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Research Article

Efficient Approach to Stability Analysis of Discrete-Time Systems with Time-Varying Delay

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This paper aims at deriving an efficient criterion for the robust stability analysis of discrete-time systems with time-varying delay. In the derivation, to obtain a larger stability region under the requirement of less computational complexity, this paper proposes a valuable method capable of establishing a less conservative stability criterion without using the free-weighting approach and an extremely augmented state. In parallel, the stabilization problem of systems with time-delayed control input is addressed in connection with the derived stability criterion.

1. Introduction

Over the last few decades, research on stability analysis of time-delay systems has rapidly accelerated owing to two main reasons. One is that such systems offer suitable mathematical models that can represent practical engineering systems with finite but uncertain signal propagation delays, such as biological systems, network systems, and nuclear reactors [1–3]. The other is that time-delay can often act as a critical factor that leads to performance degradation and instability of the systems under consideration [4–6]. Meanwhile, with the growing interest, significant progress has been made toward enlarging the feasible region of a stability criterion (referred to here as "stability region") and reducing its computational complexity. In other words, numerous investigations and research have been carried out to establish further improved stability criteria for time-delay systems by taking one of the following approaches with the use of a more attractive Lyapunov-Krasovskii functional [7–11]: the free-weighting matrix approach [11–13], the descriptor system approach [14–16], the Jensen inequality approach [17–19], or the delay-partitioning approach [20, 21].

Recently, the use of the Jensen inequality approach has attracted great attention from the control community since it requires fewer decision variables than other approaches (see [18, 22, 23] and references therein). In addition, the appearance of the reciprocally convex technique [22] has promoted the use of such an approach as a way to reduce the computational complexity and the conservatism of stability criteria. However, as reported in [24], most of the results of these studies have been confronted with a challenge to establish less conservative stability criteria in terms of performance behavior. Thus, [24] has fully exploited the zero equality terms, introduced by [25], in the process of deriving stability criteria. After that, [10, 11] have widely extended the stability criteria in accordance with the structure of Lyapunov-Krasovskii functional containing triple summation terms. However, since the use of such an augmented Lyapunov-Krasovskii function with more terms poses significant computational burdens, there is a need to explore a useful method capable of reducing the computational complexity as well as improving the performance of stability criteria.

Motivated by the above concerns, this paper is focused on deriving an efficient criterion for the robust stability analysis of discrete-time systems with time-varying delay. To be specific, the attention of this paper is paid for simultaneously achieving the following two goals: deceasing the computational complexity and improving the performance of a stability criterion. To accomplish such efficiency, this paper proposes a valuable method capable of deriving a less conservative stability criterion without using the free-weighting approach and an extremely augmented state, which plays an important role in reducing the computational complexity caused by [10, 11]. As a result, under the requirement of much less computational complexity, the stability region in the present study is enlarged to the same size as those of [10, 11]. Moreover, the stabilization problem of systems with time-delay control input is addressed in connection with the derived stability criterion. Finally, three numerical examples are provided to illustrate the efficiency of the proposed stability and stabilization criteria.

This paper is organized as follows. Section 2 provides a system description and useful properties. Section 3 introduces a Lyapunov-Krasovskii functional for deriving the stability criterion of the time-delay system. Section 4 presents the state-feedback controller for the system with delayed control input. Section 5 shows simulation results for validating the proposed results. Finally, Section 6 presents the conclusion along with a summary.

Notation. The notations $X \geq Y$ and $X > Y$ mean that $X - Y$ is positive semidefinite and positive definite, respectively. In symmetric block matrices, $(*)$ is used as an ellipsis for terms that are induced by symmetry. For any square matrix $\mathcal{Q}$, $\mathbf{He}[\mathcal{Q}] = \mathcal{Q} + \mathcal{Q}^T$. For any discrete-time function g_k, $\Delta[g_k]$ denotes its forward difference as $\Delta[g_k] = g_{k+1} - g_k$.

2. System Description and Useful Properties

Let us consider the following time-delay system:

$$x_{k+1} = Ax_k + A_d x_{k-d(k)}, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ and $x_{k-d(k)} \in \mathbb{R}^{n_x}$ are the state and the delayed state, respectively. Here, the state delay $d(k)$ is assumed to be of an interval time-varying type integer: $\underline{d} \leq d(k) \leq \overline{d}$, where $\underline{d}$ and $\overline{d}$ are known positive integers. To facilitate the derivation of the main result, we set $d_0 = 0$, $d_1 = \underline{d}$, $d_2 = d(k)$, and $d_3 = \overline{d}$ and use the following notations:

$$\begin{aligned} \Phi^1_{pq}(\eta_i) &= \sum_{i=k-d_q}^{k-d_p-1} \eta_i, \\ \Phi^2_{pq}(\eta_i) &= \sum_{j=d_p+1}^{d_q} \sum_{i=k-j}^{k-1} \eta_i, \end{aligned} \tag{2}$$

where $p \in \{0, 1, 2\}$, $q \in \{1, 2, 3 \mid q > p\}$, and η_i denotes any scalar or vector-valued function.

Property 1. For $(p, q) \in \{(0,1), (0,3), (1,3)\}$, the following properties hold:

$$\Delta\left[\Phi^1_{pq}(\eta_i)\right] = \Phi^1_{pq}(\Delta\eta_i) = \eta_{k-d_p} - \eta_{k-d_q}, \tag{3}$$

$$\Delta\left[\Phi^2_{pq}(\eta_i)\right] = \Phi^2_{pq}(\Delta\eta_i) = \Phi^1_{pq}(1)\eta_k - \Phi^1_{pq}(\eta_i), \tag{4}$$

where $\Phi^1_{pq}(1) = d_q - d_p$.

Property 2. Let us consider two time-varying parameters of the following form:

$$\begin{aligned} \theta_1 &= \frac{\Phi^1_{12}(1)}{\Phi^1_{13}(1)} \geq 0, \\ \theta_2 &= \frac{\Phi^1_{23}(1)}{\Phi^1_{13}(1)} \geq 0. \end{aligned} \tag{5}$$

Then, in the sense that $\Phi^1_{12}(1) + \Phi^1_{23}(1) = \Phi^1_{13}(1)$, it follows that $\theta_1 + \theta_2 = 1$.

Lemma 1 (see [10, 17]). *For any vector-valued function χ_i and positive-definite matrix $\mathcal{Q}$, the following inequalities hold:*

$$-\Phi^1_{pq}\left(\chi_i^T \mathcal{Q} \chi_i\right) \leq -\frac{1}{\Phi^1_{pq}(1)} \Phi^1_{pq}(\chi_i)^T \mathcal{Q} \Phi^1_{pq}(\chi_i). \tag{6}$$

