. The pair (U, R) is called a fuzzy approximation space. Based on (U, R), the fuzzy lower and the fuzzy upper approximation of $A \in F(U)$, denoted, respectively, by $\underline{R}(A)$ and $\overline{R}(A)$, are defined as follows:

$$\begin{aligned} \underline{R}(A)(x) &= \bigwedge_{y \in U} (A(y) \vee (1 - R(x, y))) \quad (x \in U), \\ \overline{R}(A)(x) &= \bigvee_{y \in U} (A(y) \wedge R(x, y)) \quad (x \in U). \end{aligned} \tag{5}$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the fuzzy rough set of A with respect to (U, R).

$\underline{R} : F(U) \to F(U)$ and $\overline{R} : F(U) \to F(U)$ are called the fuzzy lower approximation operator and the fuzzy upper approximation operator, respectively. In general, we refer to $\underline{R}$ and $\overline{R}$ as the fuzzy rough approximation operators.

Remark 6. $\overline{R}(x_1)(y) = R(y, x)$ and $\underline{R}((x_1)^c)(y) = 1 - R(y, x)$ $(x, y \in U)$.

Proposition 7 (see [30]). *Let R be a fuzzy relation on U. Then, for any $A, B \in F(U)$, $\{A_j : j \in J\} \subseteq F(U)$ and $\lambda \in I$,*

(1) $\underline{R}(\overline{1}) = \overline{1}$, $\overline{R}(\overline{0}) = \overline{0}$,

(2) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$,

(3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$,

(4) $\underline{R}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J}(\underline{R}(A_j))$, $\overline{R}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J}(\overline{R}(A_j))$,

(5) $\overline{R}(\lambda A) = \lambda \overline{R}(A)$.

Theorem 8 (see [14, 29, 30]). *Let R be a fuzzy relation on U. Then consider the following.*

(1) *R is reflexive $\Leftrightarrow$ (ILR) $\forall A \in F(U)$, $\underline{R}(A) \subseteq A$.*

$\Leftrightarrow$ (IUR) $\forall A \in F(U)$, $A \subseteq \overline{R}(A)$.

(2) *R is symmetric $\Leftrightarrow$ (ILS) $\forall (x, y) \in U \times U$, $\underline{R}((x_1)^c)(y) = \underline{R}((y_1)^c)(x)$.*

$\Leftrightarrow$ (IUS) $\forall (x, y) \in U \times U$, $\overline{R}(x_1)(y) = \overline{R}(y_1)(x)$.

(3) *R is transitive $\Leftrightarrow$ (ILT) $\forall A \in F(U)$, $\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$.*

$\Leftrightarrow$ (IUT) $\forall A \in F(U)$, $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$.

Remark 9. (1) $\forall a \in I$, $\overline{R}(\overline{a}) \subseteq \overline{a} \subseteq \underline{R}(\overline{a})$;

(2) if R is reflexive, then $\forall a \in I$, $\underline{R}(\overline{a}) = \overline{a} = \overline{R}(\overline{a})$.

Theorem 10. *Let R be a fuzzy relation on U and let τ be a fuzzy topology on U. If one of the following conditions is satisfied, then R is preorder:*

(1) *$\overline{R}$ is the closure operator of τ,*

(2) *$\underline{R}$ is the interior operator of τ.*

Proof. (1) Let $x, y, z \in U$. Put $cl_\tau(z_1)(y) = \lambda$. Note that $\underline{R}$ is the interior operator of τ. Then

$$R(x,x) = \overline{R}(x_1)(x) = cl_\tau(x_1)(x) \geq x_1(x) = 1. \quad (6)$$

Thus R is reflexive. By Remark 1, Remark 6, and Proposition 7(5),

$$\begin{aligned} R(x,y) \wedge R(y,z) &= \overline{R}(y_1)(x) \wedge \overline{R}(z_1)(y) = \overline{R}(y_1)(x) \wedge cl_\tau(z_1)(y) \\ &= \overline{R}(y_1)(x) \wedge \lambda = \lambda \overline{R}(y_1)(x) = \overline{R}(\lambda y_1)(x) \\ &= cl_\tau(\lambda y_1)(x) = cl_\tau(cl_\tau(z_1)(y)\, y_1)(x) \\ &\leq cl_\tau\left(\bigcup_{t\in U}(cl_\tau(z_1)(t)\, t_1)\right)(x) \\ &= cl_\tau(cl_\tau(z_1))(x) = cl_\tau(z_1)(x) = R(x,z). \end{aligned} \quad (7)$$

Then R is transitive. Hence R is preorder.

