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Research Article

Thermodynamics in Modified Gravity with Curvature Matter Coupling

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The first and generalized second laws of thermodynamics are studied in $f(R, L_m)$ gravity, a more general modified theory with curvature matter coupling. It is found that one can translate the Friedmann equations to the form of first law accompanied with entropy production term. This behavior is due to the nonequilibrium thermodynamics in this theory. We establish the generalized second law of thermodynamics and develop the constraints on coupling parameters for two specific models. It is concluded that laws of thermodynamics in this modified theory are more general and can reproduce the corresponding results in Einstein, $f(R)$ gravity, and $f(R)$ gravity with arbitrary as well as nonminimal curvature matter coupling.

1. Introduction

The rapid growth of observational measurements on expansion history reveals the expanding paradigm of the universe. This fact is based on accumulative observational evidences mainly from type Ia supernova and other renowned sources [1–3]. The expanding phase implicates the presence of repulsive force which compensates the attractiveness property of gravity on cosmological scales. This phenomenon may be translated as the existence of exotic matter components and most acceptable understanding for such enigma is termed as dark energy (DE) having large negative pressure. Several approaches have been proposed to explain transition from matter dominated to accelerated expansion era. The most common representative for DE is the cosmological constant having equation of state (EoS) $\omega = -1$ [4]. However, there are some serious issues to explain its mystery in any theory. Furthermore, observations show that EoS may cross the phantom divide and WMAP5 data set the bound for the value of ω in the range $-1.11 < \omega < -0.86$ [5]. In Einstein gravity, the issue of accelerated expansion is explained in the spirit of various DE models such as Chaplygin gas, quintessence, phantom, quintom, and Holographic DE models [6–10].

The other promising way to deal with the cosmic expansion is the modification of Einstein-Hilbert action where it is assumed that Einstein gravity breaks down on large scales. The $f(R)$ theory is one of the theoretical models in this context which has attained significant attention to explain the cosmic acceleration [11, 12]. In this modification, the simplest choice is to replace scalar curvature by R^n, for $n > 1$, it exhibits the de Sitter behavior for early times while for $n = -1, 3/2$, it explains the accelerated expansion [13, 14]. The $f(R)$ theory involving inverse power of scalar curvature can be a handy candidate of DE for which $f(R) = (R - (\mu/R^n))$ $(\mu > 0, n > 0)$ [15, 16]. However, this model has been ruled out due to local gravity constraints and matter instability condition. At the same time, $f(R)$ models can satisfy solar system constraints (for more general viable models see [17–19]).

In most modified gravitational theories, the Einstein gravity is generalized by changing the geometric part whereas matter part receives no attention. Bertolami et al. [20] presented the generalization of $f(R)$ theory by introducing non-minimal curvature matter coupling and it is extended to Lagrangian involving arbitrary function of matter Lagrangian density [21]. Curvature matter coupling results in non-geodesic motion of test particles and hence an extra force orthogonal to four-velocity originates [20]. The correspondence between non-minimal coupling in this theory and scalar-tensor theory has been developed in [22], and it was shown that non-minimally coupled theory would imply two scalar fields. Nesseris [23] studied matter density perturbations to constrain this theory from growth factor as well as weak lensing observations. This class of models in modified theory has been extensively studied in the literature [24–27] (for review on modified theories with curvature matter see also [28]).

Curvature matter coupling has also been considered by introducing a Lagrangian having arbitrary dependence on the trace of energy-momentum tensor named as $f(R,T)$ [29]. In this theory, motion of test particles is nongeodesic producing extra force and cosmic acceleration may result due to mutual contribution both from geometric and mater parts. Recently, this theory has been under consideration and some interesting results have been obtained [30–36]. This model is further generalized by incorporating the possible coupling between energy-momentum as well as the Ricci tensor and Lagrangian of the form $f(R,T,R_{\mu\nu}T^{\mu\nu})$ which is established [37, 38]. In a recent paper [39], we have presented the field equations for a more general case and formulated energy conditions corresponding to this modified theory. The specific functional forms of Lagrangian $f(R,T,R_{\mu\nu}T^{\mu\nu})$ have been constrained using the energy conditions and the Dolgov-Kawasaki instability.

The thermodynamic behavior of accelerating universe driven by DE is one of the major concerns in cosmology. It has been shown that Einstein field equations can be obtained using the relation $\delta Q = TdS$ by considering the proportionality of horizon area and entropy [40–43]. The relation between thermodynamics and gravity has been tested in Einstein as well as Gauss-Bonnet and Lovelock gravities [44]. Cai and Cao [45] showed that Friedmann equations in braneworld scenario can be cast to the form of first law of thermodynamics at the apparent horizon. This work is also extended in the framework of warped DGP braneworld [46] and Gauss-Bonnet Braneworld [47].

Akbar and Cai [48] discussed the first law of thermodynamics in scalar tensor and $f(R)$ gravities at the apparent horizon of FRW universe. They proposed that equilibrium thermodynamics can be achieved in these theories by incorporating the curvature contribution term to effective energymomentum tensor. However, in nonlinear theories of gravity there is an issue of non-equilibrium picture of thermodynamics which was first suggested by Eling et al. [49] and has been discussed under different circumstances [50, 51]. Cai and Cao [52] found that in scalar tensor theories thermodynamics associated with the apparent horizon of the FRW universe results in non-equilibrium description which modifies the standard Clausius relation. The non-equilibrium treatment of thermodynamics is also discussed in various modified theories [53, 54, 57, 59, 62, 63].

