

Research Article

Generalized Antiperiodic Boundary Value Problems for the Fractional Differential Equation with p -Laplacian Operator

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We discuss the existence of solutions about generalized antiperiodic boundary value problems for the fractional differential equation with p -Laplacian operator $\phi_p({}^c D_{0+}^\alpha u(t)) = f(t, u(t), u'(t))$, $0 < t < T$, $1 < \alpha \leq 2$, $u(0) + (-1)^\theta a u(T) = 0$, ${}^c D_{0+}^\beta u(0) + (-1)^\theta b {}^c D_{0+}^\beta u(T) = \lambda$, $0 < \beta < 1$, where ${}^c D_{0+}^\alpha$ is the Caputo fractional derivative, $\theta = 0, 1$, $a > 0$, $a \neq 1$, $b > 0$ and $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$. Our results are based on fixed point theorem and contraction mapping principle. Furthermore, three examples are also given to illustrate the results.

1. Introduction

Fractional differential equations arise in various areas of science and engineering, such as physics, mechanics, chemistry, and engineering. The fractional order models become more realistic and practical than the classical integer models. Due to their applications, fractional differential equations have gained considerable attentions; one can see [1–14] and references therein.

Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes. Anti-periodic problems constitute an important class of boundary value problems and have received considerable attention (see [15–19]).

In [20], Zhang considered the existence and multiplicity results of positive solutions for the following boundary value problem of fractional differential equation:

$$\begin{aligned} {}^c D_{0+}^p u(t) &= f(t, u(t)), & 0 < t < 1, \\ u(0) + u'(0) &= 0, & u(1) + u'(1) = 0, \end{aligned} \quad (1)$$

where $1 < p \leq 2$ is a real number, ${}^c D^q$ is the Caputo fractional derivative, and $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous.

In [15], the authors discussed some existence results for the following anti-periodic boundary value problem for fractional differential equations:

$$\begin{aligned} {}^c D^q x(t) &= f(t, x(t)), & t \in [0, T], & T > 0, & 3 < q \leq 4, \\ x(0) &= -x(T), & x'(0) &= -x'(T), \\ x''(0) &= -x''(T), & x'''(0) &= -x'''(T), \end{aligned} \quad (2)$$

where ${}^c D^q$ is the Caputo fractional derivative of order q ; f is a given continuous function.

In [16], the authors investigated the following anti-periodic boundary value problem for higher-order fractional differential equations:

$${}^c D^q x(t) = f(t, x(t)), \quad t \in [0, T], \quad T > 0, \quad 3 < q \leq 4,$$

$$\begin{aligned} x(0) &= -x(T), & x'(0) &= -x'(T), \\ x''(0) &= -x''(T), & x'''(0) &= -x'''(T), \end{aligned} \tag{3}$$

where ${}^c D^q$ is the Caputo fractional derivative of order q ; f is a given continuous function.

In [17], the authors investigated a class of anti-periodic boundary value problem of fractional differential equations

$$\begin{aligned} {}^c D^q x(t) &= f(t, x(t)), & t \in [0, T], & T > 0, & 1 < q \leq 2, \\ x(0) &= -x(T), & {}^c D_{0+}^p x(0) &= -{}^c D_{0+}^p u(T), & 0 < p < 1, \end{aligned} \tag{4}$$

where ${}^c D^q$ is the Caputo fractional derivative of order q ; f is a given continuous function.

In this paper, we discuss the existence of solutions about generalized anti-periodic boundary value problems for the fractional differential equation with p-Laplacian operator

$$\begin{aligned} \phi_p({}^c D_{0+}^\alpha u(t)) &= f(t, u(t), u'(t)), & 0 < t < T, & 1 < \alpha \leq 2, \\ u(0) + (-1)^\theta au(T) &= 0, \\ {}^c D_{0+}^\beta u(0) + (-1)^\theta b {}^c D_{0+}^\beta u(T) &= \lambda, & 0 < \beta < 1, \end{aligned} \tag{5}$$

where ${}^c D_{0+}^\alpha$ is the Caputo fractional derivative, $\theta = 0, 1$, $a > 0$, $a \neq 1$, $b > 0$, and $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $(1/p) + (1/q) = 1$.

