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Research Article

Application of the Homotopy Analysis Method for Solving the Variable Coefficient KdV-Burgers Equation

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The homotopy analysis method is applied to solve the variable coefficient KdV-Burgers equation. With the aid of generalized elliptic method and Fourier's transform method, the approximate solutions of double periodic form are obtained. These solutions may be degenerated into the approximate solutions of hyperbolic function form and the approximate solutions of trigonometric function form in the limit cases. The results indicate that this method is efficient for the nonlinear models with the dissipative terms and variable coefficients.

1. Introduction

To solve the nonlinear partial differential equation (NPDE) has been an attractive research topic for mathematicians and physicists. Nonlinear evolution equations with variable coefficients can describe the physical phenomenon more accurately, and it is of great significance to study how to find the solutions of nonlinear evolution equations with variable coefficients. The nonlinear partial differential equations are generally difficult to solve and their exact solutions are difficult to obtain. In recent years, some various approximate methods have been developed such as homotopy analysis method [1–13] and Adomian's decomposition method [14–17] to solve linear and nonlinear differential equations. However, the above works only studied the solutions of equations with constant coefficients. In this work, we apply the homotopy analysis method to the variable coefficients KdV-Burgers equations and obtain the approximate solution of the Jacobi elliptic function form. This method has the merits of simplicity and easy execution.

This paper is arranged in the following manner. In Section 2, we present the homotopy analysis method. In Section 3, the homotopy analysis method on the variable coefficient KdV-Burgers equation is presented. Finally, some conclusions are given.

2. Basic Idea of Homotopy Analysis Method

To explain the basic idea of the homotopy analysis method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

subject to boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2}$$

where A is the general differential operator, B is the boundary operator, $f(r)$ is the known analytic function, and Γ is the boundary of the region Ω.

Generally speaking, the operator A can be decomposed into linear part L and nonlinear part N. Equation (2) therefore can be written as

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Now we set up homotopy mapping $H(u, p) : \Omega \times [0, 1] \to R$, which satisfies

$$H(u, p) = L(u) - L(v) + p\left(L(v) + N(u) - f(r)\right), \tag{4}$$

where p is parameter, v is auxiliary function, and $L(v) + N(v) = 0$.

By (4), we obtain

$$H(u,0) = L(u) - L(v),$$
$$H(u,1) = A(u) - f(r) = 0. \quad (5)$$

As can be seen from 0 to 1 of p is the process of $L(u)-L(v)$ to $A(u) - f(r)$ of $H(u,p)$; this is the homotopy deformation.

Assume that the solution of $H(u,p) = 0$ can be written as a power series in p:

$$\widetilde{u}(x,t,p) = \sum_{i=0}^{\infty} u_i(x,t)\, p^i = u_0 + pu_1 + p^2u_2 + \cdots. \quad (6)$$

So when $p = 0$, $\widetilde{u}(x,t,0) = u_0(x,t)$ is the solution of $L(u) - L(v) = 0$; when $p \rightarrow 1$, the approximate solution of $A(u) - f(r) = 0$ is $u(x,t) = u_0 + u_1 + u_2 + \cdots$.

3. Application

In this section, we focus on the variable coefficient KdV-Burgers equation

$$u_t + uu_x + \alpha(t)\, u_{xx} + \beta(t)\, u_{xxx} = 0, \quad (7)$$

where $\alpha(t)$ and $\beta(t)$ are any function about t.

It is fascinating to observe that, when $\alpha(t)$ and $\beta(t)$ are constant, (7) becomes the well-known KdV-Burgers equation; the equation plays an important role in studying liquid with bubbles inside, the flow of liquid in elastic tubes, and the problems of turbulence [18–20]. When $\alpha(t)$ is constant, $\beta(t) = 0$, (7) becomes the well-known Burgers equation. When $\beta(t)$ is constant, $\alpha(t) = 0$, (7) becomes the well-known KdV equation. When $\beta(t) = \beta$ is constant, (7) becomes

$$u_t + uu_x + \alpha(t)\, u_{xx} + \beta u_{xxx} = 0. \quad (8)$$

Next, we applied the homotopy analysis method to study the approximate solution of (8).

In order to get the solution of (8), we lead in homotopy mapping.