We are interested to study the thermodynamic behavior in $f(R,L_m)$ gravity which is a more general modified theory with curvature matter coupling [55]. This theory can reproduce every action of non-minimal and arbitrary matter curvature coupling in $f(R)$ gravity. In previous studies on thermodynamic properties in nonlinear theories, it has been shown that non-equilibrium treatment is necessary in such type of theories [35, 36, 53, 54, 57, 59, 62, 63]. Here, we regard the non-equilibrium approach and show that the field equations in $f(R,L_m)$ gravity can be cast to the form $Td(S+\widetilde{S}) = -dE + WdV$, where $d\widetilde{S}$ is the entropy production term. Furthermore, we formulate the generalized second law of thermodynamics (GSLT) and develop constraints by utilizing some concrete examples in this theory.

The paper is organized as follows. In the next section, we formulate the field equations in $f(R,L_m)$ gravity and develop the first law of thermodynamics (FLT) at the apparent horizon of FRW universe. Section 3 investigates the validity of GSLT for some models in $f(R,L_m)$ gravity. In Section 4, we conclude our findings.

2. $f(R,\mathscr{L}_m)$ Gravity and First Law of Thermodynamics

The action of more general modified theory involving maximal arbitrary curvature matter coupling is of the form [55]

$$\mathscr{A} = \frac{1}{\kappa^2}\int f\left(R,\mathscr{L}_m\right)\sqrt{-g}dx^4, \tag{1}$$

where the function $f(R,\mathscr{L}_m)$ necessitates an arbitrary dependence on scalar curvature R and Lagrangian density $\mathscr{L}_m$ which represents the matter contents. One can recover the modified $f(R)$ theories with matter curvature coupling from action (1). Consider $f(R,L_m) = (1/2)f_1(R) + G(\mathscr{L}_m)f_2(R)$, where f_i and $G(\mathscr{L}_m)$ are arbitrary functions of R and L_m, respectively, it corresponds to $f(R)$ theory with arbitrary curvature matter coupling. If we set $f_1(R) = f(R)$, $G(L_m) = L_m$ and $f_2(R) = 1 + \lambda f_2(R)$; then it implies the nonminimally coupled $f(R)$ gravity. Moreover, the corresponding actions in pure $f(R)$ and Einstein gravities can be reproduced by fixing $f_1(R) = f(R)$, $G(L_m) = L_m$, and $f_2(R) = 1$ and $f_1(R) = R$, $G(L_m) = L_m$, $f_2(R) = 1$, respectively. The matter energy-momentum tensor is considered as follows:

$$T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathscr{L}_m\right)}{\delta g^{\mu\nu}}, \tag{2}$$

which implies that

$$T^{(m)}_{\mu\nu} = g_{\mu\nu}\mathscr{L}_m - \frac{2\partial\mathscr{L}_m}{\partial g^{\mu\nu}} \tag{3}$$

by assuming that the matter Lagrangian depends only upon the metric tensor rather than on its derivatives.

The field equations in $f(R,\mathscr{L}_m)$ gravity are

$$\begin{aligned}\frac{\kappa^2}{2}P\left(R,L_m\right)T_{\mu\nu} &= F\left(R,L_m\right)R_{\mu\nu}\\ &\quad - \frac{1}{2}\left[f\left(R,L_m\right) - P\left(R,L_m\right)L_m\right]g_{\mu\nu}\\ &\quad - \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)F\left(R,L_m\right),\end{aligned} \tag{4}$$

where we put $P(R,L_m) = f_{L_m}(R,L_m)$ and $F(R,L_m) = f_R(R,L_m)$ so that representation of equations is more convenient and subscripts R, L_m point to the derivatives with respect to scalar curvature and matter Lagrangian,

$\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, ∇_μ represents covariant derivative related to the Levi-Civita connection of the metric. One can retrieve the field equations in Einstein gravity just by replacing $f(R, L_m) = (R/2) + L_m$. The contraction of (4) implies the following relation:

$$\frac{\kappa^2}{2}P(R, L_m)T = F(R, L_m)R - 2[f(R, L_m) - P(R, L_m)L_m] + 3\Box F(R, L_m). \tag{5}$$

We can eliminate the term $\Box f_R(R, L_m)$ from (4) and (5) so that the field equations become

$$\begin{aligned}\frac{\kappa^2}{2}P(R, L_m)\left(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}\right) &= F(R, L_m)\left(R_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu}\right) \\ &\quad + \frac{1}{6}(f(R, L_m) - P(R, L_m)L_m) \\ &\quad \times g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R, L_m).\end{aligned} \tag{6}$$