3. Stability Analysis

Let $\Delta x_i = x_{i+1} - x_i$ and choose a Lyapunov-Krasovskii functional of the following form:

$$\begin{aligned} V_{1,k} &= \begin{bmatrix} x_k \\ \Phi^1_{01}(x_i) \\ \Phi^1_{13}(x_i) \end{bmatrix}^T \underbrace{\begin{bmatrix} P_1 & P_2 & P_3 \\ (*) & P_4 & P_5 \\ (*) & (*) & P_6 \end{bmatrix}}_{\triangleq \mathcal{P}} \begin{bmatrix} x_k \\ \Phi^1_{01}(x_i) \\ \Phi^1_{13}(x_i) \end{bmatrix}, \\ V_{2,k} &= \Phi^1_{01}\left(x_i^T Q_1 x_i\right) + \Phi^1_{03}\left(x_i^T Q_2 x_i\right), \\ V_{3,k} &= \Phi^1_{01}(1) \\ &\quad \cdot \Phi^2_{01}\left(\begin{bmatrix} x_i \\ \Delta x_i \end{bmatrix}^T \underbrace{\begin{bmatrix} R_{11} & 0 \\ 0 & R_{12} \end{bmatrix}}_{\triangleq \mathcal{R}_1} \begin{bmatrix} x_i \\ \Delta x_i \end{bmatrix}\right), \\ V_{4,k} &= \Phi^1_{13}(1) \\ &\quad \cdot \Phi^2_{13}\left(\begin{bmatrix} x_i \\ \Delta x_i \end{bmatrix}^T \underbrace{\begin{bmatrix} R_{21} & 0 \\ 0 & R_{22} \end{bmatrix}}_{\triangleq \mathcal{R}_2} \begin{bmatrix} x_i \\ \Delta x_i \end{bmatrix}\right), \end{aligned} \tag{7}$$

where $\mathcal{P}, Q_1, Q_2, \mathcal{R}_1$, and $\mathcal{R}_2$ are taken to be positive definite. To facilitate later steps, we define an augmented state ζ_k as

$$\zeta_k = \left[x_k^T \;\; x_{k-d_1}^T \;\; x_{k-d_2}^T \;\; x_{k-d_3}^T \;\; \Delta x_k^T \;\; \Phi^1_{01}\left(x_i^T\right) \;\; \Phi^1_{12}\left(x_i^T\right) \;\; \Phi^1_{23}\left(x_i^T\right)\right]^T \in \mathbb{R}^{n_\zeta}, \quad n_\zeta = 8n_x \tag{8}$$

and establish block entry matrices $\mathbf{e}_i$ such that $x_k = \mathbf{e}_0\zeta_k$, $x_{k-d_1} = \mathbf{e}_1\zeta_k$, $x_{k-d_2} = \mathbf{e}_2\zeta_k$, $x_{k-d_3} = \mathbf{e}_3\zeta_k$, $\Delta x_k = \mathbf{e}_4\zeta_k$, $\Phi_{01}^1(x_i) = \mathbf{e}_5\zeta_k$, $\Phi_{12}^1(x_i) = \mathbf{e}_6\zeta_k$, and $\Phi_{23}^1(x_i) = \mathbf{e}_7\zeta_k$.

Property 3 (see [25]). For symmetric matrices X_0, X_1, and X_2, the following equalities hold:

$$\begin{aligned} 0 &= \Phi_{01}^1(1)\cdot\left(\zeta_k^T\left(\mathbf{e}_0^T X_0\mathbf{e}_0 - \mathbf{e}_1^T X_0\mathbf{e}_1\right)\zeta_k \right.\\ &\quad \left. - \Phi_{01}^1\left(\Delta x_i X_0\left(\Delta x_i + 2x_i\right)\right)\right), \\ 0 &= \Phi_{13}^1(1)\cdot\left(\zeta_k^T\left(\mathbf{e}_1^T X_1\mathbf{e}_1 - \mathbf{e}_2^T X_1\mathbf{e}_2\right)\zeta_k \right.\\ &\quad \left. - \Phi_{12}^1\left(\Delta x_i X_1\left(\Delta x_i + 2x_i\right)\right)\right), \\ 0 &= \Phi_{13}^1(1)\cdot\left(\zeta_k^T\left(\mathbf{e}_2^T X_2\mathbf{e}_2 - \mathbf{e}_3^T X_2\mathbf{e}_3\right)\zeta_k \right.\\ &\quad \left. - \Phi_{23}^1\left(\Delta x_i X_2\left(\Delta x_i + 2x_i\right)\right)\right). \end{aligned} \tag{9}$$

The following theorem presents the delay- and range-dependent stability criterion for (1).

Theorem 2. *Let $d_1 = \underline{d}$ and $d_3 = \overline{d}$ be prescribed and define $\delta = d_3 - d_1$. System (1) is asymptotically stable for any time-varying $d(k)$ satisfying $d(k) \in [\underline{d}, \overline{d}]$, if there exist matrices $\{S_i\}_{i=1,\ldots,4} \in \mathbb{R}^{n_x\times n_x}$, $\{Z_i\}_{i=0,4} \in \mathbb{R}^{n_x\times n_x}$ and symmetric matrices $\mathscr{P} \in \mathbb{R}^{3n_x\times 3n_x}$, $\{Q_i\}_{i=1,2} \in \mathbb{R}^{n_x\times n_x}$, $\{R_{1i}\}_{i=1,2} \in \mathbb{R}^{n_x\times n_x}$, $\{R_{2i}\}_{i=1,2} \in \mathbb{R}^{n_x\times n_x}$, and $\{X_i\}_{i=0,1,2} \in \mathbb{R}^{n_x\times n_x}$ such that*

$$0 > \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5, \tag{10}$$

$$0 \le \left[\begin{array}{cc|cc} R_{21} & X_1 & S_1 & S_2 \\ (*) & R_{22}+X_1 & S_3 & S_4 \\ \hline (*) & (*) & R_{21} & X_2 \\ (*) & (*) & (*) & R_{22}+X_2 \end{array}\right], \tag{11}$$

where $\{\Psi_i\}_{i=1,\ldots,5}$ are defined in (13), (15), (18), (20), and (22), respectively.