To aim at (8), we set up homotopy mapping $H(u,p) : R \times I \rightarrow R$,

$$H(u,p) = L(u) - L(v) + p\left(L(v) + uu_x + \alpha(t)\, u_{xx}\right), \quad (9)$$

where $R = (-\infty, +\infty)$, $I = [0,1]$, v is the auxiliary function, and the linear operator L is expressed as $L(u) = u_t + \beta u_{xxx}$.

By using the generalized elliptic method [21], we can get that the typical KdV equation corresponding to (8),

$$v_t + vv_x + \beta v_{xxx} = 0 \quad (10)$$

has the following elliptic function solution:

$$v_1(x,t) = c_0 - \beta \frac{6k^2m^2 sn(\xi,m) + 12k^2\left(m^2-1\right) cn(\xi,m)}{sn(\xi,m) + cn(\xi,m) + dn(\xi,m)} + \beta \frac{3k^2m^4 sn^2(\xi,m) - 12k^2 cn^2(\xi,m)}{\left(sn(\xi,m) + cn(\xi,m) + dn(\xi,m)\right)^2}. \quad (11)$$

When $m \rightarrow 1$, $v_1(x,t)$ degenerates to the following solitary wave solution:

$$v_{1.1}(x,t) = c_0 - \beta \frac{6k^2 \tanh\xi}{\tanh\xi + \text{sech}\xi + \text{sech}\xi} + \beta \frac{3k^2\tanh^2\xi - 12k^2\text{sech}^2\xi}{\left(\tanh\xi + \text{sech}\xi + \text{sech}\xi\right)^2}. \quad (12)$$

When $m \rightarrow 0$, $v_1(x,t)$ degenerates to the trigonometric function solution

$$v_{1.2}(x,t) = c_0 + \frac{12k^2\beta\cos\xi}{\sin\xi + \cos\xi + 1} - \frac{12k^2\beta\cos^2\xi}{\left(\sin\xi + \cos\xi + 1\right)^2}, \quad (13)$$

where $\xi = kx + [-c_0k + \beta k^3(4m^2 - 5)]t + \xi_0$, k, c_0, and ξ_0 are any constant, m is the module, and $0 \le m \le 1$.

One can easily prove that $H(u,1) = 0$ and (8) is the same, so the solution $u(x,t)$ of (8) is the solution of $H(u,p) = 0$ when under the condition $p \rightarrow 1$.

Let

$$\widetilde{u}(x,t,p) = \sum_{i=0}^{\infty} u_i(x,t)\, p^i = u_0 + pu_1 + p^2u_2 + \cdots \quad (14)$$

be the solution of $H(u,p) = 0$; by [22] we can know that this series is uniformly convergent in the $p \in [0,1]$. Thus, it yields that

$$u = \sum_{i=0}^{\infty} u_i(x,t) = u_0 + u_1 + u_2 + \cdots. \quad (15)$$

In order to obtain the approximate solution of (8), we substitute (14) into the equation $H(u,p) = 0$. By taking the auxiliary function $v = v_1(x,t)$, and comparing the coefficients of the same power of p, one can obtain that

$$p^0 : L(u_0) = L(v) = L(v_1), \quad (16)$$

$$p^1 : L(u_1) = -L(v_1) - u_0u_{0x} - \alpha(t)\, u_{0xx} = -\alpha(t)\, u_{0xx}, \quad (17)$$

$$p^2 : L(u_2) = -u_0u_{1x} - u_1u_{0x} - \alpha(t)\, u_{1xx}. \quad (18)$$

From (16) we have

$$u_0(x,t) = v_1(x,t). \quad (19)$$

By using the Fourier transform, one can obtain the solution of (17) with the initial condition $u_1|_{t=0} = 0$ as follows:

$$u_1(x,t) = -\frac{1}{2\pi}\int_0^t \iint_{-\infty}^{+\infty} \alpha(\tau)\, v_{1\xi\xi} \times \cos\left[-\lambda^3\beta(t-\tau) + \lambda(x-\xi)\right] dx\, d\xi\, d\tau. \quad (20)$$