Equation (4) can be sorted out to develop the form of the effective Einstein field equation as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa_{\text{eff}}^2 T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\text{CM})}, \tag{7}$$

where $\kappa_{\text{eff}} = 8\pi GP(R, L_m)/2F(R, L_m)$ involves the effective gravitational coupling and $T_{\mu\nu}^{(\text{CM})}$ denotes the energy-momentum tensor associated with curvature matter coupling components defined as

$$\begin{aligned}T_{\mu\nu}^{(\text{CM})} = \frac{1}{F(R, L_m)}\Big[&\frac{1}{2}(f(R, L_m) - RF(R, L_m))g_{\mu\nu} \\ &+ (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box) \\ &\times F(R, L_m) - \frac{1}{2}P(R, L_m)L_m g_{\mu\nu}\Big].\end{aligned} \tag{8}$$

In this work, we assume the FRW metric

$$\begin{gathered}ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - k^2r^2} + r^2 d\Omega^2\right), \\ d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2\end{gathered} \tag{9}$$

which is necessarily homogenous as well as isotropic and perfect fluid is taken as matter energy-momentum tensor. In this framework, the $f(R, L_m)$ field equations take the form

$$3\left(H^2 + \frac{k}{a^2}\right) = \kappa_{\text{eff}}^2\rho_M + \rho_{\text{CM}}, \tag{10}$$

$$-2\left(\dot{H} - \frac{k}{a^2}\right) = \kappa_{\text{eff}}^2(\rho_M + p_M) + (\rho_{\text{CM}} + p_{\text{CM}}), \tag{11}$$

where ρ_m and p_m indicate the energy density and pressure of matter fluid while ρ_{CM} and p_{CM} mark energy density and pressure of components produced due to matter geometry coupling translated by the following expressions:

$$\begin{aligned}\rho_{\text{CM}} &= \frac{1}{f_R}\left[\frac{1}{2}(f - RF) - 3H\left(F_R\dot{R} + F_{L_m}\dot{L}_m\right) - \frac{1}{2}PL_m\right], \\ p_{\text{CM}} &= \frac{1}{f_R}\Big[-\frac{1}{2}(f - RF) + 2H\left(F_R\dot{R} + F_{L_m}\dot{L}_m\right) \\ &\quad + F_R\ddot{R} + F_{RR}\dot{R}^2 + 2\dot{R}\dot{L}_m \\ &\quad \times F_{RL_m} + F_{L_m}\ddot{L}_m + F_{L_mL_m}\left(\dot{L}_m\right)^2 + \frac{1}{2}PL_m\Big],\end{aligned} \tag{12}$$

where $H = \dot{a}(t)/a(t)$ is the Hubble parameter and dot in superscript indicates time derivative. Consequently, the field equations in $f(R, L_m)$ gravity can be organized as

$$\begin{aligned}-2\left(\dot{H} - \frac{k}{a^2}\right) &= \kappa_{\text{eff}}^2(\rho_m + p_m) \\ &\quad + \frac{1}{f_R}\Big[-H\left(F_R\dot{R} + F_{L_m}\dot{L}_m\right) + F_R\ddot{R} + \dot{R}^2 \\ &\quad \times F_{RR} + 2F_{RL_m}\dot{R}\dot{L}_m + F_{L_m}\ddot{L}_m \\ &\quad + F_{L_mL_m}\left(\dot{L}_m\right)^2\Big].\end{aligned} \tag{13}$$

Now, we propose to formulate the FLT in this modified theory at the apparent horizon of FRW universe. The condition $h^{\alpha\beta}\partial_\alpha\tilde{r}\partial_\beta\tilde{r} = 0$ implies the radius of dynamical apparent horizon for FRW geometry as

$$\tilde{r}_A = \left(H^2 + \frac{k}{a^2}\right)^{-1/2}, \tag{14}$$

which matches the Hubble horizon $\tilde{r}_A = 1/H$ for flat FRW universe. We differentiate (14) with respect to cosmic time which yields

$$\frac{1}{H\tilde{r}_A^3}\frac{d\tilde{r}_A}{dt} = \left(\dot{H} - \frac{k}{a^2}\right). \tag{15}$$

If we substitute (13) in the above relation and multiply the rest with $4\pi\tilde{r}_A$, it follows that

$$\begin{aligned}\frac{1}{2\pi\tilde{r}_A}\left(\frac{4\pi\tilde{r}_A F}{PG}\right)d\tilde{r}_A &= 4\pi\tilde{r}_A^3\Big[\rho_m + p_m + \frac{1}{4\pi GP} \\ &\quad \times\Big(-H\left(F_R\dot{R} + F_{L_m}\dot{L}_m\right) \\ &\quad + F_R\ddot{R} + F_{RR}\dot{R}^2 \\ &\quad + 2F_{RL_m}\dot{R}\dot{L}_m + F_{L_m}\ddot{L}_m \\ &\quad + F_{L_mL_m}\left(\dot{L}_m\right)^2\Big)\Big]Hdt,\end{aligned} \tag{16}$$