Proof. The forward difference of $V_{1,k}$ becomes

$$\begin{aligned} \Delta V_{1,k} &= \begin{bmatrix} \Delta x_k + x_k \\ \Phi_{01}^1(\Delta x_i + x_i) \\ \Phi_{13}^1(\Delta x_i + x_i) \end{bmatrix}^T \mathscr{P} \begin{bmatrix} \Delta x_k + x_k \\ \Phi_{01}^1(\Delta x_i + x_i) \\ \Phi_{13}^1(\Delta x_i + x_i) \end{bmatrix} \\ &\quad - \begin{bmatrix} x_k \\ \Phi_{01}^1(x_i) \\ \Phi_{13}^1(x_i) \end{bmatrix}^T \mathscr{P} \begin{bmatrix} x_k \\ \Phi_{01}^1(x_i) \\ \Phi_{13}^1(x_i) \end{bmatrix} \\ &= \zeta_k^T \left[\begin{array}{c} \mathbf{e}_4 \\ \mathbf{e}_0 \\ \hline \mathbf{e}_0 - \mathbf{e}_1 \\ \mathbf{e}_5 \\ \hline \mathbf{e}_1 - \mathbf{e}_3 \\ \mathbf{e}_6 + \mathbf{e}_7 \end{array}\right]^T \\ &\quad \cdot \left[\begin{array}{cc|cc|cc} P_1 & P_1 & P_2 & P_2 & P_3 & P_3 \\ P_1 & 0 & P_2 & 0 & P_3 & 0 \\ \hline P_2^T & P_2^T & P_4 & P_4 & P_5 & P_5 \\ P_2^T & 0 & P_4 & 0 & P_5 & 0 \\ \hline P_3^T & P_3^T & P_5^T & P_5^T & P_6 & P_6 \\ P_3^T & 0 & P_5^T & 0 & P_6 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{e}_4 \\ \mathbf{e}_0 \\ \hline \mathbf{e}_0 - \mathbf{e}_1 \\ \mathbf{e}_5 \\ \hline \mathbf{e}_1 - \mathbf{e}_3 \\ \mathbf{e}_6 + \mathbf{e}_7 \end{array}\right] \zeta_k, \qquad \text{from (3)} \\ &= \zeta_k \Psi_1 \zeta_k, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \Psi_1 &= \mathbf{e}_4^T P_1 \mathbf{e}_4 + \mathbf{He}\left[\mathbf{e}_4^T P_1\mathbf{e}_0 + \mathbf{e}_4^T P_2\left(\mathbf{e}_0 - \mathbf{e}_1\right) + \mathbf{e}_4^T P_2\mathbf{e}_5 \right.\\ &\quad \left. + \mathbf{e}_4^T P_3\left(\mathbf{e}_1 - \mathbf{e}_3\right) + \mathbf{e}_4^T P_3\left(\mathbf{e}_6 + \mathbf{e}_7\right)\right] \\ &\quad + \mathbf{He}\left[\mathbf{e}_0^T P_2\left(\mathbf{e}_0 - \mathbf{e}_1\right) + \mathbf{e}_0^T P_3\left(\mathbf{e}_1 - \mathbf{e}_3\right)\right] + (\mathbf{e}_0 \\ &\quad - \mathbf{e}_1)^T P_4\left(\mathbf{e}_0 - \mathbf{e}_1\right) + \mathbf{He}\left[\left(\mathbf{e}_0 - \mathbf{e}_1\right)^T P_4\mathbf{e}_5 \right.\\ &\quad \left. + \left(\mathbf{e}_0 - \mathbf{e}_1\right)^T P_5\left(\mathbf{e}_1 - \mathbf{e}_3\right) + \left(\mathbf{e}_0 - \mathbf{e}_1\right)^T P_5\left(\mathbf{e}_6 + \mathbf{e}_7\right)\right] \\ &\quad + \mathbf{He}\left[\mathbf{e}_5^T P_5\left(\mathbf{e}_1 - \mathbf{e}_3\right)\right] + \left(\mathbf{e}_1 - \mathbf{e}_3\right)^T P_6\left(\mathbf{e}_1 - \mathbf{e}_3\right) \\ &\quad + \mathbf{He}\left[\left(\mathbf{e}_1 - \mathbf{e}_3\right)^T P_6\left(\mathbf{e}_6 + \mathbf{e}_7\right)\right]. \end{aligned} \tag{13}$$

Letting $\overline{x}_i^T = \left[x_i^T \;\; \Delta x_i^T\right]$, the forward differences of $V_{2,k}$, $V_{3,k}$, and $V_{4,k}$ are, respectively, given by

$$\begin{aligned} \Delta V_{2,k} &= x_k^T Q_1 x_k - x_{k-d_1}^T Q_1 x_{k-d_1} + x_k^T Q_2 x_k \\ &\quad - x_{k-d_3} Q_2 x_{k-d_3} = \zeta_k^T \Psi_2 \zeta_k \quad \text{from (3)}, \\ \Delta V_{3,k} &= \left(\Phi_{01}^1(1)\right)^2 \cdot \overline{x}_k^T \mathscr{R}_1 \overline{x}_k - \Phi_{01}^1(1) \\ &\quad \cdot \Phi_{01}^1\left(\overline{x}_i \mathscr{R}_1 \overline{x}_i\right) \quad \text{from (4)}, \\ \Delta V_{4,k} &= \left(\Phi_{13}^1(1)\right)^2 \cdot \overline{x}_k^T \mathscr{R}_2 \overline{x}_k - \Phi_{13}^1(1) \\ &\quad \cdot \Phi_{13}^1\left(\overline{x}_i^T \mathscr{R}_2 \overline{x}_i\right) \quad \text{from (4)}, \end{aligned} \tag{14}$$

where

$$\Psi_2 = \mathbf{e}_0^T Q_1 \mathbf{e}_0 - \mathbf{e}_1^T Q_1 \mathbf{e}_1 + \mathbf{e}_0^T Q_2 \mathbf{e}_0 - \mathbf{e}_3^T Q_2 \mathbf{e}_3. \quad (15)$$

Here, with the help of (9), $\Delta V_{3,k}$ and $\Delta V_{4,k}$ can be converted into

$$\begin{aligned}
\Delta V_{3,k} &= \left(\Phi_{01}^1(1)\right)^2 \cdot \overline{x}_k^T \mathscr{R}_1 \overline{x}_k - \Phi_{01}^1(1) \\
&\quad \cdot \Phi_{01}^1\left(\overline{x}_i^T \mathscr{R}_1 \overline{x}_i\right) + \Phi_{01}^1(1) \\
&\quad \cdot \zeta_k^T \left(\mathbf{e}_0^T X_0 \mathbf{e}_0 - \mathbf{e}_1^T X_0 \mathbf{e}_1\right) \zeta_k - \Phi_{01}^1(1) \\
&\quad \cdot \Phi_{01}^1\left(\Delta x_i X_0 \left(\Delta x_i + 2x_i\right)\right), \\
\Delta V_{4,k} &= \left(\Phi_{13}^1(1)\right)^2 \cdot \overline{x}_k^T \mathscr{R}_2 \overline{x}_k - \Phi_{13}^1(1) \\
&\quad \cdot \Phi_{13}^1\left(\overline{x}_i^T \mathscr{R}_2 \overline{x}_i\right) + \Phi_{13}^1(1) \\
&\quad \cdot \zeta_k^T \left(\mathbf{e}_1^T X_1 \mathbf{e}_1 - \mathbf{e}_2^T X_1 \mathbf{e}_2\right) \zeta_k - \Phi_{13}^1(1) \\
&\quad \cdot \Phi_{12}^1\left(\Delta x_i X_1 \left(\Delta x_i + 2x_i\right)\right) + \Phi_{13}^1(1) \\
&\quad \cdot \zeta_k^T \left(\mathbf{e}_2^T X_2 \mathbf{e}_2 - \mathbf{e}_3^T X_2 \mathbf{e}_3\right) \zeta_k - \Phi_{13}^1(1) \\
&\quad \cdot \Phi_{23}^1\left(\Delta x_i X_2 \left(\Delta x_i + 2x_i\right)\right).
\end{aligned} \quad (16)$$