Similarly, one also finds the solution of (18) with the initial condition $u_2|_{t=0} = 0$ as

$$u_2(x,t) = \frac{1}{2\pi}\int_0^t \int_{-\infty}^{+\infty} \left(-v_1 u_{1\xi} - u_1 v_{1\xi} - \alpha(t) u_{1\xi\xi}\right) \times \int_{-\infty}^{+\infty} \cos\left[-\lambda^3 \beta (t-\tau) + \lambda (x-\xi)\right] dx\, d\xi\, d\tau. \quad (21)$$

From (11), (12), (13), (20), and (21) the two-degree approximate solution of (8) can be obtained as follows:

$$\begin{aligned} u_2^*(x,t) &= c_0 - \beta \frac{6k^2 m^2 sn(\xi,m) + 12k^2\left(m^2-1\right) cn(\xi,m)}{sn(\xi,m) + cn(\xi,m) + dn(\xi,m)} \\ &+ \beta \frac{3k^2 m^4 sn^2(\xi,m) - 12k^2 cn^2(\xi,m)}{\left(sn(\xi,m) + cn(\xi,m) + dn(\xi,m)\right)^2} \\ &+ \frac{1}{2\pi}\int_0^t \iint_{-\infty}^{+\infty} \left(-\alpha(\tau) v_{1\xi\xi} - v_1 u_{1\xi} - u_1 v_{1\xi} - \alpha(\tau) u_{1\xi\xi}\right) \\ &\times \cos\left[-\lambda^3\beta(t-\tau) + \lambda(x-\xi)\right] dx\, d\xi\, d\tau, \end{aligned} \quad (22)$$

where $\xi = kx + [-c_0 k + \beta k^3(4m^2 - 5)]t + \xi_0$, k, c_0, and ξ_0 are any constant, m is the module, and $0 \le m \le 1$. Consider

$$\begin{aligned} v_0 &= c_0 - \beta \frac{6k^2 m^2 sn(\xi,m) + 12k^2\left(m^2-1\right) cn(\xi,m)}{sn(\xi,m) + cn(\xi,m) + dn(\xi,m)} \\ &+ \beta \frac{3k^2 m^4 sn^2(\xi,m) - 12k^2 cn^2(\xi,m)}{\left(sn(\xi,m) + cn(\xi,m) + dn(\xi,m)\right)^2}, \\ u_1(x,t) &= -\frac{1}{2\pi}\int_0^t \iint_{-\infty}^{+\infty} \alpha(\tau) v_{1\xi\xi} \\ &\times \cos\left[-\lambda^3\beta(t-\tau) + \lambda(x-\xi)\right] dx\, d\xi\, d\tau. \end{aligned} \quad (23)$$

When $m \to 1$ and $m \to 0$, $u_2^*(x,t)$ degenerates to the following approximate solutions:

$$\begin{aligned} u_{2.1}^*(x,t) &= c_0 - \frac{6\beta k^2 \tanh\xi}{\tanh\xi + \text{sech}\xi + \text{sech}\xi} \\ &+ \beta \frac{3k^2 \tanh^2\xi - 12k^2 \text{sech}^2\xi}{\left(\tanh\xi + \text{sech}\xi + \text{sech}\xi\right)^2} \\ &- \frac{1}{2\pi}\int_0^t \iint_{-\infty}^{+\infty} \left(\alpha(\tau) u_{1\xi\xi} + v_1 u_{1\xi} + u_1 v_{1\xi} + \alpha(\tau) u_{1\xi\xi}\right) \\ &\times \cos\left[-\lambda^3\beta(t-\tau) + \lambda(x-\xi)\right] dx\, d\xi\, d\tau, \\ u_{2.2}^*(x,t) &= c_0 + \beta \frac{-6k^2 \sin\xi + 12k^2 \cos\xi}{\sin\xi + \cos\xi + 1} \\ &- \frac{12k^2 \beta \cos^2\xi}{\left(\sin\xi + \cos\xi + 1\right)^2} \\ &- \frac{1}{2\pi}\int_0^t \iint_{-\infty}^{+\infty} \cos\left[-\lambda^3\beta(t-\tau) + \lambda(x-\xi)\right] \\ &\times \left(\alpha(\tau) v_{1\xi} + v_1 u_{1\xi} + u_1 v_{1\xi} + \alpha(\tau) u_{1\xi\xi}\right) dx\, d\xi\, d\tau. \end{aligned} \quad (24)$$

By comparing the higher power coefficients of p, more higher power approximate solutions of (8) can also be obtained.