which can be expressed as

$$\begin{aligned}\frac{1}{2\pi\tilde{r}_A}d\left(\frac{AF}{2PG}\right) &= 4\pi\tilde{r}_A^3\left[\rho_m+p_m+\frac{1}{4\pi PG}\right.\\&\quad\times\left(-H\left(F_R\dot{R}+F_{L_m}\dot{L}_m\right)+F_R\ddot{R}\right.\\&\quad+F_{RR}\dot{R}^2+2F_{RL_m}\dot{R}\dot{L}_m+F_{L_m}\ddot{L}_m\\&\quad\left.\left.+F_{L_mL_m}\left(\dot{L}_m\right)^2\right)\right]Hdt\\&\quad+\frac{\tilde{r}_A}{PG}\left[F_R\dot{R}+F_{L_m}\dot{L}_m-F\partial_t\left(\ln P\right)\right]dt,\end{aligned} \tag{17}$$

where $A=4\pi\tilde{r}_A^2$ is the area of apparent horizon. In (17), we have employed the following differential:

$$d\left(\frac{AF}{2PG}\right)=\frac{4\pi\tilde{r}_AF}{PG}d\tilde{r}_A-2\pi\tilde{r}_A^2d\left(\frac{F}{PG}\right). \tag{18}$$

Furthermore, multiplying the above equation by the term $(1-(\dot{\tilde{r}}_A/2H\tilde{r}_A))$, we find

$$\begin{aligned}\frac{\left|\kappa_{\mathrm{sg}}\right|}{2\pi}d\left(\frac{AF}{2PG}\right) &= \left(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)4\pi\tilde{r}_A^3\\&\quad\times\left[\rho_m+p_m+\frac{1}{4\pi PG}\right.\\&\quad\times\left(-H\left(F_R\dot{R}+F_{L_m}\dot{L}_m\right)\right.\\&\quad+F_R\ddot{R}+F_{RR}\dot{R}^2+2F_{RL_m}\dot{R}\dot{L}_m\\&\quad\left.\left.+F_{L_m}\ddot{L}_m+F_{L_mL_m}\left(\dot{L}_m\right)^2\right)\right]H\\&\quad\times dt+\left(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)\frac{\tilde{r}_A}{PG}\\&\quad\times\left[F_R\dot{R}+F_{L_m}\dot{L}_m-F\partial_t\left(\ln P\right)\right]dt,\end{aligned} \tag{19}$$

where $T=|\kappa_{\mathrm{sg}}|/2\pi$ is the temperature of apparent horizon which includes the surface gravity $\kappa_{\mathrm{sg}}=(1/\tilde{r}_A)(1-(\dot{\tilde{r}}_A/2H\tilde{r}_A))$.

The Bekenstein-Hawking relation [40–43] $S=A/4G$ defines the horizon entropy in Einstein-gravity. In alternative theories of gravity, Wald [56] suggested that horizon entropy is associated with a Noether charge and in $f(R)$ gravity, entropy is defined as $S=AF/4G$ [53, 54, 57]. Bamba and Geng [53, 54, 57] pointed out that Wald entropy relation is similar for both metric and Palatini formalisms in $f(R)$ theory. Brustein et al. [58] demonstrated that Wald entropy is equivalent to $S=A/4G_{\mathrm{eff}}$, where G_{eff} being the effective gravitational coupling. Therefore, the Wald entropy at the apparent horizon in $f(R,L_m)$ gravity can be defined as [35, 36, 53, 54, 57, 59] $S=AF/2PG$ which reproduces the corresponding result in $f(R)$ theory involving arbitrary curvature matter coupling [59] for $f(R,L_m)=(1/2)f_1(R)+g(\mathscr{L}_m)f_2(R)$. Consequently, (19) can be rewritten as

$$\begin{aligned}TdS &= 4\pi\tilde{r}_A^3\left(\rho_m+p_m\right)Hdt-2\pi\tilde{r}_A^2\left(\rho_m+p_m\right)d\tilde{r}_A\\&\quad+\frac{TA}{2PG}\left[\tilde{r}_A^2\left(-H\left(F_R\dot{R}+F_{L_m}\dot{L}_m\right)\right.\right.\\&\quad+F_R\ddot{R}+F_{RR}\dot{R}^2+2F_{RL_m}\dot{R}\dot{L}_m\\&\quad\left.+F_{L_m}\ddot{L}_m+F_{L_mL_m}\times\left(\dot{L}_m\right)^2\right)H\\&\quad\left.+F_R\dot{R}+F_{L_m}\dot{L}_m-F\partial_t\left(\ln P\right)\right]dt.\end{aligned} \tag{20}$$

Now, we pay attention to matter energy inside a sphere of radius $\tilde{r}_A$ at apparent horizon, that is, $E=V\rho$, where $V=4/3\pi\tilde{r}_A^3$ is the volume of 3-dimensional sphere. The time derivative of energy relation gives

$$dE=4\pi\tilde{r}_A^2\rho d\tilde{r}_A+V\dot{\rho}dt. \tag{21}$$

If one considers the covariant divergence of (4), then after some manipulations it leads to

$$\nabla^{\alpha}T_{\alpha\beta}=2\nabla^{\alpha}\ln\left[P\left(R,L_m\right)\right]\frac{\partial L_m}{\partial g^{\alpha\beta}}, \tag{22}$$

which implies that the usual continuity equation does not hold in this modified theory which is similar to other modified gravities involving matter geometry coupling. The corresponding results in $f(R)$ theory with arbitrary and nonminimal matter geometry coupling can be reproduced for particular choices of Lagrangian.