(2) The proof is similar to (1). □

Proposition 11. *Let R be a fuzzy relation on U. Then, for each $A \in F(U)$ with $R_A \neq \emptyset$, consider the following.*

(1) (a) $\underline{R}(A) \supseteq A \Leftrightarrow$ (*FLO*) $\forall (x,y) \in R_A$, $1 - R(x,y) \geq A(x) \vee A(y)$.

(b) $\overline{R}(A) \subseteq A \Leftrightarrow$ (*FUO*) $\forall (x,y) \in R_A$, $R(y,x) \leq A(x) \wedge A(y)$.

(2) *If R is reflexive, then*

(a) $\underline{R}(A) = A \Leftrightarrow$ (*FLR*) $\forall (x,y) \in R_A$, $1 - R(x,y) \geq A(x) \vee A(y)$.

(b) $\overline{R}(A) = A \Leftrightarrow$ (*FUR*) $\forall (x,y) \in R_A$, $R(y,x) \leq A(x) \wedge A(y)$.

Proof. (1) (a) Necessity. Suppose that $\underline{R}(A) \supseteq A$. Note that $\forall x \in U$,

$$\bigwedge_{y\in U}(A(y) \vee (1 - R(x,y))) = (\underline{R}(A))(y) \geq A(x). \quad (8)$$

Then $\forall x, y \in U$, $A(y) \vee (1 - R(x,y)) \geq A(x)$. Since $\forall (x,y) \in R_A$, $A(x) > A(y)$, we have

$$1 - R(x,y) \geq A(x) = A(x) \vee A(y) \quad ((x,y) \in R_A). \quad (9)$$

Sufficiency. Suppose that (*FLO*) holds. Let $x \in U$. Consider the following.

(i) If $y \in (R_A)_s(x)$, then

$$A(y) \vee (1 - R(x,y)) \geq A(y) \vee (A(x) \vee A(y)) \geq A(x). \quad (10)$$

(ii) If $y \notin (R_A)_s(x)$, then $A(y) \geq A(x)$ and so

$$A(y) \vee (1 - R(x,y)) \geq A(y) \geq A(x). \quad (11)$$

Hence $\underline{R}(A)(x) = \bigwedge_{y\in U}(A(y) \vee (1 - R(x,y))) \geq A(x)$. Thus $\underline{R}(A) \supseteq A$.

(b) Necessity. Suppose that $\overline{R}(A) \subseteq A$. Note that $\forall y \in U$,

$$\bigvee_{x\in U}(A(x) \wedge R(y,x)) = \overline{R}(A)(y) \leq A(y). \quad (12)$$

Then $\forall x, y \in U$, $A(x) \wedge R(y,x) \leq A(y)$. Since $\forall (x,y) \in R_A$, $A(x) > A(y)$, we have

$$R(y,x) \leq A(y) = A(x) \wedge A(y) \quad ((x,y) \in R_A). \quad (13)$$

Sufficiency. Suppose that (*FLO*) holds. Let $y \in U$. Consider the following.

(i) If $x \in (R_A)_p(y)$, then $(x,y) \in R_A$ and so

$$A(x) \vee R(y,x) \leq A(x) \wedge (A(x) \wedge A(y)) \leq A(y). \quad (14)$$

(ii) If $x \notin (R_A)_p(y)$, then $A(x) \leq A(y)$ and so

$$A(x) \wedge R(y,x) \leq A(x) \leq A(y). \quad (15)$$

Hence $(\overline{R}(A))(y) = \bigvee_{x\in U}(A(x) \wedge R(y,x)) \leq A(y)$. Thus $\overline{R}(A) \subseteq A$.