In particular, by Lemma 1, the following inequality holds:

$$\begin{aligned}
&-\Phi_{01}^1(1) \cdot \Phi_{01}^1\left(\overline{x}_i^T \mathscr{R}_1 \overline{x}_i\right) - \Phi_{01}^1(1) \\
&\quad \cdot \Phi_{01}^1\left(\Delta x_i X_0 \left(\Delta x_i + 2x_i\right)\right) = -\Phi_{01}^1(1) \\
&\quad \cdot \Phi_{01}^1\left(\overline{x}_i^T \underbrace{\begin{bmatrix} R_{11} & X_0 \\ (*) & R_{12} + X_0 \end{bmatrix}}_{\triangleq \mathscr{X}_0} \overline{x}_i\right) \\
&\le -\Phi_{01}^1\left(\overline{x}_i\right)^T \mathscr{X}_0 \Phi_{01}^1\left(\overline{x}_i\right) \\
&= -\zeta_k^T \begin{bmatrix} \mathbf{e}_5 \\ \mathbf{e}_0 - \mathbf{e}_1 \end{bmatrix}^T \begin{bmatrix} R_{11} & X_0 \\ (*) & R_{12} + X_0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_5 \\ \mathbf{e}_0 - \mathbf{e}_1 \end{bmatrix} \zeta_k,
\end{aligned} \quad (17)$$

which implies $\Delta V_{3,k} \le \zeta_k^T \Psi_3 \zeta_k$, where

$$\begin{aligned}
\Psi_3 &= d_1^2 \left(\mathbf{e}_0^T R_{11} \mathbf{e}_0 + \mathbf{e}_4^T R_{12} \mathbf{e}_4\right) + d_1 \mathbf{e}_0^T X_0 \mathbf{e}_0 \\
&\quad - d_1 \mathbf{e}_1^T X_0 \mathbf{e}_1 - \mathbf{e}_5^T R_{11} \mathbf{e}_5 - \mathbf{He}\left[\mathbf{e}_5^T X_0 \left(\mathbf{e}_0 - \mathbf{e}_1\right)\right] \\
&\quad - \left(\mathbf{e}_0 - \mathbf{e}_1\right)^T \left(R_{12} + X_0\right)\left(\mathbf{e}_0 - \mathbf{e}_1\right).
\end{aligned} \quad (18)$$

Likewise, we can obtain

$$\begin{aligned}
&-\Phi_{13}^1(1) \cdot \Phi_{13}^1\left(\overline{x}_i^T \mathscr{R}_2 \overline{x}_i\right) - \Phi_{13}^1(1) \\
&\quad \cdot \Phi_{12}^1\left(\Delta x_i X_1 \left(\Delta x_i + 2x_i\right)\right) - \Phi_{13}^1(1) \\
&\quad \cdot \Phi_{23}^1\left(\Delta x_i X_2 \left(\Delta x_i + 2x_i\right)\right) = -\Phi_{13}^1(1) \\
&\quad \cdot \Phi_{12}^1\left(\overline{x}_i^T \underbrace{\begin{bmatrix} R_{21} & X_1 \\ (*) & R_{22} + X_1 \end{bmatrix}}_{\triangleq \mathscr{X}_1} \overline{x}_i\right) - \Phi_{13}^1(1) \\
&\quad \cdot \Phi_{23}^1\left(\overline{x}_i^T \underbrace{\begin{bmatrix} R_{21} & X_2 \\ (*) & R_{22} + X_2 \end{bmatrix}}_{\triangleq \mathscr{X}_2} \overline{x}_i\right) \\
&\le -\frac{\Phi_{13}^1(1)}{\Phi_{12}^1(1)} \cdot \Phi_{12}^1\left(\overline{x}_i\right)^T \mathscr{X}_1 \Phi_{12}^1\left(\overline{x}_i\right) - \frac{\Phi_{13}^1(1)}{\Phi_{23}^1(1)} \\
&\quad \cdot \Phi_{23}^1\left(\overline{x}_i\right)^T \mathscr{X}_2 \Phi_{23}^1\left(\overline{x}_i\right) \\
&= -\frac{1}{\theta_1} \Phi_{12}^1\left(\overline{x}_i\right)^T \mathscr{X}_1 \Phi_{12}^1\left(\overline{x}_i\right) - \frac{1}{\theta_2} \Phi_{23}^1\left(\overline{x}_i\right)^T \\
&\quad \cdot \mathscr{X}_2 \Phi_{23}^1\left(\overline{x}_i\right) \quad \text{from Property 2} \\
&= -\begin{bmatrix} \Phi_{12}^1\left(\overline{x}_i\right) \\ \Phi_{23}^1\left(\overline{x}_i\right) \end{bmatrix}^T \begin{bmatrix} \mathscr{X}_1 & \mathscr{S} \\ (*) & \mathscr{X}_2 \end{bmatrix} \begin{bmatrix} \Phi_{12}^1\left(\overline{x}_i\right) \\ \Phi_{23}^1\left(\overline{x}_i\right) \end{bmatrix} \\
&\quad - \begin{bmatrix} \sqrt{\frac{\theta_2}{\theta_1}} \Phi_{12}^1\left(\overline{x}_i\right) \\ -\sqrt{\frac{\theta_1}{\theta_2}} \Phi_{23}^1\left(\overline{x}_i\right) \end{bmatrix}^T \\
&\quad \cdot \begin{bmatrix} \mathscr{X}_1 & \mathscr{S} \\ (*) & \mathscr{X}_2 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\theta_2}{\theta_1}} \Phi_{12}^1\left(\overline{x}_i\right) \\ -\sqrt{\frac{\theta_1}{\theta_2}} \Phi_{23}^1\left(\overline{x}_i\right) \end{bmatrix} \\
&\le -\begin{bmatrix} \mathbf{e}_6 \\ \mathbf{e}_1 - \mathbf{e}_2 \\ \hline \mathbf{e}_7 \\ \mathbf{e}_2 - \mathbf{e}_3 \end{bmatrix}^T \\
&\quad \cdot \left[\begin{array}{cc|c} R_{21} & X_1 & \underbrace{\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}}_{\triangleq \mathscr{S}} \\ (*) & R_{22} + X_1 & \\ \hline \multicolumn{2}{c|}{(*)} & \begin{matrix} R_{21} & X_2 \\ (*) & R_{22} + X_2 \end{matrix} \end{array}\right] \begin{bmatrix} \mathbf{e}_6 \\ \mathbf{e}_1 - \mathbf{e}_2 \\ \hline \mathbf{e}_7 \\ \mathbf{e}_2 - \mathbf{e}_3 \end{bmatrix} \\
&\qquad \text{under (11)},
\end{aligned} \quad (19)$$

which implies $\Delta V_{4,k} \le \zeta_k^T \Psi_4 \zeta_k$, where

$$\begin{aligned}\Psi_4 &= \delta^2\left(\mathbf{e}_0^T R_{21}\mathbf{e}_0 + \mathbf{e}_4^T R_{22}\mathbf{e}_4\right) + \delta\mathbf{e}_1^T X_1 \mathbf{e}_1 - \delta\mathbf{e}_2^T X_1\mathbf{e}_2 \\ &\quad + \delta\mathbf{e}_2^T X_2\mathbf{e}_2 - \delta\mathbf{e}_3^T X_2\mathbf{e}_3 - \mathbf{e}_6^T R_{21}\mathbf{e}_6 \\ &\quad - \mathbf{He}\left[\mathbf{e}_6^T X_1\left(\mathbf{e}_1 - \mathbf{e}_2\right) + \mathbf{e}_6^T S_1 \mathbf{e}_7 + \mathbf{e}_6^T S_2\left(\mathbf{e}_2 - \mathbf{e}_3\right)\right] \\ &\quad - \left(\mathbf{e}_1 - \mathbf{e}_2\right)^T\left(R_{22} + X_1\right)\left(\mathbf{e}_1 - \mathbf{e}_2\right) \\ &\quad - \mathbf{He}\left[\left(\mathbf{e}_1 - \mathbf{e}_2\right)^T S_3\mathbf{e}_7 + \left(\mathbf{e}_1 - \mathbf{e}_2\right)^T S_4\left(\mathbf{e}_2 - \mathbf{e}_3\right)\right] \\ &\quad - \mathbf{e}_7^T R_{21}\mathbf{e}_7 - \mathbf{He}\left[\mathbf{e}_7^T X_2\left(\mathbf{e}_2 - \mathbf{e}_3\right)\right] \\ &\quad - \left(\mathbf{e}_2 - \mathbf{e}_3\right)^T\left(R_{22} + X_2\right)\left(\mathbf{e}_2 - \mathbf{e}_3\right).\end{aligned} \tag{20}$$

