Research Article

Feedback Stabilization for a Class of Nonlinear Stochastic Systems with State- and Control-Dependent Noise

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This paper mainly studies the state feedback stabilizability of a class of nonlinear stochastic systems with state- and control-dependent noise. Some sufficient conditions on local and global state feedback stabilizations are given in linear matrix inequalities (LMIs) and generalized algebraic Riccati equations (GAREs). Some obtained results improve the previous work.

1. Introduction

Stability and stabilization are two important topics in modern control theory, which are first of considered issues in the systems analysis and synthesis. It is well known that stochastic control has become a very popular research area, which has been applied to mathematical finance [1], quantum systems [2], and so forth; stochastic stability and stabilization have been studied by many researchers; we refer the reader to the celebrated book [1] for the discussions of various stabilities. A series of works on robustly exponential stability can be found in [3–6]. While the pth moment stability were discussed in [7, 8], in particular, the asymptotic mean square stability has been studied for a long time; see [9–13]. The stabilizability of linear stochastic control systems has been investigated by [9, 10, 12–17]. In recent years, the study for stabilization of nonlinear stochastic systems has attracted great attention; the methods appearring in studying this topic can be summarized as follows: GARE-based method [9, 12, 18, 19]; control Lyapunov function method [1, 3–6, 20]; passive system method [21], and spectral analysis method based on generalized Lyapunov operators [13, 16, 17]. We refer the reader to [19] for the stabilization of general nonlinear stochastic systems, where a class of new Hamilton-Jacobi inequalities were presented.

It can be seen that most of the previous works were on the systems with only the state-dependent noise.

In the present paper, we deal with a class of linearized systems with both the state- and control-dependent noise. Some sufficient conditions on local state feedback stabilization are given via LMIs and GAREs, respectively, which not only generalize but also improve the results of [18]. We also investigate the global state feedback stabilization and a sufficient condition is also given in terms of LMIs. A numerical example verifies the effectiveness of our results.

2. Problem Setting

Consider the following stochastic control system governed by Itô's differential equation:

$$dx(t) = f(x(t), u(t)) dt + \sum_{i=1}^{l} \sigma_i(x(t), u(t)) dW_i(t). \quad (1)$$

In the above, $x(t) \in \mathcal{R}^n$ is called the system state and $u(t) \in \mathcal{R}^m$ the control input. $\{W_i(t), i = 1, \ldots, l\}$ is the standard Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filter $\mathcal{F}_t = \sigma(W_i(s), 0 \le s \le t; i = 1, \ldots, l)$. Without loss of generality, we can suppose $\{W_i(t), i = 1, \ldots, l\}$ are one-dimensional. Assume $u(t)$ is an adapted and measurable process with respect to $\mathcal{F}_t$, $f(0, u) \equiv 0$, $\sigma_i(0, u) \equiv 0$; that is, $x \equiv 0$ is an equilibrium point of (1). Under very general conditions on f, σ_i ($i = 1, \ldots, l$), and

$u(t)$, stochastic control system (1) has a unique strong solution $x_{s,\xi}(t)$ for any $t \geq s \geq 0$ and the initial state $x(s) = \xi \in \mathcal{R}^n$; see [1, 22]. We first introduce the following definition.

Definition 1. We say that the equilibrium point $x \equiv 0$ of system (1) is locally asymptotically stabilizable via a linear constant state feedback $u(t) = Kx(t)$, if the solution $x \equiv 0$ of the closed-loop system,

$$dx(t) = f(x(t), Kx(t))\,dt + \sum_{i=1}^{l} \sigma_i(x(t), Kx(t))\,dW_i(t),$$
(2)

is asymptotically stable in probability [1]; that is, for any $s \geq 0$ and $\varepsilon > 0$

$$\lim_{\|\xi\| \to 0} \mathcal{P}\left(\sup_{t \geq s} \|x_{s,\xi}(t)\| > \varepsilon \right) = 0,$$
(3)

$$\lim_{\|\xi\| \to 0} \mathcal{P}\left(\lim_{t \to \infty} x_{s,\xi}(t) = 0 \right) = 1,$$
(4)

where K is a constant matrix of suitable dimension. In addition, if the solution $x \equiv 0$ of the closed-loop system (2) is asymptotically stable in the large (see, e.g., [1]), that is, if both (3) and

$$\mathcal{P}\left(\lim_{t \to \infty} x_{s,\xi}(t) = 0 \right) = 1$$
(5)

hold, then we say that the equilibrium point $x \equiv 0$ of system (1) is globally asymptotically stabilizable via a linear state feedback $u(t) = Kx(t)$.