(2) holds by (1), the reflexivity of R, and Theorem 8(1). □

4. Relationships between Fuzzy Relations and Fuzzy Topologies

4.1. Fuzzy Topology Induced by Fuzzy Relations. Let R be a fuzzy relation on U. Denote

$$\begin{gathered} \sigma_R = \{A \in F(U) : A \subseteq \underline{R}(A)\}, \\ \tau_R = \{A \in F(U) : A = \underline{R}(A)\}, \quad \theta_R = \{\underline{R}(A) : A \in F(U)\}; \\ s_R = \bigwedge_{x,y\in U, x\neq y} R(x,y), \quad t_R = \bigvee_{x,y\in U, x\neq y} R(x,y). \end{gathered} \quad (16)$$

Remark 12. Let R be a fuzzy relation on U. Then consider the following.

(1) $\tau_R \subseteq \sigma_R$, $\tau_R \subseteq \theta_R$.

(2) If R is transitive, then $\tau_R \subseteq \theta_R \subseteq \sigma_R$.

(3) If R is reflexive, then $\tau_R = \sigma_R$.

(4) If R is preorder, then $\sigma_R = \tau_R = \theta_R$.

Theorem 13 (see [30]). *Let R be a preorder fuzzy relation on U. Then consider the following.*

(1) *θ_R is a fuzzy topology on U.*

(2) *$\underline{R}$ is the interior operator of θ_R.*

(3) *$\overline{R}$ is the closure operator of θ_R.*

Theorem 14. *Let R be a fuzzy relation on U. Then*

(1) *σ_R is an Alexandrov fuzzy topology on U,*

(2) *if R is reflexive, then $\forall A \in F(U)$,*

$$\operatorname{int}_{\sigma_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq cl_{\sigma_R}(A). \quad (17)$$

(3) *$A \in (\sigma_R)^c \Leftrightarrow \overline{R}(A) \subseteq A$.*

(4) *$\forall a \in I, \overline{a} \in (\sigma_R)^c$.*

Proof. (1) By Remark 9(1), $\overline{a} \in \sigma_R$ $(a \in I)$.

Let $\{A_j : j \in J\} \subseteq \sigma_R$. Then $\forall j \in J$, $A_j \subseteq \underline{R}(A_j)$. By Proposition 7(4),

$$\bigcap_{j\in J} A_j \subseteq \bigcap_{j\in J} \underline{R}(A_j) = \underline{R}\left(\bigcap_{j\in J} A_j\right),$$
$$\bigcup_{j\in J} A_j \subseteq \bigcup_{j\in J} \underline{R}(A_j) \subseteq \underline{R}\left(\bigcup_{j\in J} A_j\right). \quad (18)$$

Hence $\bigcap_{j\in J} A_j, \bigcup_{j\in J} A_j \in \sigma_R$. So σ_R is Alexandrov.

(2) $\forall A \in F(U)$, by Proposition 7(2),

$$\begin{aligned}&\operatorname{int}_{\sigma_R}(A)\\ &= \bigcup\{B \in \sigma_R : B \subseteq A\} \subseteq \bigcup\{B \in \sigma_R : \underline{R}(B) \subseteq \underline{R}(A)\}\\ &= \bigcup\{B \in F(U) : B \subseteq \underline{R}(B) \subseteq \underline{R}(A)\} \subseteq \underline{R}(A).\end{aligned} \quad (19)$$

By Proposition 7(3),

$$cl_{\sigma_R}(A) = \left(\operatorname{int}_{\sigma_R}(A^c)\right)^c \supseteq \left(\underline{R}(A^c)\right)^c = \overline{R}(A). \quad (20)$$

By the reflexivity of R and Theorem 8(1),

$$\operatorname{int}_{\sigma_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq cl_{\sigma_R}(A). \quad (21)$$

(3) holds by Proposition 7(3).