Hence, the forward difference of V_k satisfies

$$\Delta V_k \le \zeta_k^T\left(\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4\right)\zeta_k. \tag{21}$$

Moreover, by adding $0 = \zeta_k^T \Psi_5 \zeta_k$ to (21), where

$$\Psi_5 = \mathbf{He}\left[\left(\mathbf{e}_0^T Z_0 + \mathbf{e}_4^T Z_4\right)\left(\mathbf{e}_4 + (I - A)\mathbf{e}_0 - A_d\mathbf{e}_2\right)\right], \tag{22}$$

we can obtain $\Delta V_k \le \zeta_k^T(\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5)\zeta_k$. Therefore, the stability criterion is given by (10) and (11). □

Remark 3. For the given n_x, the number of scalar variables (NSVs) used in Theorem 2 is given as $15n_x^2 + 6n_x$. Our approach leads to a significant decrease in the computational burden compared with [10], [11], and [24] that demand $27n_x^2 + 9n_x$, $90.5n_x^2 + 14.5n_x$, and $16n_x^2 + 9n_x$ scalar variables, respectively.

Remark 4. To make up for the weakness in [10, 11, 24], this paper proposes a valuable method capable of deriving a less conservative stability criterion without using the free-weighting approach and the inclusion of Δx_{k-d_1}, Δx_{k-d_2}, and Δx_{k-d_3} in the augmented state.

Remark 5. From (10), it follows that $0 > \mathbf{e}_4(\Psi_1+\Psi_2+\Psi_3+\Psi_4+\Psi_5)\mathbf{e}_4^T = P_1 + R_{12} + R_{22} + \mathbf{He}[Z_4]$. As a result, $Z_4 + Z_4^T < 0$; that is, Z_4 is nonsingular since P_1, R_{12}, and R_{22} are positive definite.

As a by-product of Theorem 2, we can obtain a robust stability criterion for

$$\begin{aligned} x_{k+1} &= Ax_k + A_d x_{k-d(k)} + Gp_k, \\ q_k &= Ex_k + E_d x_{k-d(k)}, \end{aligned} \tag{23}$$

where $p_k \in \mathbb{R}^{n_p}$ and $q_k \in \mathbb{R}^{n_q}$ such that $p_k = \Delta_k q_k$ and $\Delta_k^T \Delta_k \le I$.

Corollary 6. *Let $d_1 = \underline{d}$ and $d_3 = \overline{d}$ be prescribed and define $\delta = d_3 - d_1$. System (1) is asymptotically stable for any time-varying $d(k)$ satisfying $d(k) \in [\underline{d}, \overline{d}]$, if there exist matrices $\{S_i\}_{i=1,\dots,4} \in \mathbb{R}^{n_x \times n_x}$, $\{Z_i\}_{i=0,4} \in \mathbb{R}^{n_x \times n_x}$ and symmetric matrices $\mathcal{P} \in \mathbb{R}^{3n_x \times 3n_x}$, $\{Q_i\}_{i=1,2} \in \mathbb{R}^{n_x \times n_x}$, $\{R_{1i}\}_{i=1,2} \in \mathbb{R}^{n_x \times n_x}$, $\{R_{2i}\}_{i=1,2} \in \mathbb{R}^{n_x \times n_x}$, and $\{X_i\}_{i=0,1,2} \in \mathbb{R}^{n_x \times n_x}$ such that (11) and*

$$0 > \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6, \tag{24}$$

where $\{\Psi_i\}_{i=1,\dots,6}$ are defined in (13), (15), (18), (20), (26), and (27), respectively.

Proof. Let us redefine the augmented state ζ_k as

$$\zeta_k^T = \left[x_k^T \;\; x_{k-d_1}^T \;\; x_{k-d_2}^T \;\; x_{k-d_3}^T \;\; \Delta x_k^T \;\; \Phi_{01}^1\left(x_i^T\right) \;\; \Phi_{12}^1\left(x_i^T\right) \;\; \Phi_{23}^1\left(x_i^T\right) \;\; p_k^T\right] \tag{25}$$

and establish a block entry matrix $\mathbf{e}_8$ such that $p_k = \mathbf{e}_8\zeta_k$. Then, the term Ψ_5 is naturally converted into

$$\begin{aligned}\Psi_5 = \mathbf{He}\Big[&\left(\mathbf{e}_0^T Z_0 + \mathbf{e}_4^T Z_4\right) \\ &\cdot\left(\mathbf{e}_4 + (I - A)\mathbf{e}_0 - A_d\mathbf{e}_2 - G\mathbf{e}_8\right)\Big],\end{aligned} \tag{26}$$

and the uncertainty such that $0 \le q_k^T q_k - p_k^T p_k$ is represented as $0 \le \zeta_k^T \Psi_6 \zeta_k$, where

$$\Psi_6 = \left(E\mathbf{e}_0 + E_d\mathbf{e}_2\right)^T\left(E\mathbf{e}_0 + E_d\mathbf{e}_2\right) - \mathbf{e}_8^T\mathbf{e}_8. \tag{27}$$

As a result, $\Delta V_k \le \zeta_k^T(\Psi_1 + \cdots + \Psi_5 + \Psi_6)\zeta_k$; thus, the robust stability criterion is given by (11) and (24). □

4. Control Synthesis

Let us consider a linear system of the following form:

$$x_{k+1} = Ax_k + Bu_{k-d(k)}, \tag{28}$$

where $u_{k-d(k)} \in \mathbb{R}^{n_u}$ denotes the delayed control input. Then, under the state-feedback control law $u_k = Fx_k$, the closed-loop control system is described as follows:

$$x_{k+1} = Ax_k + A_d x_{k-d(k)}, \tag{29}$$

where $A_d = BF$ and F denotes the control gain to be designed.

Theorem 7. *Let $d_1 = \underline{d}$, $d_3 = \overline{d}$, ϵ be prescribed and define $\delta = d_3 - d_1$. The closed-loop system in (29) is asymptotically stable for any time-varying $d(k)$ satisfying $d(k) \in [\underline{d}, \overline{d}]$, if there exist* *matrices* $\{\overline{S}_i\}_{i=1,\ldots,4} \in \mathbb{R}^{n_x\times n_x}$, $\overline{W} \in \mathbb{R}^{n_x\times n_x}$, *and* $\overline{F} \in \mathbb{R}^{n_u\times n_x}$ *and symmetric matrices* $\overline{\mathscr{P}} \in \mathbb{R}^{3n_x\times 3n_x}$, $\{\overline{Q}_i\}_{i=1,2} \in \mathbb{R}^{n_x\times n_x}$, $\{\overline{R}_{1i}\}_{i=1,2} \in \mathbb{R}^{n_x\times n_x}$, $\{\overline{R}_{2i}\}_{i=1,2} \in \mathbb{R}^{n_x\times n_x}$, *and* $\{\overline{X}_i\}_{i=0,1,2} \in \mathbb{R}^{n_x\times n_x}$ *such that*

$$0 > \overline{\Psi}_1 + \overline{\Psi}_2 + \overline{\Psi}_3 + \overline{\Psi}_4 + \overline{\Psi}_5, \quad (30)$$

$$0 \le \left[\begin{array}{cc|cc} \overline{R}_{21} & \overline{X}_1 & \overline{S}_1 & \overline{S}_2 \\ (*) & \overline{R}_{22}+\overline{X}_1 & \overline{S}_3 & \overline{S}_4 \\ \hline (*) & (*) & \overline{R}_{21} & \overline{X}_2 \\ (*) & (*) & (*) & \overline{R}_{22}+\overline{X}_2 \end{array}\right], \quad (31)$$

where $\{\overline{\Psi}_i\}_{i=1,\ldots,5}$ *are defined in (34)–(38), respectively. Moreover, the control gain F can be reconstructed by* $F = \overline{F}\,\overline{W}^{-1}$.