If we take $a = b = 1$, $\theta = \lambda = 0$ and $p = 2$, then the problem (5) becomes the problem studied in [17]. In this paper, we let $a \neq 1$.

This paper is organized as follows. In Section 2, we present some background materials and preliminaries. Section 3 deals with some existence results. In Section 4, three examples are given to illustrate the results.

2. Background Materials and Preliminaries

Definition 1 (see [21]). The fractional integral of order α with the lower limit t_0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds, \quad t > t_0, \quad \alpha > 0, \tag{6}$$

where Γ is the gamma function.

Definition 2 (see [21]). Caputo's derivative of order q with the lower limit t_0 for a function f can be written as

$$\begin{aligned} {}^c D^q f(t) &= \frac{1}{\Gamma(n-q)} \int_{t_0}^t (t-s)^{n-q-1} f^{(n)}(s) ds, \\ t > t_0, \quad q > 0, \quad n &= [q] + 1. \end{aligned} \tag{7}$$

Lemma 3 (see [22]). Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then

$$I_{0+}^\alpha {}^c D_{0+}^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \tag{8}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Lemma 4. Let $y \in C[0, 1]$. Then the fractional differential equation

$$\begin{aligned} \phi_p({}^c D_{0+}^\alpha u(t)) &= y(t), & 0 < t < T, & 1 < \alpha \leq 2, \\ u(0) + (-1)^\theta au(T) &= 0, \\ {}^c D_{0+}^\beta u(0) + (-1)^\theta b {}^c D_{0+}^\beta u(T) &= \lambda \end{aligned} \tag{9}$$

has a unique solution which is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \\ &\quad - \frac{a}{[(-1)^\theta + a] \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} y(s) ds \\ &\quad + \frac{a\Gamma(2-\beta)}{[(-1)^\theta + a] T^{-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} y(s) ds \\ &\quad - \frac{a\Gamma(2-\beta)}{[1 + (-1)^\theta a] bT^{-\beta}} \\ &\quad + t \left[-\frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} y(s) ds \right. \\ &\quad \left. + \frac{(-1)^\theta \Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right]. \end{aligned} \tag{10}$$

Proof. From Lemma 3, we have

$$\begin{aligned} u(t) &= I_{0+}^\alpha (\phi_q(y(t))) + c_0 + c_1 t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(y(s)) ds + c_0 + c_1 t, \\ {}^c D_{0+}^\beta u(t) &= I_{0+}^{(\alpha-\beta)} (\phi_q(y(t))) + c_1 {}^c D_{0+}^\beta t \\ &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \phi_q(y(s)) ds \\ &\quad + c_1 \frac{t^{1-\beta}}{\Gamma(2-\beta)}. \end{aligned} \tag{11}$$

Thus,

$$\begin{aligned} u(0) &= c_0, \\ u(T) &= \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \phi_q(y(s)) ds + c_0 + c_1 T, \\ {}^c D_{0+}^\beta u(0) &= 0, \\ {}^c D_{0+}^\beta u(T) &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(y(s)) ds \\ &\quad + c_1 \frac{T^{1-\beta}}{\Gamma(2-\beta)}. \end{aligned} \tag{12}$$

By ${}^c D_{0+}^\beta u(0) + (-1)^\theta b {}^c D_{0+}^\beta u(T) = \lambda$, we have

$$c_1 = -\frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(y(s)) ds + \frac{(-1)^\theta \Gamma(2-\beta)}{bT^{1-\beta}} \lambda. \tag{13}$$

Using the boundary condition $u(0) + (-1)^\theta au(T) = 0$ and (13), we obtain

$$c_0 = -\frac{a}{[(-1)^\theta + a]\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \phi_q(y(s)) ds + \frac{a\Gamma(2-\beta)}{[(-1)^\theta + a]T^{-\beta}\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(y(s)) ds - \frac{a\Gamma(2-\beta)}{[1 + (-1)^\theta a]bT^{-\beta}} \lambda. \tag{14}$$

Thus,

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(y(s)) ds - \frac{a}{[(-1)^\theta + a]\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \phi_q(y(s)) ds + \frac{a\Gamma(2-\beta)}{[(-1)^\theta + a]T^{-\beta}\Gamma(\alpha-\beta)} \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(y(s)) ds - \frac{a\Gamma(2-\beta)}{[1 + (-1)^\theta a]bT^{-\beta}} \lambda + t \left[-\frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(y(s)) ds + \frac{(-1)^\theta \Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right]. \tag{15}$$

