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Research Article

Space-Time Fractional Diffusion-Advection Equation with Caputo Derivative

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An alternative construction for the space-time fractional diffusion-advection equation for the sedimentation phenomena is presented. The order of the derivative is considered as $0 < \beta, \gamma \leq 1$ for the space and time domain, respectively. The fractional derivative of Caputo type is considered. In the spatial case we obtain the fractional solution for the underdamped, undamped, and overdamped case. In the temporal case we show that the concentration has amplitude which exhibits an algebraic decay at asymptotically large times and also shows numerical simulations where both derivatives are taken in simultaneous form. In order that the equation preserves the physical units of the system two auxiliary parameters σ_x and σ_t are introduced characterizing the existence of fractional space and time components, respectively. A physical relation between these parameters is reported and the solutions in space-time are given in terms of the Mittag-Leffler function depending on the parameters β and γ. The generalization of the fractional diffusion-advection equation in space-time exhibits anomalous behavior.

1. Introduction

The Diffusion-Advection Equation (DAE) describes the evolution of a concentration profile due to diffusion and advection simultaneously; this equation describes physical phenomena where concentrations as mass, energy, or other physical quantities are transferred inside a physical system due to two contributions: diffusion and convection, in this equation the concentration-dependent diffusion coefficient. The fractional calculus (FC) is the generalization of derivatives and integrals to noninteger orders; the mathematical formulation was developed for Fourier, Liouville, Abel, Riemann, Lacroix, Grünwald, Riesz, among many others. Many problems in physical science electromagnetism, electrochemistry, diffusion, and general transport theory can be solved by the fractional calculus approach [1–9]. The FC may be approached via the theory of linear differential equations. FC involves nonlocal operators which can be applied to physical systems yielding new information about their behavior. Modeling as fractional order proves to be useful particulary for systems where the memory plays a significant role, in comparison with the ordinary calculus models; this is the main advantage [10–13]. The process of sedimentation of particles dispersed in a fluid is one of great practical importance, but it has always proved extremely difficult to examine theoretically. The sedimentation phenomena describe the response of the system to the action of an external force (usually centrifugal). The hydrodynamical problem of one particle falling through a fluid has been solved by Einstein, for Smoluchowski and many others [14, 15]. A random walk is a mathematical formalization of a path that consists of a succession of random steps. Random walks are related to the diffusion models and within the fractional approach it is possible to include external fields; considerations of transport in the phase space are possible within the same approach [16]. A Lévy flight, also referred to as Lévy motion, is a random walk in which the step-lengths have a probability distribution that is heavy-tailed. When defined as a walk in a space of dimension greater than one, the steps are defined in terms of the step-lengths, which have a certain probability distribution; the steps made are in isotropic random directions [17]. Scher and Montroll [18] in their description of anomalous transit-time dispersion in amorphous solids considered a system where the traditional methods proved to fail. The authors presented a stochastic transport model for the current; the dynamics considered continuous time random walk approach for a variety of physical quantities in numerous experimental realizations. Jespersen et al. in [19] considered Lévy flights subject to external force fields; the authors presented a Riesz/Weyl form of the DAE; the corresponding Fokker-Planck equation contains a fractional derivative in space. Mainardi et al. in [20] present the interpretation of the corresponding Green function as a probability density for the particular cases of space fractional, time fractional, and neutral fractional diffusion; the fundamental space-time fractional diffusion equation is obtained from the standard diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative of order $0 < \alpha \leq 2$ and the first-order time derivative with a Caputo derivative of order $0 < \beta \leq 2$. Recently, the fractional kinetic equations of the diffusion, DAE, and Fokker-Planck type are presented in [21]; the equations are derived from basic random walk models. Metzler and Compte in [22] consider advection processes following anomalous statistics on an effective transport level within the framework of fractional kinetic equations. Liu et al. [23] present the space fractional Fokker-Planck equation, the authors use the Riemann-Liouville and Grünwald-Letnikov definitions of fractional derivatives, the Fokker-Planck equation is transformed into a system of ordinary differential equations, and numerical results are presented. In the works of the authors mentioned above the pass from an ordinary derivative to a fractional one is direct, to be consistent with the dimensionality of the fractional differential equations (FDE) in the work [24]; the authors have proposed a systematic way to construct FDE for the physical systems analyzing the dimensionality of the ordinary derivative operator and trying to bring it to a fractional derivative operator consistently. Following [24], in this work we propose an alternative procedure for constructing the fractional DAE; the order of the derivative being considered is $0 < \beta, \gamma \leq 1$ for space-time domain, respectively. The main purpose in this paper was to show an alternative solution of the DAE using the fractional derivative of Caputo type; this representation preserves the physical units of the system for any value taken by the exponent of the fractional derivative. This alternative solution in the range $0 < \beta, \gamma \leq 1$ describes Levy flights (non-Markovian version) and the phenomena of subdiffusion for space-time domain, respectively. The paper is organized as follows: second section introduction to fractional calculus, third section the fractional DAE, and the fourth section the conclusions.