In FRW framework, the energy density meets the following equation

$$\dot{\rho}+3H\left(\rho+p\right)=2\partial_t\left[P\left(R,L_m\right)\right]\left.\frac{\partial L_m}{\partial g^{\mu\nu}}\right|_{g^{00}}. \tag{23}$$

Using this equation in (21), we obtain

$$dE=4\pi\tilde{r}_A^2\rho d\tilde{r}_A-3V\left(\rho+p\right)Hdt+2V\partial_t\left[\ln P\right]\left.\frac{\partial L_m}{\partial g^{\mu\nu}}\right|_{g^{00}}dt. \tag{24}$$

Hence, (20) can be represented as

$$\begin{aligned}T_hdS_h &= -dE+WdV+\frac{TA}{2PG}\\&\quad\times\left[\tilde{r}_A^2H\left(-H\left(F_R\dot{R}+F_{L_m}\dot{L}_m\right)+F_R\ddot{R}\right.\right.\\&\quad+F_{RR}\dot{R}^2+2F_{RL_m}\dot{R}\dot{L}_m+F_{L_m}\ddot{L}_m\\&\quad\left.+F_{L_mL_m}\left(\dot{L}_m\right)^2\right)+F_R\dot{R}+F_{L_m}\dot{L}_m\\&\quad\left.-F\partial_t\left(\ln P\right)\right]dt\\&\quad+T\frac{4\pi V}{|\kappa|}\partial_t\left(\ln P\right)\left.\frac{\partial L_m}{\partial g^{\mu\nu}}\right|_{g^{00}}dt,\end{aligned} \tag{25}$$

where $W = (1/2)(\rho - p)$ is the work density. Equation (25) shows that the field equations in this theory does not obey the standard form of FLT, $TdS = -dE + WdV$ which is satisfied in Einstein, Lovelock, and Gauss-Bonnet gravities. The above relation involves additional terms which are produced due to the non-equilibrium representation of thermodynamics. Thus the FLT in this theory can be represented as

$$T_h dS_h + T_h d\widetilde{S}_h = -dE + WdV, \tag{26}$$

where

$$\begin{aligned} d\widetilde{S}_h = \frac{-A}{2PG} \Big[\widetilde{r}_A^2 H \Big(&- H \left(F_R \dot{R} + F_{L_m} \dot{L}_m \right) \\ &+ F_R \ddot{R} + F_{RR} \dot{R}^2 + 2F_{RL_m} \dot{R} \dot{L}_m \\ &+ F_{L_m} \ddot{L}_m + F_{L_m L_m} \left(\dot{L}_m \right)^2 \Big) \\ &+ F_R \dot{R} + F_{L_m} \dot{L}_m - F \partial_t (\ln P) \Big] dt \\ &- \frac{4\pi V}{|\kappa|} \partial_t (\ln P) \left. \frac{\partial L_m}{\partial g^{\mu\nu}} \right|_{g^{00}} dt \end{aligned} \tag{27}$$

denotes the entropy production term which is a result of matter geometry coupling in more general $f(R)$ gravity at the apparent horizon of FRW universe.

One can retrieve $d\widetilde{S}$ in pure $f(R)$ theory for the Lagrangian $f(R, L_m) = (1/2)f(R) + L_m$ which is identical with the results in [53, 54, 57]. It is remarked that we establish the FLT in more general modified gravity which recovers the corresponding laws in $f(R)$ theory involving arbitrary as well as non-minimal gravitational coupling. For $f(R, L_m) = (1/2)f_1(R) + g(\mathscr{L}_m)f_2(R)$, we can get the respective FLT in $f(R)$ theory having arbitrary coupling which is also formulated in [59]. If one defines effective entropy as a sum of horizon entropy and entropy production term $S_{\text{eff}} = S_h + \widetilde{S}$, then FLT can also be represented in the form $T_h dS_{\text{eff}} = -dE + WdV$.

3. GSLT in $f(R, L_m)$ Gravity

Here, we investigate the validness of GSLT in the context of $f(R, L_m)$ gravity at the apparent horizon. It states that the sum of the horizon entropy and entropy of ordinary matter fluid components always increases in time. The validity of GSLT has been tested in modified gravities including $f(R)$, $f(\mathscr{T})$, $f(R,T)$, and $f(R)$ theories involving matter geometry coupling. It would be interesting to examine the GSLT in this modified theory which can reproduce other curvature matter coupled modified gravities. The entropy associated with the apparent horizon is given by $S_h = AF/2PG$ and its time derivative is

$$\dot{S}_h = \frac{2\pi \widetilde{r}_A}{PG} \left[\widetilde{r}_A \left(\dot{F} - F \frac{d}{dt} \ln [P] \right) + 2\dot{\widetilde{r}}_A F \right]. \tag{28}$$

The evolution of S_h multiplied with T_h can be found as

$$T_h \dot{S}_h = \frac{1}{PG} \left(1 - \frac{\dot{\widetilde{r}}_A}{2H\widetilde{r}_A} \right) \left[\widetilde{r}_A \left(\dot{F} - F \frac{d}{dt} \ln [P] \right) + 2\dot{\widetilde{r}}_A F \right]. \tag{29}$$