It is well known [1] that if there exists a neighborhood U of the origin, a Lyapunov function $V(t, x) \in C_2^0(\{t > 0\} \times U)$, $V(t, x) > 0$ in domain $\{t > 0\} \times U$, which has an infinitesimal upper limit, that is,

$$\lim_{\substack{\|x\| \to 0 \\ t > 0}} \sup V(t, x) = 0,$$
(6)

satisfying

$$\mathcal{L}V(t, x) = \frac{\partial V(t, x)}{\partial t} + f'(x, Kx)\frac{\partial V(t, x)}{\partial x}$$
$$+ \frac{1}{2}\sum_{i=1}^{l} \sigma_i'(x(t), Kx(t))$$
(7)
$$\times \frac{\partial^2 V(t, x)}{\partial x^2}\sigma_i(x(t), Kx(t)) < 0,$$

then the solution $x(t) \equiv 0$ of system (2) is asymptotically stable in probability. If $V(t, x)$ also admits the following property,

$$\inf_{t > 0} V(t, x) \longrightarrow \infty, \quad \text{as } \|x\| \longrightarrow \infty,$$
(8)

then the solution $x(t) \equiv 0$ of system (2) is asymptotically stable in the large. $\mathcal{L}$ is the so-called infinitesimal generator of (2).

Now, suppose $f(x, u)$ and $\sigma_i(x, u)$ can be linearized as

$$f(x, u) = Ax + Bu + H_0(x, u), \qquad H_0(0, u) = 0,$$
$$\sigma_i(x, u) = C_i x + D_i u + H_i(x, u), \qquad H_i(0, u) = 0, \quad (9)$$
$$i = 1, \ldots, l,$$

Respectively; then the linearized stochastic system of (1) is as

$$dx = \left(Ax + Bu + H_0(x, u) \right) dt$$
$$+ \sum_{i=1}^{l} \left(C_i x + D_i u + H_i(x, u) \right) dW_i,$$
(10)

where $A, B, C_i, D_i, i = 1, \ldots, l$, are constant matrices. In what follows, we will discuss the stabilization of (10).

3. Locally Asymptotic Stabilization

3.1. Main Results. In this section, we obtain two theorems on locally asymptotic stabilization of (10) as follows.

Theorem 2. *Suppose*

$$\lim_{\|x\| \to 0} \frac{\|H_i(x, Kx)\|}{\|x\|} = 0, \quad \forall K \in \mathcal{R}^{m \times n},$$
(11)
$$i = 0, 1, \ldots, l,$$

and the following LMI,

$$\begin{bmatrix} PA' + AP + Y'B' + BY & C_1 P + D_1 Y & C_2 P + D_2 Y & \cdots & C_l P + D_l Y \\ PC_1' + Y'D_1' & -P & 0 & \cdots & 0 \\ PC_2' + Y'D_2' & 0 & -P & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ PC_l' + Y'D_l' & 0 & 0 & \cdots & -P \end{bmatrix} < 0,$$
(12)

has a solution $P > 0$, $Y \in \mathcal{R}^{m \times n}$; then the equilibrium point $x \equiv 0$ of system (10) is locally asymptotically stabilizable in probability with control law

$$u(t) = YP^{-1}x(t).\tag{13}$$

The following theorem is another description for locally asymptotic stabilization in probability via GARE.

Theorem 3. *Under the condition of* (11), *if for any* $R > 0$, $Q > 0$, *GARE,*

$$PA + A'P + \sum_{i=1}^{l} C_i' P C_i - \left(PB + \sum_{i=1}^{l} C_i' P D_i \right) \tag{14}$$
$$\times \left(R + \sum_{i=1}^{l} D_i' P D_i \right)^{-1} \left(B'P + \sum_{i=1}^{l} D_i' P C_i \right) + Q = 0,$$

has a positive solution $P > 0$, *then system* (10) *is locally asymptotically stabilizable in probability with control law*

$$u(t) = -\left(R + \sum_{i=1}^{l} D_i' P D_i \right)^{-1} \left(B'P + \sum_{i=1}^{l} D_i' P C_i \right) x(t). \tag{15}$$

To prove our main results, we first consider the linear constant coefficient stochastic control system

$$dx = Fx\, dt + \sum_{i=1}^{l} G_i x dW_i. \tag{16}$$

System (16) is said to be asymptotically mean square stable if, for any $x(s) = \xi$, $\lim_{t \to \infty} E\|x_{s,\xi}(t)\|^2 = 0$.