2. Fundamentals of Fractional Calculus

The most commonly used definitions in fractional calculus are Riemann-Liouville (RL), Grünwald-Letnnikov definition (GL), and Caputo fractional derivative (CFD) [3].

The RL definition of the fractional derivative for $(q > 0)$ is

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_a^t \frac{f(\eta)}{(t-\eta)^{q-m+1}} d\eta, \quad m-1 < q < m. \tag{1}$$

The GL definition is formulated as $(q\varepsilon R)$

$$ {}_aD_t^q f(t) = \lim_{j\to 0} \frac{1}{j^q} \sum_{k=0}^{(t-a)/j} (-1)^k \binom{q}{k} f(t-kj), \tag{2}$$

where j is the time increment.

The CFD for a function $f(t)$ is defined as follows:

$$ {}_0^C D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}} d\eta, \tag{3}$$

where $n = 1, 2, \ldots \in N$ and $n-1 < \gamma \leq n$. In this case, $0 < \gamma \leq 1$ is the order of the CFD. In this definition the derivative of a constant is zero and the initial conditions for the fractional order differential equations take the same form as for the integer-order ones taking a known physical interpretation, which is very suitable in engineering [4, 6, 9, 24]. In this paper we will use the CFD.

Laplace transform to CFD is given by [3]

$$ L\left[{}_0^C D_t^\gamma f(t)\right] = S^\gamma F(S) - \sum_{k=0}^{m-1} S^{\gamma-k-1} f^{(k)}(0). \tag{4}$$

The Mittag-Leffler function has caused extensive interest among physicists due to its vast potential of applications describing realistic physical systems with memory and delay. The inverse Laplace transform requires the introduction of the Mittag-Leffler function defined by the series expansion as [3]

$$ E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha m + \beta)}, \quad (\alpha > 0), \ (\beta > 0), \tag{5}$$

where Γ is the gamma function, when $\alpha, \beta = 1$; from (5) we obtain e^t, the exponential function as a special case of the Mittag-Leffler function.

This function will often appear with the argument $t = -at^\alpha$; its Laplace transform then is given as

$$ L\left[E_{\alpha,\beta}(-at^\alpha)\right] = L\left[\sum_{m=0}^{\infty} \frac{(-a)^\alpha t^{m\alpha}}{\Gamma(\alpha m+\beta)}\right] = \frac{s^\alpha}{s(s^\alpha + a)}, \quad (p > 0). \tag{6}$$

Some common Laplace transforms are

$$ \begin{aligned} \frac{1}{s^\alpha + a} &= t^{\alpha-1} E_{\alpha,\alpha}(-at^\alpha), \\ \frac{a}{s(s^\alpha + a)} &= 1 - E_\alpha(-at^\alpha). \end{aligned} \tag{7}$$

3. Fractional Diffusion-Advection Equation

The DAE [21] is represented by

$$ D\frac{\partial^2 C(x,t)}{\partial x^2} + \lambda \frac{\partial C(x,t)}{\partial x} - \frac{\partial C(x,t)}{\partial t} = 0, \tag{8}$$

where C is the concentration, D is the diffusion coefficient, and λ is the drift velocity; this equation considered only the distribution of one cartesian component x.

To be consistent with dimensionality and following [24] we introduce an auxiliary parameter σ_x and σ_t as follows:

$$\frac{d}{dx} \longrightarrow \frac{1}{\sigma_x^{1-\beta}} \cdot \frac{d^\beta}{dx^\beta}, \quad n-1 < \beta \leq n, \tag{9}$$

$$\frac{d}{dt} \longrightarrow \frac{1}{\sigma_t^{1-\gamma}} \cdot \frac{d^\gamma}{dt^\gamma}, \quad n-1 < \gamma \leq n; \tag{10}$$

this is true if the parameter σ_x has dimensions of length (inverse meters) and σ_t has the dimension of time (inverse seconds).