□

3. Main Results

Let $X = C^1([0, 1], R)$ denote the Banach space of continuous functions $u(t)$ and $u'(t)$ from $[0, 1] \rightarrow R$ endowed with the norm defined by

$$\|u\|_1 = \max \{ \|u\|, \|u'\| \}, \tag{16}$$

where

$$\|u\| = \sup_{0 \leq t \leq 1} |u(t)|, \quad \|u'\| = \sup_{0 \leq t \leq 1} |u'(t)|. \tag{17}$$

Define an operator $F : X \rightarrow X$ as

$$Fu(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(f(s, u(s), u'(s))) ds - \frac{a}{[(-1)^\theta + a]\Gamma(\alpha)} \times \int_0^T (T-s)^{\alpha-1} \phi_q(f(s, u(s), u'(s))) ds + \frac{a\Gamma(2-\beta)}{[(-1)^\theta + a]T^{-\beta}\Gamma(\alpha-\beta)} \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(f(s, u(s), u'(s))) ds - \frac{a\Gamma(2-\beta)}{[1 + (-1)^\theta a]bT^{-\beta}} \lambda + t \left[-\frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(f(s, u(s), u'(s))) ds + \frac{(-1)^\theta \Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right]. \tag{18}$$

From (18), we conclude that

$$F'u(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \phi_q(f(s, u(s), u'(s))) ds - \frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(f(s, u(s), u'(s))) ds + \frac{(-1)^\theta \Gamma(2-\beta)}{bT^{1-\beta}} \lambda. \tag{19}$$

Then (5) has a solution if and only if the operator F has a fixed point.

Theorem 5. Let $f : [0, T] \times R_+ \times R_+ \rightarrow R_+$ be continuous. Assume that f meets the following condition: there exist $d \geq 0, \gamma \geq 0$ such that

$$f(t, x(t), x'(t)) \leq \gamma \phi_p(|x(t)|), \quad \text{for } t \in [0, T], \|x\|_1 \leq d, \tag{20}$$

$$\begin{aligned} \phi_q(\gamma) \leq & \frac{1}{2} \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} \right. \\ & + \frac{aT^\alpha\Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} + \frac{T^\alpha\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \\ & \left. + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right)^{-1}. \end{aligned} \quad (21)$$

Then the problem (5) has at least one solution on $[0, T]$ for

$$0 \leq \lambda \leq \frac{d}{2} \left(\frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \right)^{-1}. \quad (22)$$

Proof. From $f(t, u(t), u'(t)) \in C([0, 1] \times R_+ \times R_+, R_+)$, we know that T is continuous.

Let

$$B_d = \{u \mid \|u\|_1 \leq d, u \in X\}. \quad (23)$$

For $u \in B_d$, we have

$$\begin{aligned} |Fu(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & + \frac{a}{|(-1)^\theta + a|\Gamma(\alpha)} \\ & \times \int_0^T (T-s)^{\alpha-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & + \frac{a\Gamma(2-\beta)}{|(-1)^\theta + a|T^{-\beta}\Gamma(\alpha-\beta)} \\ & \times \int_0^T (T-s)^{\alpha-\beta-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & + \left| \frac{a\Gamma(2-\beta)}{[1+(-1)^\theta a]bT^{-\beta}} \lambda \right| \\ & + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \right. \\ & \times \int_0^T (T-s)^{\alpha-\beta-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & \left. + \left| \frac{(-1)^\theta\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right| \right] \end{aligned}$$