The dynamical equation relating the entropy of matter sources inside the horizon S_{in} and temperature T_{in} to the density and pressure in the horizon is given by

$$T_{\text{in}} dS_{\text{in}} = d\left(\rho_m V\right) + p_m dV. \tag{30}$$

The evolution of entropy inside the horizon can be found using (23) as

$$\begin{aligned} T_{\text{in}} \dot{S}_{\text{in}} = {} & 4\pi \widetilde{r}_A^2 \left(\rho_m + p_m \right) \left(\dot{\widetilde{r}}_A - H\widetilde{r}_A \right) \\ & + 2V \partial_t (\ln [P]) \left. \frac{\partial L_m}{\partial g^{\mu\nu}} \right|_{g^{00}}, \end{aligned} \tag{31}$$

where matter energy density and pressure can be evaluated from (4) which yields for FRW spacetime as

$$\begin{aligned} \rho_m &= \frac{2}{\kappa^2 P} \left[\frac{1}{2} \left(PL_m - f \right) - 3\left(\dot{H} + H^2 \right) F + 3H\dot{F} \right], \\ p_m &= \frac{2}{\kappa^2 P} \left[\frac{1}{2} \left(f - PL_m \right) + \left(\dot{H} + 3H^2 + \frac{2\kappa^2}{a^2} \right) F \right. \\ &\qquad \left. - 2H\dot{F} - \ddot{F} \right]. \end{aligned} \tag{32}$$

Substituting ρ_m and p_m in (31), we obtain

$$\begin{aligned} T_{\text{in}} \dot{S}_{\text{in}} = {} & 8\pi \widetilde{r}_A^2 \left(\dot{\widetilde{r}}_A - H\widetilde{r}_A \right) \frac{2}{\kappa^2 P} \left[2\left(\frac{\kappa}{a^2} - \dot{H} \right) F + H\dot{F} - \ddot{F} \right] \\ & + 2V \partial_t (\ln P) \left. \frac{\partial L_m}{\partial g^{\mu\nu}} \right|_{g^{00}}. \end{aligned} \tag{33}$$

Now, we proceed to establish the GSLT in this modified theory which requires $(\dot{T}_h \dot{S}_h + \dot{T}_{\text{in}} \dot{S}_{\text{in}}) \geqslant 0$. The temperature of matter and energy components inside the horizon is assumed in proportion to temperature of apparent horizon, that is, $T_{\text{in}} = bT_h$, where $0 < b < 1$. In fact, it is natural to consider such proportionality relation which results in local equilibrium by setting the proportionality constant b as unity. In general, the horizon temperature does not match to that of fluid components inside the horizon and this difference makes the spontaneous flow of energy between the horizon and fluid contents so that local thermal equilibrium is no longer preserved [60]. Moreover, the mutual matter geometry coupling in this theory may also play its role in energy flow and systems must go through some interaction for some period of time before achieving the thermal equilibrium. Thus, the GSLT requires $T_h \dot{S}_{\text{tot}} = \dot{T}_h (\dot{S}_h + \dot{S}_{\text{in}}) \geqslant 0$, using (29) and (33), we have

$$\begin{aligned} T_h \dot{S}_{\text{tot}} = {} & \frac{1}{2PG} \left(H^2 + \frac{\kappa}{a^2} \right)^{-5/2} \\ & \times \Big[2H \left(\frac{\kappa}{a^2} - \dot{H} \right) \Big\{ \frac{\kappa}{a^2} + (1 - 2b)\dot{H} \\ & \qquad + 2(1 - b) \times H^2 \Big\} F \\ & - \left(\frac{\kappa}{a^2} + H^2 \right) \left(\frac{\kappa}{a^2} + \dot{H} + 2H^2 \right) \partial_t (\ln P) F \end{aligned}$$

$$+\left\{\frac{\kappa}{a^2}\left(\frac{\kappa}{a^2}+\dot{H}+3H^2\right)+(1-2b)\dot{H}H^2 +2(1-b)H^4\right\}\left(F_R\dot{R}+F_{L_m}\dot{L}_m\right) +2bH\left(\dot{H}+H^2\right) \times\left\{F_R\ddot{R}+F_{RR}\dot{R}^2+2F_{RL_m}\dot{R}\dot{L}_m +F_{L_m}\ddot{L}_m+F_{L_mL_m}\times\left(\dot{L}_m\right)^2\right\} +\frac{16\pi bPG}{3}\left(\frac{\kappa}{a^2}+H^2\right)\partial_t(\ln P) \times\left.\frac{\partial L_m}{\partial g^{\mu\nu}}\right|_{g^{00}}\Bigg]\geqslant 0, \quad (34)$$

which is a constraint to meet GSLT in this modified theory and it counts on the basis of different choices in Lagrangian. This condition is more comprehensive and one can deduce the corresponding results in Einstein, $f(R)$, and $f(R)$ theories having non-minimal and arbitrary curvature matter coupling. For $f(R,L_m)=(1/2)f_1(R)+g(\mathscr{L}_m)f_2(R)$, we get the GSLT in $f(R)$ gravity with arbitrary matter geometry coupling and if $g(L_m)=L_m$, $f_2(R)=1+\lambda g(R)$, then the corresponding result in $f(R)$ gravity with non-minimal coupling can be found [59].