The fractional representation of (8) is

$$\frac{D}{\sigma_x^{2(1-\beta)}} \frac{\partial^{2\beta} C(x,t)}{\partial x^{2\beta}} + \frac{\lambda}{\sigma_x^{1-\beta}} \frac{\partial^\beta C(x,t)}{\partial x^\beta} - \frac{1}{\sigma_t^{1-\gamma}} \frac{\partial^\gamma C(x,t)}{\partial t^\gamma} = 0. \tag{11}$$

The order of the derivative being considered is $0 < \beta$, $\gamma \leq 1$ for the fractional DAE in space and time domain, respectively.

3.1. Fractional Space Diffusion-Advection Equation. Considering (11) and assuming that the space derivative is fractional (9) and the time derivative is ordinary, the spatial fractional equation is

$$\frac{\partial^{2\beta} C(x,t)}{\partial x^{2\beta}} + \frac{\lambda}{D} \sigma_x^{1-\beta} \frac{\partial^\beta C(x,t)}{\partial x^\beta} - \frac{1}{D} \sigma_x^{2(1-\beta)} \frac{\partial C(x,t)}{\partial t} = 0. \tag{12}$$

A particular solution of this equation may be found in the form

$$C(x,t) = C_0 e^{-\omega t} u(x), \tag{13}$$

where ω is the natural frequency and C_0 is a constant.

Substituting (13) in (12) we obtain

$$\frac{d^{2\beta} u(x)}{dx^{2\beta}} + \frac{\lambda}{D} \sigma_x^{1-\beta} \frac{d^\beta u(x)}{dx^\beta} + \frac{\omega}{D} \sigma_x^{2(1-\beta)} u(x) = 0. \tag{14}$$

The solution of (14) may be obtained applying (4). Taking the solution (13) we have

$$C(x,t) = C_0 e^{-\omega t} \cdot E_\beta \left(-\frac{\lambda}{2D} \sigma_x^{1-\beta} x^\beta \right) \cdot E_{2\beta} \left(-\left[\frac{\omega}{D} - \frac{\lambda^2}{4D^2} \right] \sigma_x^{2(1-\beta)} x^{2\beta} \right). \tag{15}$$

For underdamped case, with $(\omega/D - \lambda^2/4D^2) = 0$, $\lambda = 2\sqrt{\omega D}$. Considering $\lambda = 2\sqrt{\omega D}$ and $C(0) = C_0$, in (15), $\vec{K}_0^2 = \omega/D$ is the wave vector and $\alpha^2 = \lambda/2D$ is the damping factor.

When $\beta = 1$, from (15) we have

$$C(x,t) = C_0 e^{-\omega t} \cdot e^{-(\lambda/2D)x} \cdot \cos\left(\sqrt{\frac{\omega}{D} - \frac{\lambda^2}{4D^2}} x \right); \tag{16}$$

(16) represents the classical case in the underdamped case. From (16) we see that there is a relation between β and σ_x given by

$$\beta = \left(\frac{\omega}{D} - \frac{\lambda^2}{4D^2} \right)^{1/2} \sigma_x, \quad 0 < \sigma_x \leq \frac{1}{(\omega/D - \lambda^2/4D^2)^{1/2}}. \tag{17}$$

Then, the solution (15) for the underdamped case $\lambda < 2\sqrt{\omega D}$ or $\alpha < \vec{K}_0$ takes the form

$$C(\tilde{x},t) = C_0 e^{-\omega t} \cdot E_\beta \left(-\frac{\lambda}{2D\sqrt{\omega/D - \lambda^2/4D^2}} \beta^{1-\beta} \tilde{x}^\gamma \right) \cdot E_{2\beta} \left(-\beta^{2(1-\beta)} \tilde{x}^{2\beta} \right), \tag{18}$$

where $\tilde{x} = ((\omega/D) - (\lambda^2/4D^2))^{1/2} x$ is a dimensionless parameter.

Due to the condition $\lambda < 2\sqrt{\omega D}$ we can choose an example

$$\frac{\lambda}{2D\sqrt{(\omega/D) - (\lambda^2/4D^2)}} = \frac{1}{3}, \qquad 0 \leq \frac{\lambda}{2D\sqrt{(\omega/D) - (\lambda^2/4D^2)}} < \infty. \tag{19}$$

So, the solution (15) takes its final form

$$C(\tilde{x},t) = C_0 e^{-\omega t} \cdot E_\beta \left(-\frac{1}{3} \beta^{1-\beta} \tilde{x}^\beta \right) \cdot E_{2\beta} \left(-\beta^{2(1-\beta)} \tilde{x}^{2\beta} \right). \tag{20}$$

Figure 1 shows the concentration in the space for different values of β arbitrarily chosen.