Proof. First of all, let us consider a nonsingular matrix $\overline{\mathscr{W}}$ of the following form:

$$\overline{\mathscr{W}} = \left[\mathbf{e}_0^T\overline{W}\ \ \mathbf{e}_1^T\overline{W}\ \ \mathbf{e}_2^T\overline{W}\ \ \mathbf{e}_3^T\overline{W}\ \ \mathbf{e}_4^T\overline{W}\ \ \mathbf{e}_5^T\overline{W}\ \ \mathbf{e}_6^T\overline{W}\ \ \mathbf{e}_7^T\overline{W}\right], \quad (32)$$

which satisfies that $\mathbf{e}_i\overline{\mathscr{W}} = \overline{W}\mathbf{e}_i$ for all i. Further, define

$$\overline{\mathscr{P}} = \begin{bmatrix} \overline{W} & 0 & 0 \\ 0 & \overline{W} & 0 \\ 0 & 0 & \overline{W} \end{bmatrix}^T \mathscr{P} \begin{bmatrix} \overline{W} & 0 & 0 \\ 0 & \overline{W} & 0 \\ 0 & 0 & \overline{W} \end{bmatrix} = \begin{bmatrix} \overline{P}_1 & \overline{P}_2 & \overline{P}_3 \\ (*) & \overline{P}_4 & \overline{P}_5 \\ (*) & (*) & \overline{P}_6 \end{bmatrix} \quad (33)$$

and let $Z_0 = W$ and $Z_4 = \epsilon W$, where $W = \overline{W}^{-1}$, and ϵ is a scalar variable. Then, pre- and postmultiplying Ψ_i by $\overline{\mathscr{W}}^T$ and $\overline{\mathscr{W}}$ yields $\overline{\Psi}_i = \overline{W}^T\Psi_i\overline{W}$:

$$\begin{aligned}
\overline{\Psi}_1 &= \begin{bmatrix} \mathbf{e}_4+\mathbf{e}_0 \\ \mathbf{e}_0-\mathbf{e}_1+\mathbf{e}_5 \\ \mathbf{e}_1-\mathbf{e}_3+\mathbf{e}_6+\mathbf{e}_7 \end{bmatrix}^T \overline{\mathscr{P}} \begin{bmatrix} \mathbf{e}_4+\mathbf{e}_0 \\ \mathbf{e}_0-\mathbf{e}_1+\mathbf{e}_5 \\ \mathbf{e}_1-\mathbf{e}_3+\mathbf{e}_6+\mathbf{e}_7 \end{bmatrix} - \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_5 \\ \mathbf{e}_6+\mathbf{e}_7 \end{bmatrix}^T \overline{\mathscr{P}} \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_5 \\ \mathbf{e}_6+\mathbf{e}_7 \end{bmatrix} = \mathbf{e}_4^T\overline{P}_1\mathbf{e}_4 \\
&+ \mathbf{He}\left[\mathbf{e}_4^T\overline{P}_1\mathbf{e}_0 + \mathbf{e}_4^T\overline{P}_2(\mathbf{e}_0-\mathbf{e}_1) + \mathbf{e}_4^T\overline{P}_2\mathbf{e}_5 + \mathbf{e}_4^T\overline{P}_3(\mathbf{e}_1-\mathbf{e}_3) + \mathbf{e}_4^T\overline{P}_3(\mathbf{e}_6+\mathbf{e}_7)\right] \\
&+ \mathbf{He}\left[\mathbf{e}_0^T\overline{P}_2(\mathbf{e}_0-\mathbf{e}_1) + \mathbf{e}_0^T\overline{P}_3(\mathbf{e}_1-\mathbf{e}_3)\right] + (\mathbf{e}_0-\mathbf{e}_1)^T\overline{P}_4(\mathbf{e}_0-\mathbf{e}_1) + \mathbf{He}\left[(\mathbf{e}_0-\mathbf{e}_1)^T\overline{P}_4\mathbf{e}_5 \right.\\
&\left. + (\mathbf{e}_0-\mathbf{e}_1)^T\overline{P}_5(\mathbf{e}_1-\mathbf{e}_3) + (\mathbf{e}_0-\mathbf{e}_1)^T\overline{P}_5(\mathbf{e}_6+\mathbf{e}_7)\right] + \mathbf{He}\left[\mathbf{e}_5^T\overline{P}_5(\mathbf{e}_1-\mathbf{e}_3)\right] \\
&+ (\mathbf{e}_1-\mathbf{e}_3)^T\overline{P}_6(\mathbf{e}_1-\mathbf{e}_3) + \mathbf{He}\left[(\mathbf{e}_1-\mathbf{e}_3)^T\overline{P}_6(\mathbf{e}_6+\mathbf{e}_7)\right],
\end{aligned} \quad (34)$$

$$\overline{\Psi}_2 = \mathbf{e}_0^T\underbrace{\overline{W}^TQ_1\overline{W}}_{=\overline{Q}_1}\mathbf{e}_0 - \mathbf{e}_1^T\overline{Q}_1\mathbf{e}_1 + \mathbf{e}_0^T\underbrace{\overline{W}^TQ_2\overline{W}}_{=\overline{Q}_2}\mathbf{e}_0 - \mathbf{e}_3^T\overline{Q}_2\mathbf{e}_3, \quad (35)$$

$$\begin{aligned}
\overline{\Psi}_3 &= d_1^2\left(\mathbf{e}_0^T\underbrace{\overline{W}^TR_{11}\overline{W}}_{=\overline{R}_{11}}\mathbf{e}_0 + \mathbf{e}_4^T\underbrace{\overline{W}^TR_{12}\overline{W}}_{=\overline{R}_{12}}\mathbf{e}_4\right) + d_1\mathbf{e}_0^T\underbrace{\overline{W}^TX_0\overline{W}}_{=\overline{X}_0}\mathbf{e}_0 - d_1\mathbf{e}_1^T\overline{X}_0\mathbf{e}_1 - \mathbf{e}_5^T\overline{R}_{11}\mathbf{e}_5 \\
&- \mathbf{He}\left[\mathbf{e}_5^T\overline{X}_0(\mathbf{e}_0-\mathbf{e}_1)\right] - (\mathbf{e}_0-\mathbf{e}_1)^T\overline{R}_{12}(\mathbf{e}_0-\mathbf{e}_1) - (\mathbf{e}_0-\mathbf{e}_1)^T\overline{X}_0(\mathbf{e}_0-\mathbf{e}_1),
\end{aligned} \quad (36)$$