Lemma 4 (see [23]). *System* (16) *is asymptotically mean square stable if and only if the following Lyapunov-type inequality,*

$$PF + F'P + \sum_{i=1}^{l} G_i' P G_i < 0, \tag{17}$$

has at least one solution $P > 0$.

Lemma 5 (see [13]). *System* (16) *is asymptotically mean square stable if and only if its dual system,*

$$dx = F'x\, dt + \sum_{i=1}^{l} G_i' x dW_i, \tag{18}$$

is asymptotically mean square stable.

Proof of Theorem 2. By Schur's complement, LMI (12) is equivalent to the following inequality:

$$PA' + AP + Y'B' + BY \tag{19}$$
$$+ \sum_{i=1}^{l} (C_i P + D_i Y) P^{-1} (C_i P + D_i Y)' < 0,$$

having a pair of solutions $P > 0$, $Y \in \mathcal{R}^{m \times n}$. Let $Y = KP$; then (19) becomes

$$(A + BK)P + P(A + BK)' \tag{20}$$
$$+ \sum_{i=1}^{l} (C_i + D_i K) P (C_i + D_i K)' < 0.$$

By Lemma 4, (20) implies

$$dx = (A + BK)'x\, dt + \sum_{i=1}^{l} (C_i + D_i K)' x dW_i, \tag{21}$$

to be asymptotically mean square stable, which yields

$$dx = (A + BK)x\, dt + \sum_{i=1}^{l} (C_i + D_i K) x dW_i, \tag{22}$$

to be also asymptotically mean square stable from Lemma 5. Again, by Lemma 4, there exists at least one solution $P_1 > 0$ satisfying

$$P_1(A + BK) + (A + BK)'P_1 \tag{23}$$
$$+ \sum_{i=1}^{l} (C_i + D_i K)' P_1 (C_i + D_i K) < 0.$$

Take the Lyapunov function $V(x) = x'P_1 x$, $u(t) = Kx(t) = YP^{-1}x(t)$; then, for system (10),

$$\mathcal{L}V(x) = \sum_{i=1}^{l} (C_i x + D_i Kx + H_i(x, Kx))'$$
$$\times P_1 (C_i x + D_i Kx + H_i(x, Kx))$$
$$+ 2 \langle Ax + BKx + H_0(x, Kx), P_1 x \rangle$$
$$= x' \left[P_1(A + BK) + (A + BK)'P_1 \right.$$
$$\left. + \sum_{i=1}^{l} (C_i + D_i K)' P_1 (C_i + D_i K) \right] x \tag{24}$$
$$+ 2H_0'(x, Kx) P_1 x$$
$$+ 2 \sum_{i=1}^{l} H_i'(x, Kx) P_1 (C_i + D_i K) x$$
$$+ \sum_{i=1}^{l} H_i'(x, Kx) P_1 H_i(x, Kx).$$

Let

$$-Q := P_1(A + BK) + (A + BK)'P_1 \tag{25}$$
$$+ \sum_{i=1}^{l} (C_i + D_i K)' P_1 (C_i + D_i K);$$

then, by (23), $Q > 0$. So

$$
\begin{aligned}
x' &\left[P_1 (A + BK) + (A + BK)' P_1 \right. \\
&\left. + \sum_{i=1}^{l} (C_i + D_i K)' P_1 (C_i + D_i K) \right] x \\
&= -x'Qx \leq -\lambda_{\min}(Q) \|x\|^2.
\end{aligned}
\tag{26}
$$

By the given condition (11), for any $\epsilon > 0$, there exists $\delta > 0$, such that when $\|x\| < \delta$, $\|H_i(x, Kx)\| \leq \epsilon \|x\|$, $i = 0, 1, \ldots, l$. So

$$
\begin{aligned}
2H_0' &(x, Kx) P_1 x + 2 \sum_{i=1}^{l} H_i'(x, Kx) P_1 (C_i + D_i K) x \\
&+ \sum_{i=1}^{l} H_i'(x, Kx) P_1 H_i(x, Kx) \\
&\leq \left(2\epsilon + 2\epsilon \sum_{i=1}^{l} \|C_i + D_i K\| + l\epsilon^2 \right) \\
&\quad \times \|P_1\| \cdot \|x\|^2 \leq \frac{1}{2} \lambda_{\min}(Q) \|x\|^2.
\end{aligned}
\tag{27}
$$