(4) holds by (3) and Remark 9(1). □

Definition 15. Let R be a fuzzy relation on U. σ_R is called the fuzzy topology induced by R on U.

Definition 16. Let R be a fuzzy relation on U. R is called pseudoconstant, if there exists $a \in I$ such that, for any $x, y \in U$,

$$R(x, y) = \begin{cases} 1, & \text{if } x = y, \\ a, & \text{if } x \neq y. \end{cases} \quad (22)$$

We write R by a^* or $a^*_{U\times U}$.

Obviously, every pseudoconstant fuzzy relation is an equivalence fuzzy relation.

Remark 17. (1) $\forall a, b \in I$, $a \leq b$ implies $a^* \subseteq b^*$.

(2) $\forall a \in I$, $\sigma_{a^*} = \tau_{a^*} = \theta_{a^*}$.

(3) $\sigma_{0^*} = F(U)$, $\sigma_{1^*} = \{\overline{a} : a \in I\}$.

(4) $\forall R \in F(U \times U)$, $s^*_R \subseteq R \subseteq t^*_R$.

The following theorem gives the topological structure of fuzzy approximation spaces.

Theorem 18. *Let (U, R) be a fuzzy approximation space. Then*

(1) $\sigma_R = \sigma_{1^*} \cup \{A \in F(U) : \forall (x, y) \in R_A, A(x) \vee A(y) \leq 1 - R(x, y)\}$,

(2) $\sigma_{t^*_R} \subseteq \sigma_R \subseteq \sigma_{s^*_R}$.

Proof. (1) holds by Proposition 11(1) and Remark 17(3).

(2) holds by Remark 17(1). □

4.2. Fuzzy Relations Induced by Fuzzy Topologies

Definition 19. Let σ be a fuzzy topology on U. Define

$$R_\sigma(x, y) = cl_\sigma(y_1)(x) \quad (x, y \in U). \quad (23)$$

Then R_σ is called the fuzzy relation induced by σ on U.

Theorem 20. *Let σ be a fuzzy topology on U and let R_σ be the fuzzy relation induced by σ on U. Then*

(1) *R_σ is reflexive,*

(2) *If $\overline{a} \in \sigma^c$ whenever $a \in I$, then $\forall A \in F(U)$,*

$$\underline{R_\sigma}(A) \subseteq \operatorname{int}_\sigma(A) \subseteq A \subseteq cl_\sigma(A) \subseteq \overline{R_\sigma}(A). \quad (24)$$

Proof. (1) $\forall x \in U$,

$$R_\sigma(x, x) = cl_\sigma(x_1)(x) \geq (x_1)(x) = \overline{1}. \quad (25)$$

Then R_σ is reflexive.

(2) $\forall A \in F(U)$, by Remark 1 and Proposition 7(2),

$$\begin{aligned} cl_\sigma(A) &= cl_\sigma\left(\bigcup_{y\in U}(A(y)\, y_1)\right)\\ &= \bigcup_{y\in U} cl_\sigma(A(y)\, y_1) = \bigcup_{y\in U} cl_\sigma(A(y) \cap y_1)\\ &\subseteq \bigcup_{y\in U}(cl_\sigma(A(y)) \cap cl_\sigma(y_1))\\ &= \bigcup_{y\in U}(A(y) \cap cl_\sigma(y_1)).\end{aligned} \quad (26)$$

Then $\forall x \in U$,

$$cl_\sigma(A)(x) \le \bigvee_{y\in U}(A(y)(x) \wedge cl_\sigma(y_1)(x)) = \bigvee_{y\in U}(A(y) \wedge R_\sigma(x,y)) = \overline{R_\sigma}(A)(x). \tag{27}$$

Hence $cl_\sigma(A) \subseteq \overline{R_\sigma}(A)$.