In the overdamped case, $\alpha > \vec{K}_0$ or $\lambda > 2\sqrt{\omega D}$, the solution of (15) has the form

$$\widetilde{C}(x,t) = \widetilde{C}_0 e^{-\omega t} \cdot E_\beta \left(-\frac{\lambda}{2D} \sigma_x^{1-\beta} x^\beta \right) \cdot E_\beta \left(-\left[\frac{\lambda^2}{4D^2} - \frac{\omega}{D} \right]^{1/2} \sigma_x^{1-\beta} x^\beta \right). \tag{21}$$

When $\beta = 1$, from (21) we have

$$\widetilde{C}(x,t) = \widetilde{C}_0 e^{-\omega t} \cdot e^{-(\lambda/2D)(1+\sqrt{1-(4\omega D/\lambda^2)})x}, \tag{22}$$

where $\widetilde{C}(0) = \widetilde{C}_0$ is the initial concentration in $(x = 0, t = 0)$. The solution (22) represents the change of the concentration $\widetilde{C}(x,t)$. This represents the classical case.

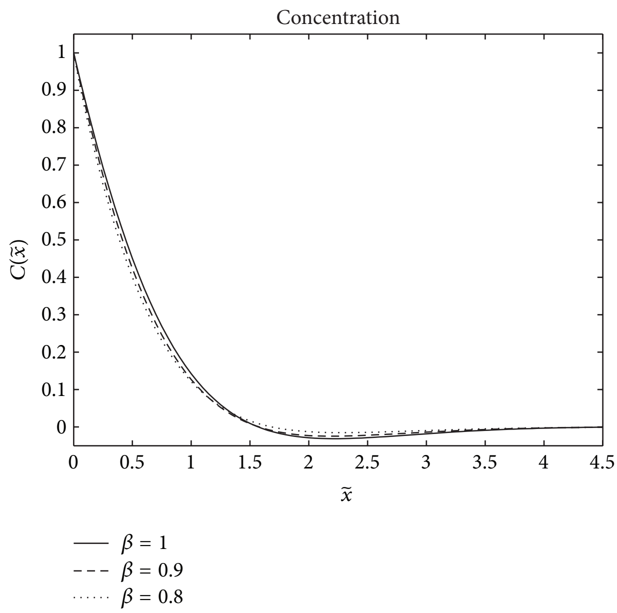

FIGURE 1: Concentration in the space. Simulation for (20), $\beta = 1$, $\beta = 0.9$, and $\beta = 0.8$.

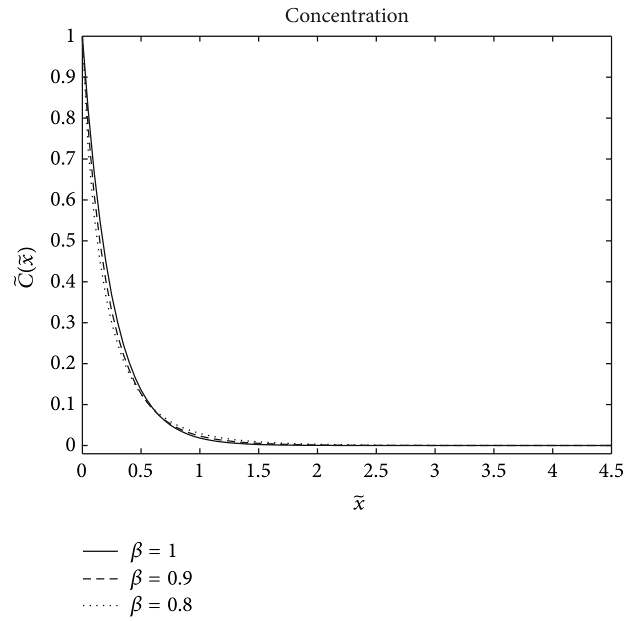

FIGURE 2: Concentration in the space. Simulation for (26), $\beta = 1$, $\beta = 0.9$, and $\beta = 0.8$.