If we take ϵ sufficiently small, such that

$$
2\epsilon + 2\epsilon \sum_{i=1}^{l} \|C_i + D_i K\| + l\epsilon^2 \leq \frac{\lambda_{\min}(Q)}{2 \|P_1\|},
\tag{28}
$$

then (26) together with (27) gives

$$
\mathscr{L}V(x) \leq -\frac{1}{2} \lambda_{\min}(Q) \|x\|^2 < 0,
\tag{29}
$$

for $x \neq 0$. Therefore, the system (10) is locally asymptotically stabilizable in probability with control law

$$
u(t) = Kx(t) = YP^{-1}x(t).
\tag{30}
$$

The proof of Theorem 2 is completed. □

Remark 6. If there is a constant matrix K of suitable dimension such that system (22) is asymptotically mean square stable, then the following control system,

$$
dx = (Ax + Bu)\,dt + \sum_{i=1}^{l} (C_i x + D_i u)\,dW_i,
\tag{31}
$$

is called stabilizable in mean square sense [9, 12, 13].

Proof of Theorem 3. Note that if we let

$$
K = -\left(R + \sum_{i=1}^{l} D_i' P D_i \right)^{-1} \left(B'P + \sum_{i=1}^{l} D_i' P C_i \right),
\tag{32}
$$

then GARE (14) can be written as

$$
\begin{aligned}
P &(A + BK) + (A + BK)' P \\
&+ \sum_{i=1}^{l} (C_i + D_i K)' P (C_i + D_i K) \\
&= -Q - K'RK < 0.
\end{aligned}
\tag{33}
$$

By repeating the proof of Theorem 2, Theorem 3 is easy to be proved. □

In fact, Theorems 2 and 3 are equivalent; this can be seen from the following proposition.

Proposition 7. *If, for some $Q > 0$, $R > 0$, GARE (14) has a positive solution $P > 0$, then LMI (12) is feasible with at least a pair of solutions $P_1 > 0$, $Y \in \mathscr{R}^{m \times n}$; conversely, if LMI (12) has a pair of solutions $P_1 > 0$, $Y \in \mathscr{R}^{m \times n}$, then, for any $Q > 0$, $R > 0$, GARE (14) has a unique positive solution $P > 0$.*

Proof. If, for some $Q > 0$, $R > 0$, GARE (14) has a positive solution $P > 0$, then from (33) together with Lemma 4, system (22) is asymptotically mean square stable. Accordingly, system (21) is also asymptotically mean square stable by Lemma 5. Again, by Lemma 4, there exists $P_1 > 0$, such that

$$
\begin{aligned}
P_1 &(A + BK)' + (A + BK) P_1 \\
&+ \sum_{i=1}^{l} (C_i + D_i K) P_1^{-1} (C_i + D_i K)' < 0.
\end{aligned}
\tag{34}
$$

Let $Y = KP_1$; then (34) follows

$$
\begin{aligned}
P_1 A' &+ AP_1 + Y'B' + BY \\
&+ \sum_{i=1}^{l} (C_i P_1 + D_i Y) P_1^{-1} (C_i P_1 + D_i Y)' < 0.
\end{aligned}
\tag{35}
$$

By Schur's complement, $P_1 > 0$ and Y are also the solutions of (12). Conversely, if (12) has a pair of solutions $P_1 > 0$, $Y \in \mathscr{R}^{m \times n}$, then, from the same discussion as above, system (22) is mean square stable. So (31) is stabilizable in mean square sense. From [9, 13], for any $Q > 0$, $R > 0$, GARE (14) has a unique positive solution $P > 0$. □