By Proposition 7(3),

$$\mathrm{int}_\sigma(A) = (cl_\sigma(A^c))^c \supseteq \left(\overline{R_\sigma}(A^c)\right)^c = \underline{R_\sigma}(A). \tag{28}$$

So

$$\underline{R_\sigma}(A) \subseteq \mathrm{int}_\sigma(A) \subseteq A \subseteq cl_\sigma(A) \subseteq \overline{R_\sigma}(A). \tag{29}$$

□

Theorem 21. *Let R be a preorder fuzzy relation, let σ_R be the fuzzy topology by R on U, and let R_{σ_R} be the fuzzy relation induced by σ_R on U. Then $R_{\sigma_R} = R$.*

Proof. By Remark 6, Remark 12(4), and Theorem 13(3),

$$R(x,y) = \overline{R}(y_1)(x) = cl_{\theta_R}(y_1)(x) = cl_{\sigma_R}(y_1)(x) = R_{\sigma_R}(x,y) \quad (x, y \in U). \tag{30}$$

Then $R_{\sigma_R} = R$. □

4.3. (CC) Axiom. The following condition for a fuzzy topology σ on U is called (CC) axiom in [31],

(CC) axiom: for any $\lambda \in I$ and $A \in F(U)$,

$$cl_\sigma(\lambda A) = \lambda cl_\sigma(A). \tag{31}$$

Proposition 22. *Let σ be a fuzzy topology on U. If σ satisfied the (CC) axiom, then*

(1) *$\overline{R_\sigma}$ is the closure operator of σ,*

(2) *R_σ is a preorder fuzzy relation on U,*

(3) *$\forall a \in I$, $\overline{a} \in \sigma$,*

(4) *σ is Alexandrov.*

Proof. (1) $\forall A \in F(U)$, by Remark 1 and (CC) axiom,

$$cl_\sigma(A) = cl_\sigma\left(\bigcup_{y\in U}(A(y)y_1)\right) = \bigcup_{y\in U} cl_\sigma(A(y)y_1) = \bigcup_{y\in U}(A(y)cl_\sigma(y_1)). \tag{32}$$

Then $\forall x \in U$,

$$cl_\sigma(A)(x) = \bigvee_{y\in U}(A(y)(x) \wedge cl_\sigma(y_1)(x)) = \bigvee_{y\in U}(A(y) \wedge R_\sigma(x,y)) = \overline{R_\sigma}(A)(x). \tag{33}$$

Thus $\overline{R_\sigma}(A) = cl_\sigma(A)$. So $\overline{R_\sigma}$ is the closure operator of σ.

(2) holds by (1) and Theorem 10.

(3) $\forall a \in I$, by (2), Proposition 7(3), and Remark 9(2),

$$\mathrm{int}_\sigma(\overline{a}) = (cl_\sigma(\overline{a}^c))^c = \left(\overline{R_\sigma}(\overline{a}^c)\right)^c = \underline{R_\sigma}(\overline{a}) = \overline{a}. \tag{34}$$

Then $\overline{a} \in \sigma$.

(4) By (1) and Proposition 7(3), $\underline{R_\sigma}$ is the interior operator of σ.

Let $\{A_j : j \in J\} \subseteq \sigma$. Then $\forall j \in J$, $\mathrm{int}(A_j) = A_j$. By Proposition 7(4),

$$\bigcap_{j\in J} A_j = \bigcap_{j\in J}\mathrm{int}_\sigma(A_j) = \bigcap_{j\in J}\underline{R_\sigma}(A_j) = \underline{R_\sigma}\left(\bigcap_{j\in J} A_j\right) = \mathrm{int}_\sigma\left(\bigcap_{j\in J} A_j\right). \tag{35}$$

So $\bigcap_{j\in J} A_j \in \sigma$. Hence σ is Alexandrov. □

Proposition 23. *Let R be a preorder fuzzy relation on U. Then σ_R satisfies (CC) axiom.*

Proof. For any $\lambda \in I$ and $A \in F(U)$, by Proposition 7(6) and Proposition 22,

$$cl_{\sigma_R}(\lambda A) = \overline{R}(\lambda A) = \lambda \overline{R}(A) = \lambda cl_{\sigma_R}(A). \tag{36}$$