Take into account the fact that the relation between β and σ_x is

$$\beta = \left(\frac{\lambda^2}{4D^2} - \frac{\omega}{D}\right)^{1/2} \sigma_x, \quad 0 < \sigma_x \leq \frac{1}{\left((\lambda^2/4D^2) - (\omega/D)\right)^{1/2}}. \tag{23}$$

The solution (21) takes the form

$$\widetilde{C}(\tilde{x}, t) = \widetilde{C}_0 e^{-\omega t} \cdot E_\beta\left(-\frac{\lambda}{2D\sqrt{(\lambda^2/4D^2) - (\omega/D)}} \beta^{1-\beta} \tilde{x}^\beta\right) \cdot E_\beta\left(-\beta^{1-\beta} \tilde{x}^\beta\right), \tag{24}$$

where $\tilde{x} = ((\lambda^2/4D^2) - (\omega/D))^{1/2} x$ is a dimensionless parameter.

Due to the condition $\lambda > 2\sqrt{\omega D}$, we have the following range of values:

$$\frac{\lambda}{2D\sqrt{(\lambda^2/4D^2) - (\omega/D)}} = 2,$$
$$1 < \frac{\lambda}{2D\sqrt{(\lambda^2/4D^2) - (\omega/D)}} < \infty. \tag{25}$$

Then, the solution (21) can be written in its final form

$$\widetilde{C}(\tilde{x}, t) = \widetilde{C}_0 e^{-\omega t} \cdot E_\beta\left(-2\beta^{1-\beta} \tilde{x}^\beta\right) \cdot E_\beta\left(-\beta^{1-\beta} \tilde{x}^\beta\right). \tag{26}$$

Figure 2 shows the concentration in the space for different values of β arbitrarily chosen.

3.2. Fractional Time Diffusion-Advection Equation. Considering (11) and assuming that the time derivative is fractional (10) and the space derivative is ordinary, the temporal fractional equation is

$$\frac{\partial^\gamma C(x, t)}{\partial t^\gamma} - \lambda \sigma_t^{1-\gamma} \frac{\partial C(x, t)}{\partial x} - D\sigma_t^{1-\gamma} \frac{\partial^2 C(x, t)}{\partial x^2} = 0. \tag{27}$$

A particular solution of this equation may be found in the form

$$C(x, t) = C_0 e^{i\vec{k}x} u(t), \tag{28}$$

where $\vec{k}$ is the wave vector in the x direction and C_0 is a constant.

Substituting (28) into (27) we obtain

$$\frac{d^\gamma u(x)}{dt^\gamma} + \left(D\vec{k}^2 - i\lambda\vec{k}\right) \sigma_t^{1-\gamma} u(t) = 0; \tag{29}$$

if $\vec{\omega}$ is the natural frequency with real and imaginary parts δ and φ, we have

$$\delta = D\vec{k}^2,$$
$$\varphi = \lambda\vec{k}, \tag{30}$$

where

$$\vec{\omega}^2 = \left(D\vec{k}^2 - i\lambda\vec{k}\right), \tag{31}$$

$$\vec{\tilde{\omega}}^2 = \left(D\vec{k}^2 - i\lambda\vec{k}\right) \sigma_t^{1-\gamma}, \tag{32}$$

$$\vec{\tilde{\omega}}^2 = \vec{\omega}^2 \sigma_t^{1-\gamma}, \tag{33}$$

$\vec{\tilde{\omega}}^2$ is the natural frequency in the medium in presence of fractional time components, and $\vec{\omega}^2$ is the natural frequency without its presence.

Substituting (33) in (29) gives

$$\frac{d^\gamma u(t)}{dt^\gamma} + \vec{\tilde{\omega}}^2 u(t) = 0, \tag{34}$$

The solution is written as

$$u(t) = E_\gamma\left(-\vec{\tilde{\omega}}^2 t^\gamma\right). \tag{35}$$

The particular solution of the equation (35) as follows

$$C(x,t) = C_0 e^{i\vec{k}x} E_\gamma\left(-\vec{\tilde{\omega}}^2 t^\gamma\right), \tag{36}$$

where $E_\gamma(-\vec{\tilde{\omega}}^2 t^\gamma)$ is the Mittag-Leffler function.