$$\begin{aligned} & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(\gamma) |u(s)| ds \\ & + \frac{a}{|1-a|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \phi_q(\gamma) |u(s)| ds \\ & + \frac{a\Gamma(2-\beta)}{|1-a|T^{-\beta}\Gamma(\alpha-\beta)} \\ & \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(\gamma) |u(s)| ds \\ & + \frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}} \lambda \\ & + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \right. \\ & \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(\gamma) |u(s)| ds \\ & \left. + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right] \\ & \leq \left(\frac{T^\alpha\phi_q(\gamma)}{\Gamma(\alpha+1)} + \frac{aT^\alpha\phi_q(\gamma)}{|1-a|\Gamma(\alpha+1)} \right. \\ & + \frac{aT^\alpha\Gamma(2-\beta)\phi_q(\gamma)}{|1-a|\Gamma(\alpha-\beta+1)} \\ & \left. + \frac{T^\alpha\Gamma(2-\beta)\phi_q(\gamma)}{\Gamma(\alpha-\beta+1)} \right) \|u\|_1 \\ & + \frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}} \lambda + \frac{\Gamma(2-\beta)}{bT^{-\beta}} \lambda, \\ |F'u(t)| \leq & \frac{1}{\Gamma(\alpha-1)} \\ & \times \int_0^t (t-s)^{\alpha-2} |\phi_q(f(s, u(s), u'(s)))| ds \\ & + \frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \\ & \times \int_0^T (T-s)^{\alpha-\beta-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \\ & \leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \phi_q(\gamma) |u(s)| ds \\ & + \frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(\gamma) |u(s)| ds \\ & + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \end{aligned}$$

$$\leq \left(\frac{T^{\alpha-1} \phi_q(\gamma)}{\Gamma(\alpha)} + \frac{T^{\alpha-1} \Gamma(2-\beta) \phi_q(\gamma)}{\Gamma(\alpha-\beta+1)} \right) \|u\|_1 + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda. \tag{24}$$

This, together with (21) and (22), yields that

$$\|Fu(t)\|_1 \leq d. \tag{25}$$

Hence, $F(B_d)$ is uniformly bounded.

Next we show that F is equicontinuous.

For any $0 \leq t_1 \leq t_2 \leq 1$, $u \in B_d$, we have

$$\begin{aligned} & |Fu(t_2) - Fu(t_1)| \\ & \leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} \phi_q(f(s, u(s), u'(s))) ds \right. \\ & \quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \phi_q(f(s, u(s), u'(s))) ds \right| \\ & \quad + \left[\frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \right. \\ & \quad \left. \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(f(s, u(s), u'(s))) ds \right. \\ & \quad \left. + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right] (t_2 - t_1) \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \phi_q(\gamma) |u(s)| ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} \phi_q(\gamma) |u(s)| ds \\ & \quad + \left[\frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(\gamma) |u(s)| ds \right. \\ & \quad \left. + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right] (t_2 - t_1) \\ & \leq \frac{\phi_q(\gamma) d}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| ds \\ & \quad + \frac{\phi_q(\gamma) d}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} ds \\ & \quad + \left[\frac{T^{\alpha-1} \phi_q(\gamma) d \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right. \\ & \quad \left. + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right] (t_2 - t_1), \end{aligned}$$

$$\begin{aligned} & |F'u(t_2) - F'u(t_1)| \\ & \leq \left| \frac{1}{\Gamma(\alpha-1)} \int_0^{t_2} (t_2 - s)^{\alpha-2} \phi_q(f(s, u(s), u'(s))) ds \right. \\ & \quad \left. - \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} \phi_q(f(s, u(s), u'(s))) ds \right| \\ & \leq \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \phi_q(\gamma) |u(s)| ds \\ & \quad + \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha-2} \phi_q(\gamma) |u(s)| ds \\ & \leq \frac{\phi_q(\gamma) d}{\Gamma(\alpha-1)} \int_0^{t_1} |(t_2 - s)^{\alpha-2} - (t_1 - s)^{\alpha-2}| ds \\ & \quad + \frac{\phi_q(\gamma) d}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha-2} ds. \end{aligned} \tag{26}$$

Thus, we conclude that F is equicontinuous on B_d , and

$$F : B_d \longrightarrow B_d \text{ is completely continuous.} \tag{27}$$

By Schauder fixed point theorem we know that there exists a solution for the boundary value problem (5). \square

Theorem 6. Let $f : [0, 1] \times R \times R \rightarrow R$ be continuous. Assume that f meets the following condition: there exist $l_1 > 0$, $l_2 > 0$ such that

$$|\phi_q(f(t, u, u')) - \phi_q(f(t, v, v'))| \leq l_1 |u - v| + l_2 |u' - v'|, \tag{28}$$

$$\begin{aligned} & \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha \Gamma(2-\beta)}{|1-a| \Gamma(\alpha-\beta+1)} \right. \\ & \quad + \frac{aT^\alpha}{|1-a| \Gamma(\alpha+1)} + \frac{T^{\alpha-1} \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \\ & \quad \left. + \frac{T^\alpha \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) (l_1 + l_2) < 1. \end{aligned} \tag{29}$$

Then the problem (5) has a unique solution on $[0, 1]$ for any $\lambda \geq 0$.