□

Theorem 24. *Let σ be a fuzzy topology on U, let R_σ be the fuzzy relation induced by σ on U, and let σ_{R_σ} be the fuzzy topology induced by R_σ on U. If σ satisfies (CC) axiom, then $\sigma_{R_\sigma} = \sigma$.*

Proof. By Theorem 20(1), R_σ is reflexive. $\forall x, y, z \in U$, put $cl_\sigma(z_1)(y) = \lambda$. By (CC) axiom and Remark 1,

$$\lambda cl_\sigma(y_1) = cl_\sigma(\lambda y_1) = cl_\sigma(cl_\sigma(z_1)(y)y_1) \subseteq cl_\sigma\left(\bigcup_{t\in U}(cl_\sigma(z_1)(t)t_1)\right) = cl_\sigma(cl_\sigma(z_1)) = cl_\sigma(z_1). \tag{37}$$

Then

$$\begin{aligned} R_\sigma(x,y) \wedge R_\sigma(y,z) &= cl_\sigma(y_1)(x) \wedge cl_\sigma(z_1)(y) = cl_\sigma(y_1)(x) \wedge \lambda \\ &= \lambda \wedge cl_\sigma(y_1)(x) = (\lambda cl_\sigma(y_1))(x) \\ &\le cl_\sigma(z_1)(x) = R_\sigma(x,z). \end{aligned} \tag{38}$$

Thus R_σ is transitive and so R_σ is preorder.

$\forall A \in F(U)$, by Remark 12 and Theorem 13(3),

$$cl_{\sigma_{R_\sigma}}(A) = cl_{\theta_{R_\sigma}}(A) = \overline{R_\sigma}(A). \quad (39)$$

By (CC) axiom and Proposition 22, $\overline{R_\sigma}(A) = cl_\sigma(A)$. So $cl_{\sigma_{R_\sigma}}(A) = cl_\sigma(A)$. Thus

$$\begin{aligned} \text{int}_{\sigma_{R_\sigma}}(A) &= \left(cl_{\sigma_{R_\sigma}}(A^c)\right)^c \\ &= \left(cl_\sigma(A^c)\right)^c = \text{int}_\sigma(A). \end{aligned} \quad (40)$$

Hence $\sigma_{R_\sigma} = \sigma$. □

Theorem 25. *Let τ be a fuzzy topology on U. Then the following are equivalent.*

(1) *τ satisfies (CC) axiom.*

(2) *There exists a preorder fuzzy relation ρ on U such that $\overline{\rho}$ is the closure operator of τ.*

(3) *There exists a preorder fuzzy relation ρ on U such that $\underline{\rho}$ is the interior operator of τ.*

(4) *$\overline{R_\tau}$ is the closure operator of τ.*

(5) *$\underline{R_\tau}$ is the interior operator of τ.*

Proof. (1) ⇒ (2). Suppose that τ satisfies (CC) axiom. Pick $\rho = R_\tau$. By Proposition 22, $\overline{\rho}$ is the closure operator of τ. By Theorem 10, $\overline{\rho}$ is preorder.

(2) ⇒ (3). Let $\overline{\rho}$ be the closure operator of τ for some preorder fuzzy relation ρ on U. $\forall A \in F(U)$, by Proposition 7(3),

$$\underline{\rho}(A) = \left(\overline{\rho}(A^c)\right)^c = \left(cl_\tau(A^c)\right)^c = \text{int}_\tau(A). \quad (41)$$

Thus $\underline{\rho}$ is the interior operator of τ.

(3) ⇒ (5). Let $\underline{\rho}$ be the interior operator of τ for some preorder fuzzy relation ρ on U.

By Remark 6,

$$\begin{aligned} \rho(x, y) &= 1 - \underline{\rho}\left((y_1)^c\right)(x) = 1 - \text{int}_\tau\left((y_1)^c\right)(x) \\ &= cl_\tau(y_1)(x) = R_\tau(x, y) \quad (x, y \in U). \end{aligned} \quad (42)$$

Then $\rho = R_\tau$. Thus $\underline{R_\tau}$ is the interior operator of τ.