Remark 8. Although Theorem 2 is equivalent to Theorem 3, it seems that Theorem 2 is more convenient in actual use than Theorem 3, because we can easily test whether or not LMI (12) is feasible by existing convex optimization tools; see [10, 24]. However, we would like to point out that if GARE (14) has a positive solution $P > 0$, by applying Theorem 10 of [9], P must solve the following semidefinite programming problem:

$$
\max \operatorname{Tr}(P),
\tag{36}
$$

subject to

$$
\begin{bmatrix}
PA + A'P + \sum_{i=1}^{l} C_i'PC_i + Q & PB + \sum_{i=1}^{l} C_i'PD_i \\
B'P + \sum_{i=1}^{l} D_i'PC_i & R + \sum_{i=1}^{l} D_i'PD_i
\end{bmatrix} \leq 0,
$$

$$
R + \sum_{i=1}^{l} D_i'PD_i > 0, \quad P > 0. \tag{37}
$$

The semidefinite programming problem (36)-(37), as LMI (12), can also be verified via some convex optimization tools [10, 24].

3.2. Comparison with the Existing Results. In (10), if we let $D_i = 0, H_i(x, u) \equiv 0$ for $i = 1, \ldots, l$, then the linearized system of (1) becomes

$$
dx = \left(Ax + Bu + H_0(x, u) \right) dt + \sum_{i=1}^{l} C_i x dW_i. \tag{38}
$$

By means of GARE-based method, the following result was obtained in [18].

Theorem 9. *If for any real matrix $M \geq 0$, there exists a constant $\lambda > 0$, such that*

$$
\sum_{i=1}^{l} C_i'MC_i \leq \lambda M. \tag{39}
$$

Moreover, suppose $\lim_{\|x\| \to 0}(\|H_0(x, Kx)\|/\|x\|) = 0, \forall K \in \mathscr{R}^{m \times n}$; (A, B) is controllable; $(Q^{1/2}, A)$ is observable with $Q \geq 0$; then system (38) is locally asymptotically stabilizable in probability with the control law

$$
u(t) = -\frac{1}{2} R^{-1} B' Px(t), \quad R > 0, \tag{40}
$$

where $P > 0$ is a unique solution of GARE

$$
PA + A'P + \sum_{i=1}^{l} C_i'PC_i - PBR^{-1}B'P + Q = 0. \tag{41}
$$

Based on Theorem 9, we give the following remarks.

Remark 10. It is not convenient to use Theorem 9 in practice, because the condition (39) is difficult to verify for all real nonnegative symmetric matrices.

Remark 11. Checking the proof of Theorem 9 in [18], we can find that Theorem 9 of [18] required that the smallest eigenvalue of Q should be larger than zero; that is, $Q > 0$; so $(Q^{1/2}, A)$ is certainly observable.

GARE (41) is a special case of (14). We should point out that (39) and the controllability of (A, B) are only sufficient but not necessary conditions for the existence of positive solutions of GARE (41) with $Q > 0$, $R > 0$; see [25] and the following counterexample.

Example 12. In GARE (41), we set $l = 1, R = 1, Q = I$, and

$$
A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \tag{42}
$$

$$
C_1 = \begin{bmatrix} 0 & 0 \\ n & n \end{bmatrix}, \quad n > 0.
$$

In this case, GARE (41) reduces to

$$
PA + A'P + C_1'PC_1 - PBB'P + I = 0. \tag{43}
$$

It is easy to test that

$$
dx = (Ax + Bu) dt + C_1 x dW_1 \tag{44}
$$

is stabilizable in mean square sense. By [9, 13], (43) must have a unique positive definite solution $P > 0$. However, (39) is not satisfied; this can be seen by setting

$$
M = \begin{bmatrix} 0 & 0 \\ 0 & m_{22} \end{bmatrix}, \quad m_{22} > 0. \tag{45}
$$

Considering Proposition 7, Theorem 2 not only has computational advantage but also generalizes and improves Theorem 9 given in [18].

Remark 13. In general, feedback stabilizing control laws are not unique; for example, in Theorem 9, except for $u(t) = -(1/2)R^{-1}B'Px(t)$, $u(t) = -R^{-1}B'Px(t)$ is another locally feedback stabilizing control law of system (38).