In this setting, we limit our discussion to hypothesis of thermal equilibrium so that energy would not flow in the system and horizon temperature is more or less equal to temperature inside the horizon. This situation would correspond to the case of late times where universe components and horizon would have interacted for long time [61] while its existence for early or intermediate times would be ambiguous. Though the assumption of thermal equilibrium is limiting in some sense to avoid the non-equilibrium complexities but it has widely been accepted to study the GSLT [53, 54, 57, 62, 63]. In [35, 36], we have discussed the GSLT under non-equilibrium picture of thermodynamics in $f(R,T)$ gravity. Here, we consider the case of thermal equilibrium with $b=1$ so that horizon temperature is equal to that of fluid components inside the horizon.

To illustrate the validity of GSLT in $f(R,L_m)$ gravity, we take concrete model in this modified theory [59]

$$f(R,L_m)=\lambda\exp\left(\frac{1}{2\lambda}R+\frac{1}{\lambda}L_m\right), \quad (35)$$

where $\lambda>0$ is an arbitrary constant, and if $(R/2\lambda)+(L_m/\lambda)\ll 1$, then $f(R,T)\approx\lambda+R/2+L_m+\cdots$ represents the ΛCDM model. Substituting model (35) in (34), it implies that

$$T_h\dot{S}_{\text{tot}}=\frac{1}{4G}\left(H^2+\frac{\kappa}{a^2}\right)^{-5/2} \times\Bigg[2H\left(\frac{\kappa}{a^2}-\dot{H}\right)^2-\left(\frac{\kappa}{a^2}+H^2\right) \times\left(\frac{\kappa}{a^2}+\dot{H}-2H^2\right)\partial_t\left(\frac{1}{2\lambda}(R+2L_m)\right) +\left\{\frac{\kappa}{a^2}\left(\frac{\kappa}{a^2}+\dot{H}+3H^2\right)-\dot{H}H^2\right\} \times\partial_t\left(\frac{1}{2\lambda}(R+2L_m)\right)+2H\left(\dot{H}+H^2\right) \times\left(\exp\left(\frac{1}{2\lambda}(R+2L_m)\right)\right)^{-1} \times\partial_{tt}\exp\left(\frac{1}{2\lambda}(R+2L_m)\right)+\frac{32\pi G}{3}\left(\frac{\kappa}{a^2}+H^2\right) \times\partial_t\left(\frac{1}{2\lambda}(R+2L_m)\right)\times\left.\frac{\partial L_m}{\partial g^{\mu\nu}}\right|_{g^{00}}\Bigg]\geqslant 0, \quad (36)$$

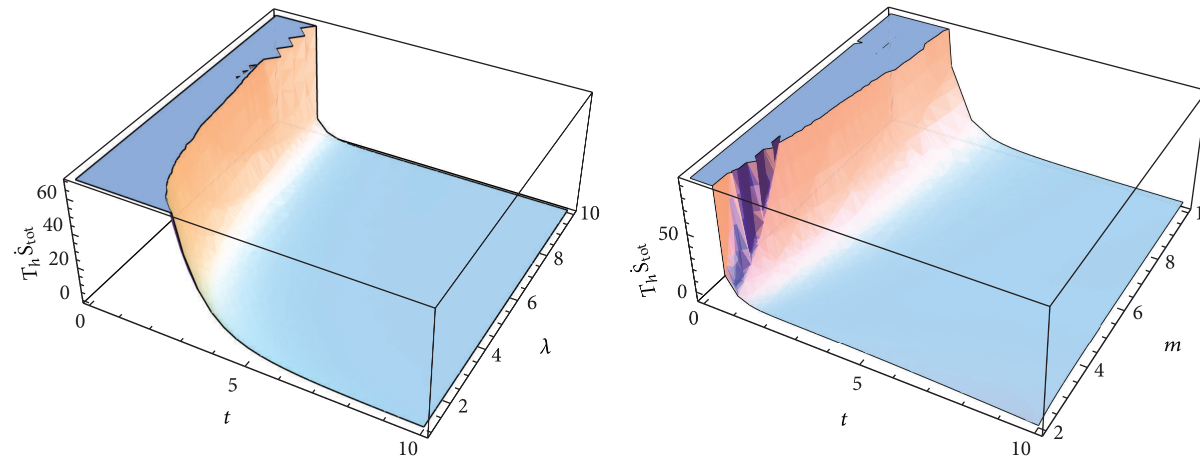
which is a constraint to validate GSLT for model (35). To be more explicit about the above constraint, we choose $L_m=\rho$ and set power law cosmology $a(t)=a_0t^m$, $(m>1)$ with $\rho=\rho_0t^{-3m}$. We plot GSLT for flat FRW geometry shown in Figure 1. To get more insights of these conditions, we consider the model of the form $f(R,L_m)=\alpha R+\beta R^2+\gamma L_m$, where α, β and γ are arbitrary constants, and hence the GSLT becomes

$$T_h\dot{S}_{\text{tot}}=\frac{1}{2\gamma G}\left(H^2+\frac{\kappa}{a^2}\right)^{-5/2} \times\Bigg[2H\left(\frac{\kappa}{a^2}-\dot{H}\right)^2(\alpha+2\beta R) +2\beta\left\{\frac{\kappa}{a^2}\left(\frac{\kappa}{a^2}+\dot{H}-3H^2\right)-\dot{H}H^2\right\}\dot{R} +4\beta H\left(\dot{H}+H^2\right)\ddot{R}\Bigg]\geqslant 0. \quad (37)$$

For the flat FRW universe, the above condition depends upon the parameters α, β, and γ which can be satisfied if $\beta<0$ and $(\alpha,\gamma)>0$. The validity of GSLT for the second model is shown in Figure 2.