(5) ⇒ (4) holds by Proposition 7(3).

(4) ⇒ (1). For any $\lambda \in I$ and $A \in F(U)$, by Proposition 7(6),

$$cl_\tau(\lambda A) = \overline{R_\tau}(\lambda A) = \lambda \overline{R_\tau}(A) = \lambda cl_\tau(A). \quad (43)$$

Thus τ satisfies (CC) axiom. □

Theorem 26. *Let*

$$\Sigma = \{R : R \textit{ is a preorder fuzzy relation on } U\},$$
$$\Gamma = \{\sigma : \sigma \textit{ is a fuzzy topology on } U \textit{ satisfying (CC) axiom}\}. \quad (44)$$

Then there exists a one-to-one correspondence between Σ and Γ.

Proof. $f : \Sigma \to \Gamma$ and $g : \Gamma \to \Sigma$ are defined as follows:

$$\begin{aligned} f(R) &= \sigma_R \quad (R \in \Sigma), \\ g(\sigma) &= R_\sigma \quad (\sigma \in \Gamma). \end{aligned} \quad (45)$$

By Theorem 21,

$$g \circ f = i_\Sigma, \quad (46)$$

where $g \circ f$ is the composition of f and g and i_Σ is the identity mapping on Γ.

By Theorem 24,

$$f \circ g = i_\Gamma, \quad (47)$$

where $f \circ g$ is the composition of g and f and i_Γ is the identity mapping on Σ.

Hence f and g are two one-to-one correspondences. This proves that there exists a one-to-one correspondence between Σ and Γ. □

5. Fuzzy Approximating Spaces

As can be seen from Section 4, a fuzzy relation yields a fuzzy topology. In this section, we consider the reverse problem; that is, when can the given fuzzy topology coincide with the fuzzy topology induced by some fuzzy relation?

Definition 27. Let (U, σ) be a fuzzy topological space. If there exists a fuzzy relation on U such that $\sigma_R = \sigma$, then (U, σ) is called a fuzzy approximating space.

Theorem 28. *Let (U, σ) be a fuzzy topological space. If one of the following conditions is satisfied, then (U, τ) is a fuzzy approximating space.*

(1) *τ satisfies (CC) axiom.*

(2) *There exists a preorder fuzzy relation ρ on U such that $\overline{\rho}$ is the closure operator of τ.*

(3) *There exists a preorder fuzzy relation ρ on U such that $\underline{\rho}$ is the interior operator of τ.*

(4) *$\overline{R_\tau}$ is the closure operator of τ.*

(5) *$\underline{R_\tau}$ is the interior operator of τ.*

Proof. These hold by Theorems 24 and 25. □

Theorem 29. *Let (U, σ) be a fuzzy topological space. Then (U, σ) is a fuzzy approximating space if and only if there exists a fuzzy relation R such that*

$$\begin{aligned} \sigma = \sigma_{1^*} \cup \{A \in F(U) : \forall (x, y) \in R_A, \\ A(x) \vee A(y) \leq 1 - R(x, y)\}. \end{aligned} \quad (48)$$

Proof. This holds by Theorem 18(1). □

Example 30. $\{\overline{a} : a \in I\}$ is a fuzzy approximating space.

6. Conclusions

In this paper, relationships between fuzzy relations and fuzzy topology were discussed. The fact that there exists a one-to-one correspondence between the set of all preorder fuzzy relations and the set of all fuzzy topologies which satisfy (CC) axiom was proved. We introduced the concept of fuzzy approximating spaces and gave decision conditions that a fuzzy topological space is a fuzzy approximating space.

The results of this paper illustrate that fuzzy relations can be researched by means of topology. We hope that one can find applications of topological properties of fuzzy rough sets in information sciences. In future work, we will do similar exploration of fuzzy neighborhood spaces like this paper.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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