$$\begin{aligned}
\overline{\Psi}_4 &= \delta^2\left(\mathbf{e}_0^T\underbrace{\overline{W}^TR_{21}\overline{W}}_{=\overline{R}_{21}}\mathbf{e}_0 + \mathbf{e}_4^T\underbrace{\overline{W}^TR_{22}\overline{W}}_{=\overline{R}_{22}}\mathbf{e}_4\right) + \delta\mathbf{e}_1^T\underbrace{\overline{W}^TX_1\overline{W}}_{=\overline{X}_1}\mathbf{e}_1 - \delta\mathbf{e}_2^T\overline{X}_1\mathbf{e}_2 + \delta\mathbf{e}_2^T\underbrace{\overline{W}^TX_2\overline{W}}_{=\overline{X}_2}\mathbf{e}_2 \\
&- \delta\mathbf{e}_3^T\overline{X}_2\mathbf{e}_3 - \mathbf{e}_6^T\overline{R}_{21}\mathbf{e}_6 - \mathbf{He}\left[\mathbf{e}_6^T\overline{X}_1(\mathbf{e}_1-\mathbf{e}_2) + \mathbf{e}_6^T\underbrace{\overline{W}^TS_1\overline{W}}_{=\overline{S}_1}\mathbf{e}_7\right] - \mathbf{He}\left[\mathbf{e}_6^T\underbrace{\overline{W}^TS_2\overline{W}}_{=\overline{S}_2}(\mathbf{e}_2-\mathbf{e}_3)\right] \\
&- (\mathbf{e}_1-\mathbf{e}_2)^T\overline{R}_{22}(\mathbf{e}_1-\mathbf{e}_2) - (\mathbf{e}_1-\mathbf{e}_2)^T\overline{X}_1(\mathbf{e}_1-\mathbf{e}_2) \\
&- \mathbf{He}\left[(\mathbf{e}_1-\mathbf{e}_2)^T\underbrace{\overline{W}^TS_3\overline{W}}_{=\overline{S}_3}\mathbf{e}_7 + (\mathbf{e}_1-\mathbf{e}_2)^T \cdot \underbrace{\overline{W}^TS_4\overline{W}}_{=\overline{S}_4}(\mathbf{e}_2-\mathbf{e}_3)\right] - \mathbf{e}_7^T\overline{R}_{21}\mathbf{e}_7 \\
&- \mathbf{He}\left[\mathbf{e}_7^T\overline{X}_2(\mathbf{e}_2-\mathbf{e}_3)\right] - (\mathbf{e}_2-\mathbf{e}_3)^T\overline{R}_{22}(\mathbf{e}_2-\mathbf{e}_3) - (\mathbf{e}_2-\mathbf{e}_3)^T\overline{X}_2(\mathbf{e}_2-\mathbf{e}_3),
\end{aligned} \quad (37)$$

$$\overline{\Psi}_5 = \mathbf{He}\left[\overline{\mathscr{W}}^T\left(\mathbf{e}_0^T Z_0 + \mathbf{e}_4^T Z_4\right) \cdot \left(\overline{W}\mathbf{e}_4 + (I - A)\overline{W}\mathbf{e}_0 - A_d\overline{W}\mathbf{e}_2\right)\right] = \mathbf{He}\left[\left(\mathbf{e}_0^T + \epsilon\mathbf{e}_4^T\right) \cdot \left(\overline{W}\mathbf{e}_4 + \left(\overline{W} - A\overline{W}\right)\mathbf{e}_0 - B\underbrace{F\overline{W}}_{=\overline{F}}\mathbf{e}_2\right)\right]. \quad (38)$$

In other words, the stabilization condition is given by

$$0 > \sum_{i=1}^{5}\overline{\mathscr{W}}^T\Psi_i\overline{\mathscr{W}} = \sum_{i=1}^{5}\overline{\Psi}_i, \quad (39)$$

which becomes (30). Here, (30) implies $\mathbf{e}_7(\sum_{i=1}^{5}\overline{\Psi}_i)\mathbf{e}_7^T = -\overline{R}_{21} = -\overline{W}^T R_{21}\overline{W} < 0$; that is, $\overline{W}$ is nonsingular and thus $\overline{\mathscr{W}}$ becomes also nonsingular. Next, pre- and postmultiplying (11) by $\text{diag}(\overline{W}^T, \overline{W}^T, \overline{W}^T, \overline{W}^T)$ and its transpose yields (31). □

5. Numerical Examples

Three numerical examples are considered in order to illustrate the effectiveness of the obtained results.

Example 1 (stability analysis). Let us consider a delayed discrete-time system used in [17]:

$$x_{k+1} = \begin{bmatrix} 0.80 & 0.00 \\ 0.05 & 0.90 \end{bmatrix} x_k + \begin{bmatrix} -0.1 & 0.0 \\ -0.2 & -0.1 \end{bmatrix} x_{k-d(k)}, \quad (40)$$

where $\underline{d} \le d(k) \le \overline{d}$. For (40) with various $\underline{d}$, Table 1 lists the maximum allowable upper bounds (MAUBs) of $d(k)$, obtained by Theorem 2 and different methods. From Table 1, it can be seen that the stability criteria established in [10, 11] and Theorem 2 offer the most improved of the results. In particular, it is noteworthy that Theorem 2 provides the same delay bounds (i.e., stability region) as those of [10, 11] under the requirement of much less computational complexity with respect to the number of scalar variables (NSVs), as mentioned in Remark 3. That is, in contrast with [10, 11, 24], Theorem 2 offers a more efficient approach in terms of both performance and computational complexity.

Example 2 (robust stability analysis). Consider the following uncertain discrete-time system with time-varying delay used in [24]:

$$x_{k+1} = \begin{bmatrix} 0.8 & 0.0 \\ 0.0 & 0.9 \end{bmatrix} x_k + \begin{bmatrix} -0.1 & 0.0 \\ -0.1 & -0.1 \end{bmatrix} x_{k-d(k)} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix} p_k, \quad (41)$$
$$q_k = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix} x_k.$$

Table 1: Example 1: MAUB of $d(k)$ for various $\underline{d}$.

$\underline{d}$	2	6	10	15	20	NSV
Theorem 1 [17]	13	14	17	21	25	21
Theorem 1 [18]	14	16	18	21	25	18
Proposition 1 [23]	17	18	20	23	27	38
Corollary 1 [24]	19	20	21	24	27	82
Corollary 3 [11]	22	22	23	25	28	391
Theorem 2 [10]	22	22	23	25	28	126
Theorem 2	22	22	23	25	28	72

Table 2: Example 2: MAUB of α for various $d(k) \in [\underline{d}, \overline{d}]$.

$d(k) \in [\underline{d}, \overline{d}]$	$[2, 7]$	$[5, 10]$	$[10, 15]$	$[20, 25]$	NSV
Theorem 1 [24]	0.2007	0.1554	0.1144	0.0957	82
Corollary 6	0.2013	0.1555	0.1155	0.0961	72

Table 3: Example 3: MAUB of $d(k)$ and control gains F when $\underline{d} = 1$.