4. Conclusion

This work studies the variable coefficients KdV-Burgers equations by using the homotopy analysis method, and the two-degree approximate solution of the Jacobi elliptic function form is obtained, which can degenerate to solitary wave approximate solution and trigonometric function approximate solution in the limit case. Our results show that the homotopy analysis method is applicable to the variable solution equations; how to apply this method to high-degree and high-dimensional system remains to be further studied.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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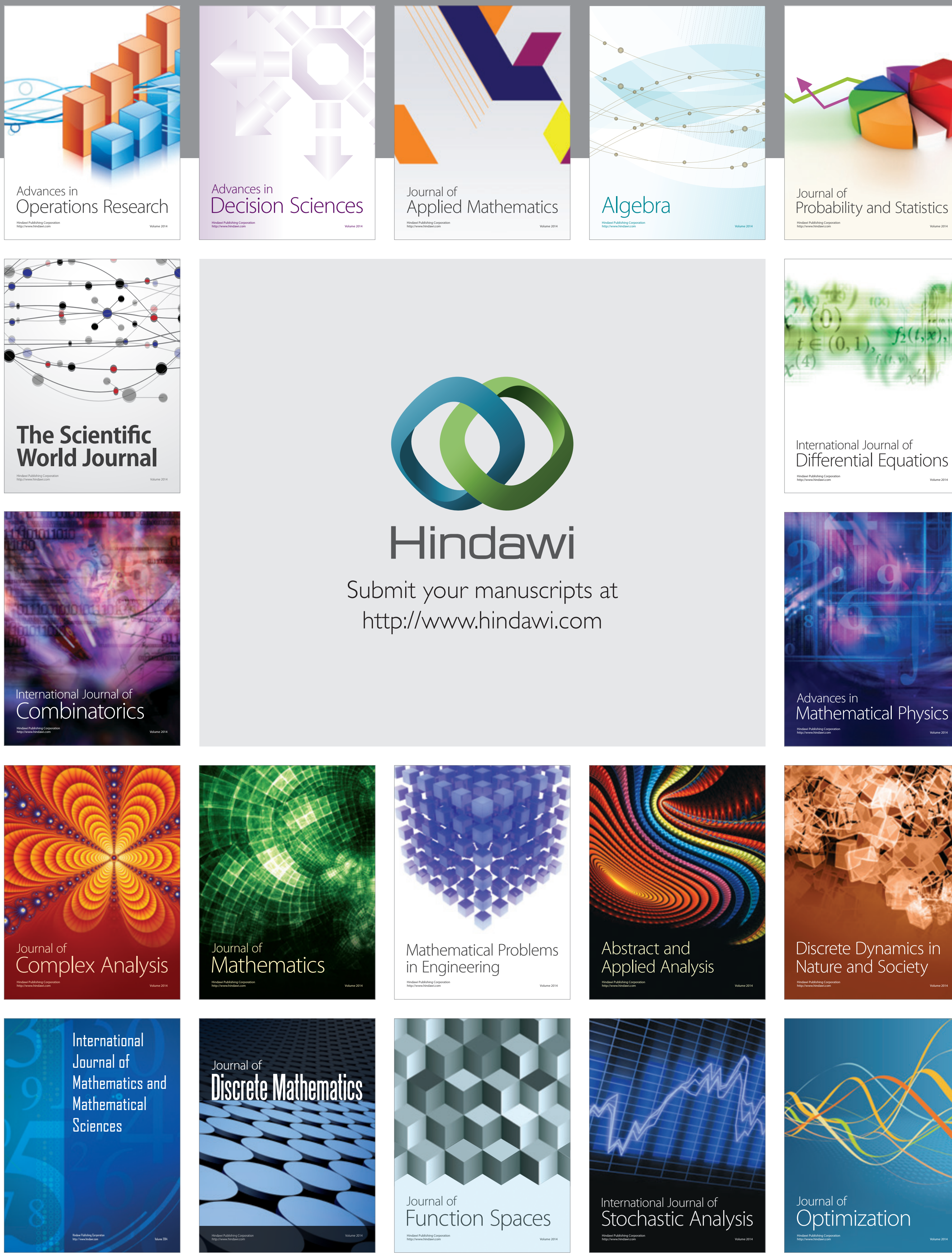
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