4. Globally Asymptotic Stabilization

Theorem 14. *Suppose there exists a scalar $\lambda > 0$, such that, for any $K \in \mathscr{R}^{m \times n}$ and $x \in \mathscr{R}^n$,*

$$
H_i(x, Kx) H_i'(x, Kx) \leq \lambda xx', \quad i = 0, 1, \ldots, l, \tag{46}
$$

and the following LMI,

$$\begin{bmatrix} P\left(A + \dfrac{1}{2}I + \dfrac{(2l+1)\lambda}{2}I\right)' + \left(A + \dfrac{1}{2}I + \dfrac{(2l+1)\lambda}{2}I\right)P + Y'B' + BY & \sqrt{2}\left(C_1 P + D_1 Y\right) & \sqrt{2}\left(C_2 P + D_2 Y\right) & \cdots & \sqrt{2}\left(C_l P + D_l Y\right) \\ \sqrt{2}\left(PC_1' + Y'D_1'\right) & -P & 0 & \cdots & 0 \\ \sqrt{2}\left(PC_2' + Y'D_2'\right) & 0 & -P & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \sqrt{2}\left(PC_l' + Y'D_l'\right) & 0 & 0 & \cdots & -P \end{bmatrix} < 0, \quad (47)$$

has solutions $P > 0$, $Y \in \mathcal{R}^{m\times n}$; then the equilibrium point $x \equiv 0$ of system (10) is globally asymptotically stabilizable with the control law

$$u(t) = YP^{-1}x(t). \tag{48}$$

Proof. Similar to the proof of Theorem 2, by Schur's complement, (47) is equivalent to the fact that there exist $P > 0$, $Y \in \mathcal{R}^{m\times n}$, such that

$$\begin{aligned} &P\left(A + \frac{1}{2}I + \frac{(2l+1)\lambda}{2}I\right)' \\ &+ \left(A + \frac{1}{2}I + \frac{(2l+1)\lambda}{2}I\right)P + Y'B' + BY \\ &+ 2\sum_{i=1}^{l}\left(C_i P + D_i Y\right)P^{-1}\left(C_i P + D_i Y\right)' < 0. \end{aligned} \tag{49}$$

Let $Y = KP$; then (49) implies that there exists a solution $P_1 > 0$ to

$$\begin{aligned} &P_1\left(A + BK + \frac{1}{2}I + \frac{(2l+1)\lambda}{2}I\right) \\ &+ \left(A + BK + \frac{1}{2}I + \frac{(2l+1)\lambda}{2}I\right)'P_1 \\ &+ 2\sum_{i=1}^{l}\left(C_i + D_i K\right)'P_1\left(C_i + D_i K\right) < 0. \end{aligned} \tag{50}$$

Still take the Lyapunov function $V(x) = x'P_1 x$, $u(t) = Kx(t) = YP^{-1}x(t)$; then $V(x)$ satisfies (8) and (24). It is well known that

$$2H_0'(x, Kx)P_1 x \leq H_0'(x, Kx)P_1 H_0(x, Kx) + x'P_1 x. \tag{51}$$

By (46), it concludes

$$H_0'(x, Kx)P_1 H_0(x, Kx) \leq \lambda x'P_1 x. \tag{52}$$

So

$$2H_0'(x, Kx)P_1 x \leq (\lambda + 1)x'P_1 x. \tag{53}$$

Similarly,

$$\begin{aligned} &2\sum_{i=1}^{l}H_i'(x, Kx)P_1\left(C_i + D_i K\right)x \\ &\leq \sum_{i=1}^{l}H_i'(x, Kx)P_1 H_i(x, Kx) \\ &\quad + \sum_{i=1}^{l}x'\left(C_i + D_i K\right)'P_1\left(C_i + D_i K\right)x \\ &\leq (\lambda l)x'P_1 x + \sum_{i=1}^{l}x'\left(C_i + D_i K\right)'P_1\left(C_i + D_i K\right)x, \\ &\sum_{i=1}^{l}H_i'(x, Kx)P_1 H_i(x, Kx) \leq (\lambda l)x'P_1 x. \end{aligned} \tag{54}$$

Repeating the same procedure as in Theorem 2, we can prove $\mathcal{L}V(x) < 0$ for all $x \in \mathcal{R}^n$. The theorem is shown. $\qquad\square$

Remark 15. Obviously, (11) and (46) do not imply each other, which motivates us to search for other less conservative conditions in the future.

5. Numerical Example

In this section, we present the following numerical example to illustrate the effectiveness of our main results.