When $\gamma = 1$, we have, $\vec{\omega} = \delta - i\varphi$, substituting this expression in (30) we have

$$(\delta - i\varphi)^2 = \delta^2 - 2i\delta\varphi - \varphi^2, \tag{37}$$

where

$$\delta^2 - 2i\delta\varphi - \varphi^2 = D\vec{k}^2 - i\lambda\vec{k}; \tag{38}$$

solving for φ we obtain

$$\varphi = \frac{\lambda\vec{k}}{2\delta}; \tag{39}$$

and for δ

$$\delta = \vec{k}\sqrt{D}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{\lambda^2}{\vec{k}^2 D^2}}\right]^{1/2}; \tag{40}$$

substitutitng (40) into (39) we have

$$\varphi = \frac{\lambda}{2} \cdot \frac{1}{\sqrt{D}\left[(1/2) \pm (1/2)\sqrt{1 + \left(\lambda^2/\vec{k}^2 D^2\right)}\right]^{1/2}}. \tag{41}$$

Equations (40) and (41) describe the real and imaginary part of the natural frequency in terms of the wave vector k, the diffusion coefficient D, and the drift velocity λ. From (36) we have

$$C(x,t) = C_0 e^{i\vec{k}x} E_1(-\vec{\omega}t). \tag{42}$$

The solution to (27) which follows from (42) is given by

$$C(x,t) = \Re\left[\widetilde{C}_0 \cdot e^{-\vec{\omega}t + i\vec{k}x}\right], \tag{43}$$

where $\vec{\omega} = \delta - i\varphi$, is the natural frequency, δ and φ is given by (40) and (41), respectively. The equation (42) represent the classical case.

In these case exists a physical relation between the natural frequency $\vec{\omega}$, the parameter σ_t and the period T_0 given by the order γ of the fractional differential equation

$$\gamma = \vec{\omega}\sigma_t = \frac{\sigma_t}{T_0}, \quad 0 < \sigma_t \le T_0. \tag{44}$$

We can use this relation to write (43) as

$$C(x,\tilde{t}) = C_0 \cdot e^{i\vec{k}x} \cdot E_\gamma\left(-\gamma^{1-\gamma}\tilde{t}^\gamma\right), \tag{45}$$

where $\tilde{t} = \vec{\omega}t$ is a dimensionless parameter.

When $\gamma = 1/2$, $\vec{\tilde{\omega}}^2 = \vec{\omega}^2\sigma_t^{1/2}$, from (36) we have

$$C(x,t) = C_0 e^{i\vec{k}x} E_{1/2}\left(-\vec{\omega}^2\sigma_t^{1/2}t^{1/2}\right), \tag{46}$$

$$u(t) = E_{1/2}\left(-\vec{\omega}^2\sigma_t^{1/2}t^{1/2}\right), \tag{47}$$

where

$$E_{1/2}(z) = e^{z^2}\operatorname{erfc}(-z) = e^{\vec{\omega}^4\sigma_t t}\operatorname{erfc}\left(-\vec{\omega}^2\sqrt{\sigma_t t}\right), \tag{48}$$

and the $\operatorname{erfc}(z)$ denotes the error function defined as

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt. \tag{49}$$

For large values of z, the error function can be approximated as

$$\operatorname{erfc}(z) \simeq \frac{1}{\sqrt{\pi z}} e^{-z^2}; \tag{50}$$

substituting (48) into (46) leads to the solution

$$C(x,t) = C_0 e^{i\vec{k}x} e^{\vec{\omega}^4\sigma_t t} erfc\left(\vec{\omega}^2\sqrt{\sigma_t t}\right). \tag{51}$$

At asymptotically large times and using (49) we have

$$C(x,t) \approx \left(\frac{C_0}{\vec{\omega}^2\sqrt{\pi\sigma_t t}} t^{-\gamma}\right) e^{i\vec{k}x}. \tag{52}$$

The solution (52) represents plane waves with time decaying amplitude. The asymptotic behavior of Mittag-Leffler function [25], for $z = -\vec{\tilde{\omega}}^2 t^\gamma$ is

$$E_\gamma\left(-\vec{\tilde{\omega}}^2 t^\gamma\right) \approx \frac{1}{\Gamma(1-\gamma)} \cdot \frac{1}{\vec{\tilde{\omega}}^2 t^\gamma}. \tag{53}$$

Then, substituting (53) in (36) gives

$$C(x,t) \approx \left(\frac{C_0}{\vec{\tilde{\omega}}^2\Gamma(1-\gamma)} t^{-\gamma}\right) e^{i\vec{k}x}, \tag{54}$$

where $\vec{\tilde{\omega}}^2 = \vec{\omega}\sigma_t^{1-\gamma}$ is the natural frequency in the medium in presence of fractional time components, the parameter σ_t characterizes these structures (components that show an intermediate behavior between a system conservative and dissipative) of the fractional time operator. In this case, (52) and (54) represent the time evolution of the concentration and the amplitude which exhibits an algebraic decay for $t \to \infty$. The fractional differentiation with respect to time can be interpreted as an existence of memory effects which correspond to intrinsic dissipation in the system [6]. Figure 3 shows the simulation of (45) for $\gamma = 0.9$ arbitrarily chosen and Figure 4 shows the concentration in the time for different values of γ also chosen arbitrarily.