Proof. From (18) and (19), we have, for $u_1, u_2 \in X$,

$$\begin{aligned} & |Fu_2(t) - Fu_1(t)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\phi_q(f(s, u_2(s), u_2'(s))) \\ & \quad - \phi_q(f(s, u_1(s), u_1'(s)))| ds \end{aligned}$$

$$\begin{aligned}
& + \frac{a}{|(-1)^\theta + a| \Gamma(\alpha)} \\
& \times \int_0^T (T-s)^{\alpha-1} \left| \phi_q \left(f \left(s, u_2(s), u_2'(s) \right) \right. \right. \\
& \quad \left. \left. - \phi_q \left(f \left(s, u_1(s), u_1'(s) \right) \right) \right| ds \\
& + \frac{a\Gamma(2-\beta)}{|(-1)^\theta + a| T^{-\beta} \Gamma(\alpha-\beta)} \\
& \times \int_0^T (T-s)^{\alpha-\beta-1} \left| \phi_q \left(f \left(s, u_2(s), u_2'(s) \right) \right. \right. \\
& \quad \left. \left. - \phi_q \left(f \left(s, u_1(s), u_1'(s) \right) \right) \right| ds \\
& + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \right. \\
& \quad \times \int_0^T (T-s)^{\alpha-\beta-1} \left| \phi_q \left(f \left(s, u_2(s), u_2'(s) \right) \right. \right. \\
& \quad \left. \left. - \phi_q \left(f \left(s, u_1(s), u_1'(s) \right) \right) \right| ds \right] \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[l_1 |u_2(s) - u_1(s)| \right. \\
& \quad \left. + l_2 |u_2'(s) - u_1'(s)| \right] ds \\
& + \frac{a}{|(-1)^\theta + a| \Gamma(\alpha)} \\
& \times \int_0^T (T-s)^{\alpha-1} \left[l_1 |u_2(s) - u_1(s)| \right. \\
& \quad \left. + l_2 |u_2'(s) - u_1'(s)| \right] ds \\
& + \frac{a\Gamma(2-\beta)}{|(-1)^\theta + a| T^{-\beta} \Gamma(\alpha-\beta)} \\
& \times \int_0^T (T-s)^{\alpha-\beta-1} \left[l_1 |u_2(s) - u_1(s)| \right. \\
& \quad \left. + l_2 |u_2'(s) - u_1'(s)| \right] ds \\
& + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \right. \\
& \quad \times \int_0^T (T-s)^{\alpha-\beta-1} \left[l_1 |u_2(s) - u_1(s)| \right. \\
& \quad \left. + l_2 |u_2'(s) - u_1'(s)| \right] ds \left. \right] \\
& \leq \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{a\Gamma(2-\beta)}{|(-1)^\theta + a| T^{-\beta} \Gamma(\alpha-\beta)} \\
& \times \int_0^T (T-s)^{\alpha-\beta-1} ds \\
& + \frac{a}{|(-1)^\theta + a| \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \\
& \left. + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} ds \right] \right) \\
& \times (l_1 + l_2) \|u_2 - u_1\|_1 \\
& \leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha \Gamma(2-\beta)}{|1-a| \Gamma(\alpha-\beta+1)} \right. \\
& \quad \left. + \frac{aT^\alpha}{|1-a| \Gamma(\alpha+1)} + \frac{T^\alpha \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) \\
& \times (l_1 + l_2) \|u_2 - u_1\|_1, \\
& |F'u_2(t) - F'u_1(t)| \\
& \leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left| \phi_q \left(f \left(s, u_2(s), u_2'(s) \right) \right. \right. \\
& \quad \left. \left. - \phi_q \left(f \left(s, u_1(s), u_1'(s) \right) \right) \right| ds \\
& + \frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \\
& \times \int_0^T (T-s)^{\alpha-\beta-1} \left| \phi_q \left(f \left(s, u_2(s), u_2'(s) \right) \right. \right. \\
& \quad \left. \left. - \phi_q \left(f \left(s, u_1(s), u_1'(s) \right) \right) \right| ds \\
& \leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left[l_1 |u_2(s) - u_1(s)| \right. \\
& \quad \left. + l_2 |u_2'(s) - u_1'(s)| \right] ds \\
& + \frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} \left[l_1 |u_2(s) - u_1(s)| \right. \\
& \quad \left. + l_2 |u_2'(s) - u_1'(s)| \right] ds \\
& \leq \left(\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \right. \\
& \quad \left. + \frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} ds \right) \\
& \times (l_1 + l_2) \|u_2 - u_1\|_1 \\
& \leq \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1} \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) (l_1 + l_2) \|u_2 - u_1\|_1.
\end{aligned}$$