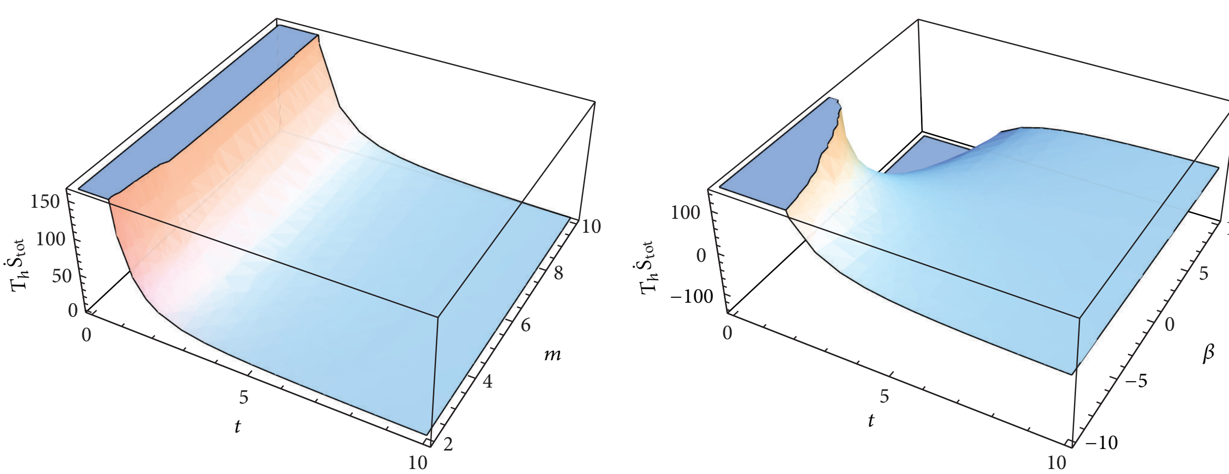
4. Conclusion

In this paper, we have discussed thermodynamic properties in more general modified gravity characterized by curvature matter coupling. We have found that Friedmann equations can be cast to the fundamental form of FLT $T_hdS_{\text{eff}}=\delta Q$, where T_h is the horizon temperature and S_{eff} is the effective entropy which consists of two factors $S_{\text{eff}}=S_h+\widetilde{S}$, the first denotes the horizon entropy satisfying the usual FLT and other factor is the entropy production term. It is remarked that entropy production term appears due to the non-equilibrium description in $f(R,L_m)$ gravity. This indicates that one may need non-equilibrium treatment of thermodynamics in this theory. The entropy production term appears to be more general which can reproduce the corresponding results in Einstein, pure $f(R)$ and $f(R)$ gravity involving non-minimal and arbitrary curvature matter coupling.



(a) (b)

FIGURE 1: Evolution of GSLT for the Lagrangian $f(R, L_m) = \lambda \exp((1/2\lambda)R + (1/\lambda)L_m)$, plot (a) shows the bound on parameter λ for $m = 10$, whereas in plot (b) we set $\lambda = 2$ to find bounds on m. It is evident that GSLT is valid if $\lambda > 0$, $m > 1$.



(a) (b)

FIGURE 2: Evolution of GSLT for $f(R, L_m) = \alpha R + \beta R^2 + \gamma L_m$, plot (a) shows the bound on parameter m for $\beta = -5$, whereas in plot (b) we set $m = 10$ to find bounds on β. We choose $\alpha = \gamma = 2$.

Moreover, in modified theories with explicit matter gravity coupling, we have nondiminishing covariant divergence of the energy-momentum tensor which demonstrates the energy flow due to mutual interaction of matter and gravity. The non-equilibrium treatment can be translated due to energy flow on the apparent horizon of FRW universe. We have also examined the time evolution of total entropy including the horizon entropy and entropy inside the horizon. The proportionality relation between temperature of apparent horizon and temperature associated with matter contents inside the horizon is assumed to develop the GSLT. We have also found constraints on two specific gravitational models $f(R, L_m) = \lambda \exp((1/2\lambda)R + (1/\lambda)L_m)$ and $f(R, L_m) = \alpha R + \beta R^2 + \gamma L_m$ to secure the GSLT in this theory. We have checked the validity of GSLT for flat FRW geometry with local thermal equilibrium and constrained the coupling parameters. For the first model, GSLT is satisfied if $\lambda > 0$ and $m > 1$, whereas for the second case it requires $\beta < 0$ and $(\alpha, \gamma) > 0$.

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