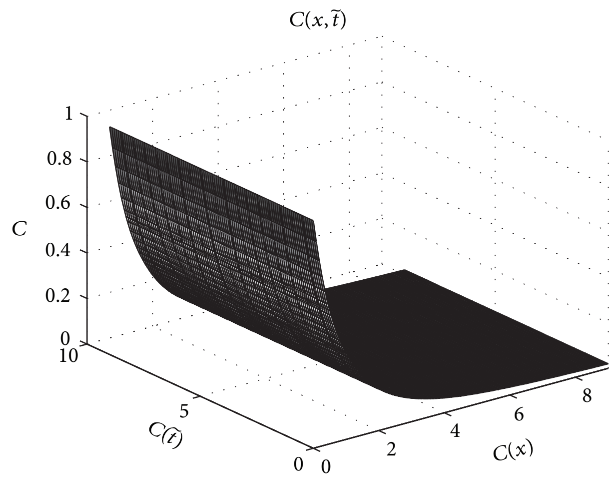

Figure 3: Simulation for (45), $\gamma = 0.9$.

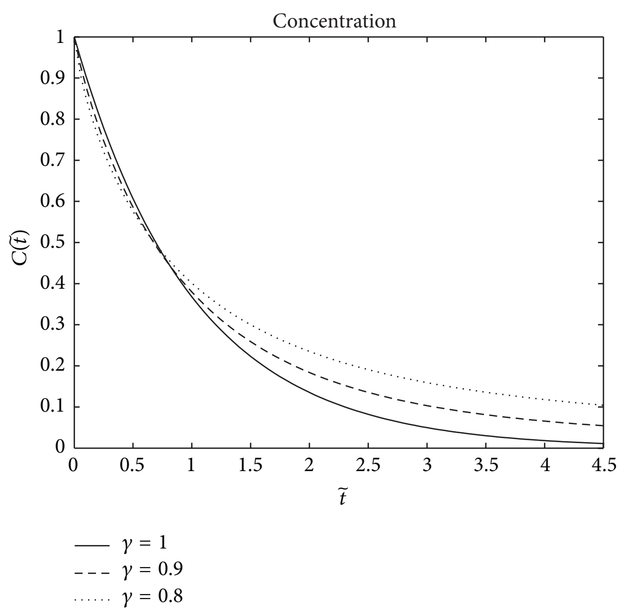

Figure 4: Concentration in the time. Simulation for (45), $\gamma = 1$, $\gamma = 0.9$, and $\gamma = 0.8$.

3.3. Fractional Space-Time Diffusion-Advection Equation. Now considering (11) and assuming that the space and time derivatives are fractional, the order of the time-space fractional differential equation is $0 < \beta, \gamma \le 1$, σ_x has dimension of length, and σ_t has dimension of time. Figures 5 and 6 show the simulation of (20)–(45) and (26)–(45) for different values of β and γ arbitrarily chosen.

4. Conclusions

We present the analysis of the DAE from the point of view of fractional calculus. FDE have been examined separately; the fractional spatial derivative and the temporal fractional derivative have finally shown the numerical simulations where both derivatives are taken in simultaneous form. The parameters σ_x and σ_t are introduced characterizing

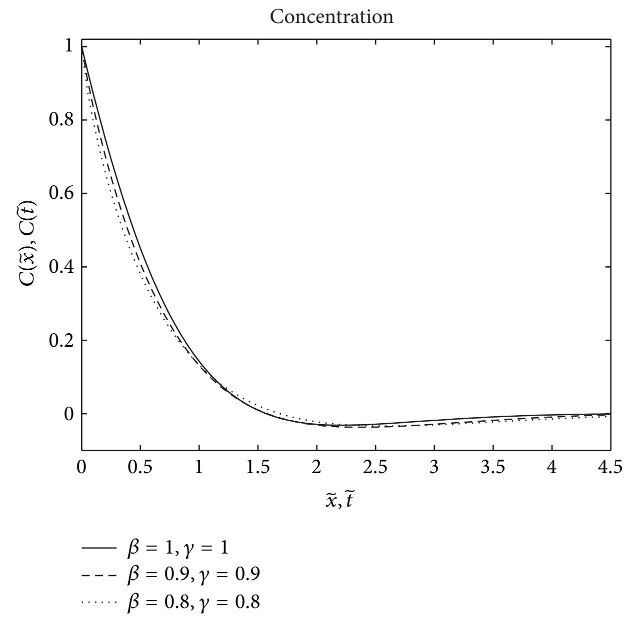

Figure 5: Concentration in the space and time. Simulation for (20) and (45) for ($\beta = 1, \gamma = 1$), ($\beta = 1, \gamma = 0.9$), and ($\beta = 1, \gamma = 0.8$).