Thus,

$$\begin{aligned} \|Fu_2 - Fu_1\|_1 \leq & \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha\Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} \right. \\ & + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \\ & \left. + \frac{T^\alpha\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) (l_1 + l_2) \|u_2 - u_1\|_1. \end{aligned} \tag{31}$$

It follows from (29) that F is a contraction. Thus, the conclusion of the theorem follows from the contraction mapping principle. \square

Theorem 7. Let $1 < p < 2$. Assume that f meets the following condition: there exist $d_1 > 0, d_2 > 0, d > 0$ such that

$$|f(t, u, u')| \leq M, \tag{32}$$

$$\begin{aligned} |f(t, u, u') - f(t, v, v')| \leq & d_1 |u - v| + d_2 |u' - v'|, \\ & \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha\Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} \right. \\ & + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \\ & \left. + \frac{T^\alpha\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) (q-1) M^{(q-2)} (d_1 + d_2) < 1. \end{aligned} \tag{33}$$

Then the problem (5) has unique solution on $[0, 1]$ for

$$\begin{aligned} 0 \leq \lambda \leq & \left(\frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \right)^{-1} \\ & \times \left[d - \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right. \right. \\ & + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} \\ & + \frac{aT^\alpha\Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} \\ & \left. \left. + \frac{T^\alpha\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) \phi_q(M) \right]. \end{aligned} \tag{34}$$

Proof. Let

$$B_d = \{u \mid \|u\|_1 \leq d, u \in X\}, \tag{35}$$

where

$$\begin{aligned} d \geq & \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} \right. \\ & + \frac{aT^\alpha\Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} + \frac{T^\alpha\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \left. \right) \phi_q(M) \\ & + \frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}} \lambda + \frac{\Gamma(2-\beta)}{bT^{-\beta}} \lambda + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda. \end{aligned} \tag{36}$$

By (18) and (19), we have, for $u \in B_d$,

$$\begin{aligned} |Fu(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & + \frac{a}{|(-1)^\theta + a|\Gamma(\alpha)} \\ & \times \int_0^T (T-s)^{\alpha-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & + \frac{a\Gamma(2-\beta)}{|(-1)^\theta + a|T^{-\beta}\Gamma(\alpha-\beta)} \\ & \times \int_0^T (T-s)^{\alpha-\beta-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & + \left| \frac{a\Gamma(2-\beta)}{[1+(-1)^\theta a]bT^{-\beta}} \lambda \right| \\ & + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \right. \\ & \times \int_0^T (T-s)^{\alpha-\beta-1} |\phi_q(f(s, u(s), u'(s)))| ds \\ & \left. + \left| \frac{(-1)^\theta\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right| \right] \\ \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(M) ds \\ & + \frac{a}{|1-a|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \phi_q(M) ds \\ & + \frac{a\Gamma(2-\beta)}{|1-a|T^{-\beta}\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(M) ds \\ & + \frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}} \lambda \\ & + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \right. \\ & \left. \times \int_0^T (T-s)^{\alpha-\beta-1} \phi_q(M) ds + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \lambda \right] \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} \right. \\ &\quad \left. + \frac{aT^\alpha\Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} + \frac{T^\alpha\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) \phi_q(M) \\ &\quad + \frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}}\lambda + \frac{\Gamma(2-\beta)}{bT^{-\beta}}\lambda, \end{aligned} \quad (37)$$