Example 1. Consider the following two-dimensional nonlinear stochastic system:

$$\begin{aligned} dx = &\left(Ax + Bu + H_0(x, u)\right)dt \\ &+ \left(C_1 x + D_1 u + H_1(x, u)\right)dW_1, \end{aligned} \tag{55}$$

with

$$A = \begin{bmatrix} -1 & 2 \\ 0.5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 2 & 0.5 \\ 0 & 1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{56}$$
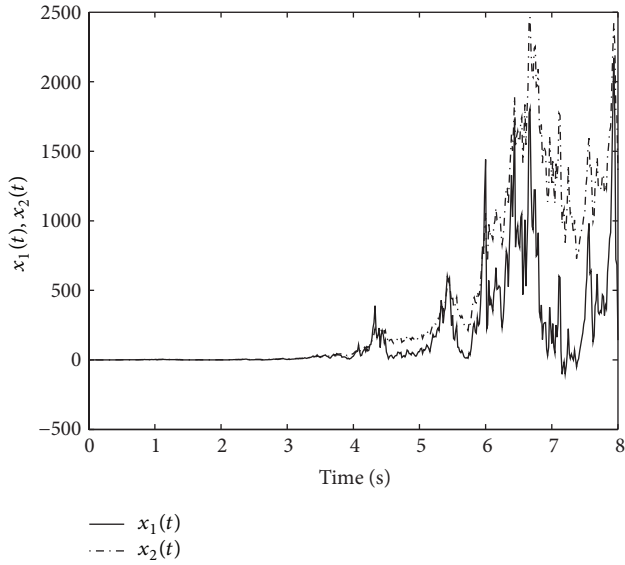
$$H_0(x, u) = 2xu, \qquad H_1(x, u) = -xu.$$

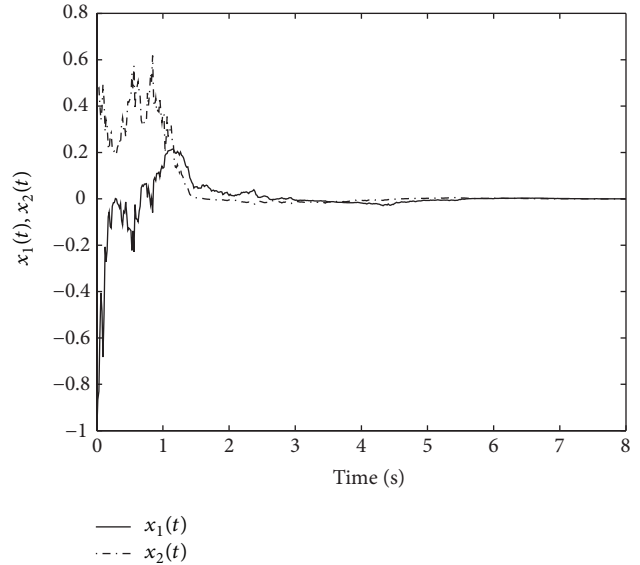FIGURE 1: The state responses of the unforced system ($u = 0$).

FIGURE 2: The state responses of the controlled system ($u = Kx$).

Obviously, $H_0(x, u)$ and $H_1(x, u)$ satisfy condition (11). According to Theorem 2, a feasible solution is derived by solving LMI (12):

$$P = \begin{bmatrix} 5.1894 & 0.4095 \\ 0.4095 & 3.8125 \end{bmatrix},$$

$$Y = \begin{bmatrix} -6.7947 & -6.5819 \end{bmatrix}. \tag{57}$$

Therefore, the control gain matrix is

$$K = YP^{-1} = \begin{bmatrix} -1.1831 & -1.5993 \end{bmatrix}. \tag{58}$$

The state responses of the unforced system ($u = 0$) and the controlled system ($u = Kx$) are shown in Figures 1 and 2, respectively. From Figure 2, it can be found that the controlled system can achieve stability by using the proposed controller.

6. Conclusion

In this paper, we have studied the feedback stabilizability of nonlinear stochastic systems with state- and control-dependent noise. Some sufficient conditions on stabilization have been derived in terms of LMIs and GAREs. A numerical example is presented to show the validity of the obtained results.

Notations

$\mathscr{S}_n$:	The set of all $n \times n$ symmetric matrices
A':	Transpose of a matrix A
$A \geq 0$ $(A > 0)$:	Positive semidefinite (positive definite) symmetric matrix A
I:	Identity matrix
$\text{Tr}(P)$:	Trace of a square matrix P
$C_2^0(\{t > 0\} \times U)$:	Class of functions $V(t, x)$ twice continuously differential with respect to $x \in U$ and once continuously differential with respect to $t > 0$ except possibly at the point $x = 0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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