Methods	$\overline{d}$	Control gains	NSV
Theorem 3 [26]	4	$F = \begin{bmatrix} 110.6827 & 34.6980 \end{bmatrix}$	BMI problem
Theorem 3 [10]	7	$F = \begin{bmatrix} 98.5858 & 24.0621 \end{bmatrix}$	131
Theorem 7	7	$F = \begin{bmatrix} 100.5577 & 24.4503 \end{bmatrix}$	70

For each $\underline{d} \le d(k) \le \overline{d}$, the MAUBs of α such that (23) is robustly asymptotically stable are listed in Table 2, obtained by Corollary 6 and different methods. That is, from Table 2, we can see that the robust stability criterion given in Corollary 6 is much more efficient than the ones in [24] from the viewpoint of both performance and computational complexity.

Example 3 (control synthesis). Consider the following discrete-time system transformed from the continuous-time model of an inverted pendulum (refer to [7]):

$$x_{k+1} = \begin{bmatrix} 1.0078 & 0.0301 \\ 0.5202 & 1.0078 \end{bmatrix} x_k + \begin{bmatrix} -0.0001 \\ -0.0053 \end{bmatrix} u_{k-d(k)}. \quad (42)$$

The goal of this example is to design the control $u_k = Fx_k$ that stabilizes (42) with $1 \le d(k) \le \overline{d}$, such that the closed-loop system is asymptotically stable. Table 3 shows the MAUBs of $d(k)$ and the corresponding control gains, obtained by Theorem 7 ($\epsilon = 100$). From Table 3, we can see that the proposed stabilization condition is significantly valuable in the sense that it requires less computational complexity as well as providing larger MAUB than that of [26]. Meanwhile, based on the obtained control gain, Figure 1 shows the state responses of (42) with $x_k = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ for $k \in \{-7, \ldots, 0\}$ and $d(k) = 6 \cdot |\lceil \sin(\pi k/2) \rceil| + 1 \in [1, 7]$. Here, it can be found that the state converges to zero as time goes to infinity.

Remark 8. In Example 3, the matrix W is given by

$$W = \overline{W}^{-1} = \begin{bmatrix} -24.0367 & -5.9185 \\ -5.7376 & -1.5868 \end{bmatrix}, \quad \text{for } \epsilon > 0. \quad (43)$$

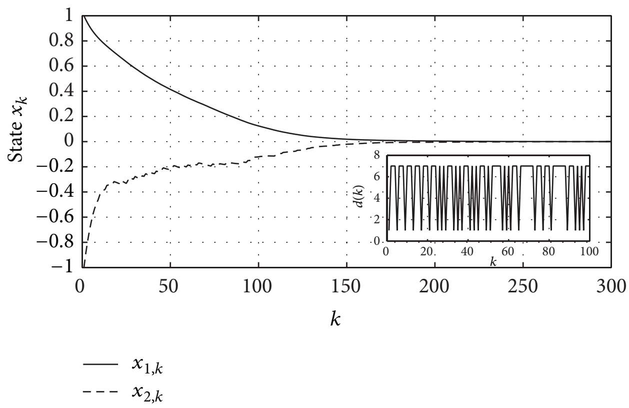

FIGURE 1: State responses $x_k = [x_{1,k} \ \ x_{2,k}]^T$ and time-varying delay $d(k) = 6 \cdot |\lceil \sin(\pi k/2) \rceil| + 1$.

Thus, as mentioned in Remark 4, the following relation is satisfied: $\mathbf{He}[Z_4] = \mathbf{He}[\epsilon W] < 0$.

6. Concluding Remarks

In this paper, the problem of deriving an efficient stability criterion is investigated for discrete-time systems with time-varying delay. The main feature herein is that the conservatism of a stability criterion is reduced in spite of the requirement of less computational complexity. In addition, the stabilization problem of systems with time-delayed control input is addressed in the LMI framework.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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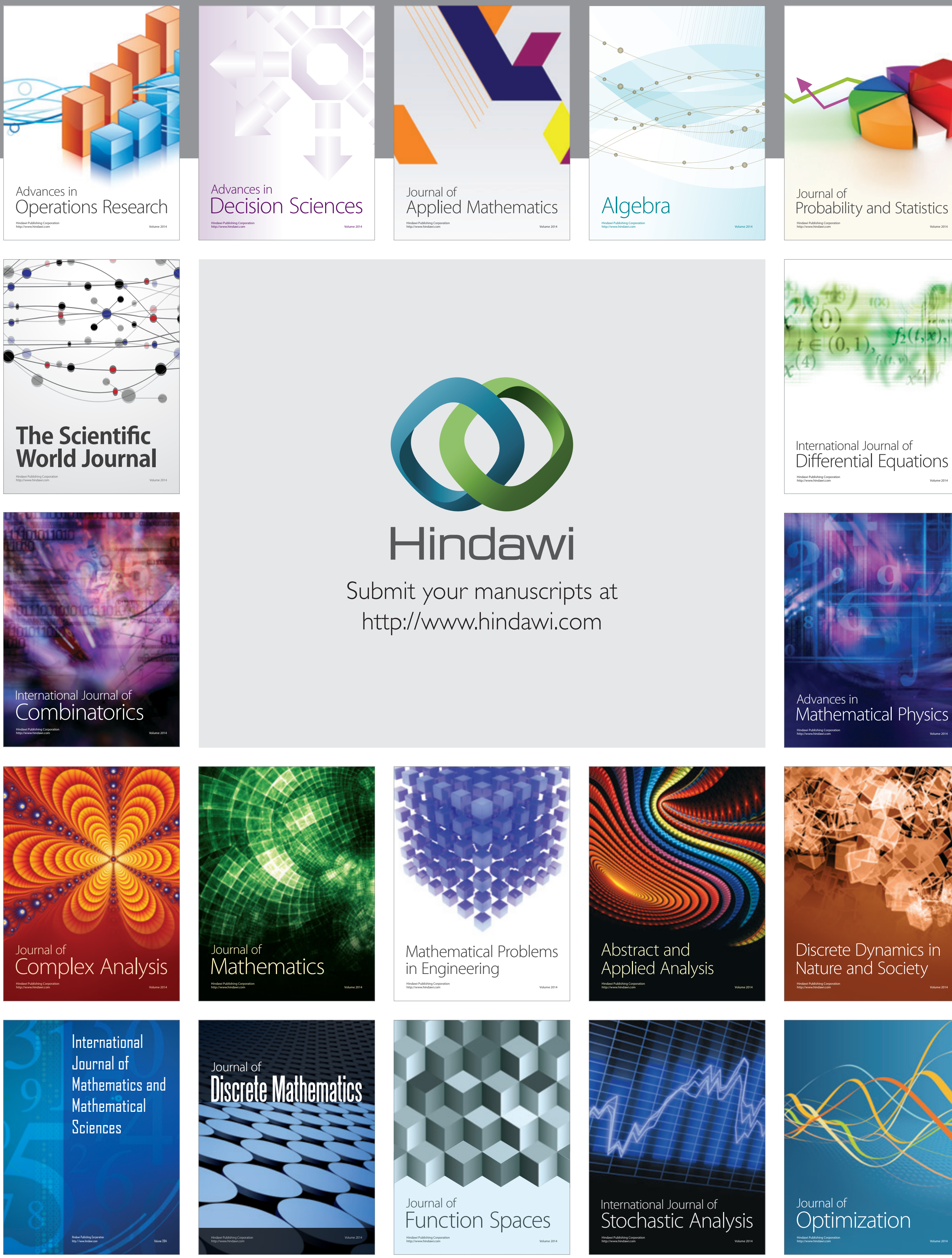

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