the existence of the fractional space and time components, respectively; these parameters represent components that show an intermediate behavior between a conservative and dissipative system. The general solutions of the fractional DAE depending only on the parameters β and γ are given in the form of the multivariate Mittag-Leffler functions preserving the physical units of the system studied.

It was shown in the spatial case that the solution (15) corresponds to the spatial generalized solution of the DAE, in the case $\beta = 1$; (16) represents the classical case in the underdamped case. In the overdamped case, (24) for $\beta = 1$ represents the overdamped case.

For the temporal case (36) shows the solution for the fractional time DAE. When $\gamma = 1$, the natural frequency $\vec{\omega}$ is represented by $\vec{\omega} = \delta - i\varphi$, where δ and φ are given by (40) and (41), respectively, and describes the real and imaginary part of the natural frequency in terms of the wave vector k, the diffusion coefficient D, and the drift velocity λ; (43) represents the classical case. In this case exists a physical relation between the parameter σ_t and the period T_0 given by the order γ of the fractional differential equation, relation (44). An important measure for sedimentation processes is the speed with which the major part of the tracer is setting down; in the subdiffusive case, the drift velocity λ is only an effective velocity and it is explicitly time dependent; (52) and (54) represent the time evolution of the concentration; these equations involve the drift velocity and the amplitude which exhibit an algebraic decay for $t \to \infty$ and exhibit anomalous slow diffusion (subdiffusion).

The solutions (15) and (36) correspond to space-time generalized solutions of the DAE. Figures 5 and 6 show the case where both derivatives are considered (time-space) simultaneous; besides, it was shown that when β and γ are less than 1, the concentration behaves like a concentration

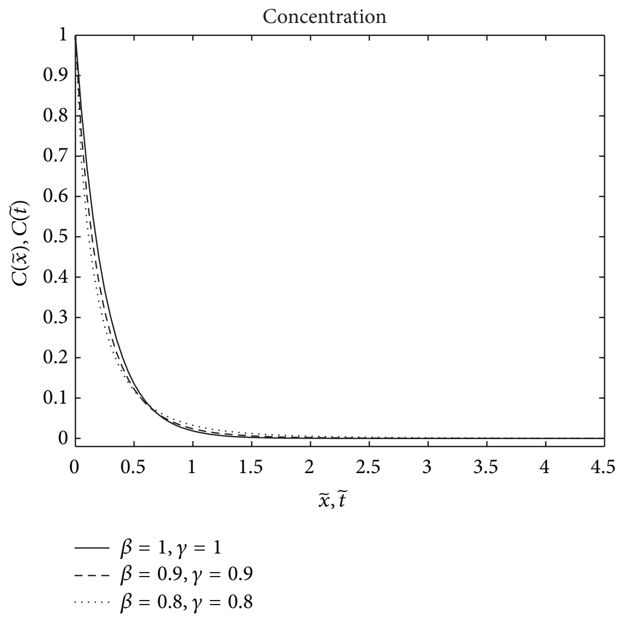

FIGURE 6: Concentration in the space and time. Simulation for (26) and (45) for ($\beta = 1, \gamma = 1$), ($\beta = 0.9, \gamma = 0.9$), and ($\beta = 0.8, \gamma = 0.8$).

with spatial-temporal-decaying amplitude with respect to time t and the space x. The numerical simulations represent a nonlocal concentration interpreted as an existence of memory effects which correspond to intrinsic dissipation characterized by the exponent of the fractional derivatives β and γ in the system and related to concentration in a fractal space-time geometry.

This alternative solution of DAE using the fractional derivative of Caputo type in the range $0 < \beta, \gamma \leq 1$ may contribute to the study and interpretation of the electrokinetic phenomena, the models of porous electrodes, the groundwater dynamics, and the description of anomalous complex processes (mass transport by diffusion and convection considering the formation of different intermediate species).

In the case of Caputo's derivative it is defined in the range of $1 < \beta, \gamma \leq 2$ which allows describing superdiffusion phenomena (including ballistic diffusion); the Levy flights model and subwave phenomena will be made in a future paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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