This, together with (36), yields that

$$\|Fu(t)\|_1 \leq d. \quad (38)$$

Hence,

$$F : B_d \longrightarrow B_d. \quad (39)$$

In view of $1 < p < 2$, we have $q > 2$. Thus, by the following property of p-Laplacian operator:

if $p > 2$, $|x|, |y| \leq c$, then $|\varphi_p(x) - \varphi_p(y)| \leq (p-1)c^{(p-2)}|x-y|$; we have, for $u \in B_d$,

$$\begin{aligned} &|Fu_2(t) - Fu_1(t)| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| \phi_q \left(f(s, u_2(s), u_2'(s)) \right. \right. \\ &\quad \left. \left. - \phi_q \left(f(s, u_1(s), u_1'(s)) \right) \right) \right| ds \\ &\quad + \frac{a}{|(-1)^\theta + a|\Gamma(\alpha)} \\ &\quad \times \int_0^T (T-s)^{\alpha-1} \left| \phi_q \left(f(s, u_2(s), u_2'(s)) \right. \right. \\ &\quad \left. \left. - \phi_q \left(f(s, u_1(s), u_1'(s)) \right) \right) \right| ds \\ &\quad + \frac{a\Gamma(2-\beta)}{|(-1)^\theta + a|T^{-\beta}\Gamma(\alpha-\beta)} \\ &\quad \times \int_0^T (T-s)^{\alpha-\beta-1} \left| \phi_q \left(f(s, u_2(s), u_2'(s)) \right. \right. \\ &\quad \left. \left. - \phi_q \left(f(s, u_1(s), u_1'(s)) \right) \right) \right| ds \\ &\quad + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \right. \\ &\quad \left. \times \int_0^T (T-s)^{\alpha-\beta-1} \left| \phi_q \left(f(s, u_2(s), u_2'(s)) \right. \right. \right. \\ &\quad \left. \left. - \phi_q \left(f(s, u_1(s), u_1'(s)) \right) \right) \right] ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{(q-1)M^{(q-2)}}{\Gamma(\alpha)} \\ &\quad \times \int_0^t (t-s)^{\alpha-1} \left| f(s, u_2(s), u_2'(s)) \right. \\ &\quad \left. - f(s, u_1(s), u_1'(s)) \right| ds \\ &\quad + \frac{a(q-1)M^{(q-2)}}{|(-1)^\theta + a|\Gamma(\alpha)} \\ &\quad \times \int_0^T (T-s)^{\alpha-1} \left| f(s, u_2(s), u_2'(s)) \right. \\ &\quad \left. - f(s, u_1(s), u_1'(s)) \right| ds \\ &\quad + \frac{a(q-1)M^{(q-2)}\Gamma(2-\beta)}{|(-1)^\theta + a|T^{-\beta}\Gamma(\alpha-\beta)} \\ &\quad \times \int_0^T (T-s)^{\alpha-\beta-1} \left| f(s, u_2(s), u_2'(s)) \right. \\ &\quad \left. - f(s, u_1(s), u_1'(s)) \right| ds \\ &\quad + t \left[\frac{(q-1)M^{(q-2)}\Gamma(2-\beta)}{T^{1-\beta}\Gamma(\alpha-\beta)} \right. \\ &\quad \left. \times \int_0^T (T-s)^{\alpha-\beta-1} \left| f(s, u_2(s), u_2'(s)) \right. \right. \\ &\quad \left. \left. - f(s, u_1(s), u_1'(s)) \right) \right] ds \\ &\leq \frac{(q-1)M^{(q-2)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[d_1 |u_2(s) - u_1(s)| \right. \\ &\quad \left. + d_2 |u_2'(s) - u_1'(s)| \right] ds \\ &\quad + \frac{a(q-1)M^{(q-2)}}{|(-1)^\theta + a|\Gamma(\alpha)} \\ &\quad \times \int_0^T (T-s)^{\alpha-1} \left[d_1 |u_2(s) - u_1(s)| \right. \\ &\quad \left. + d_2 |u_2'(s) - u_1'(s)| \right] ds \\ &\quad + \frac{a(q-1)M^{(q-2)}\Gamma(2-\beta)}{|(-1)^\theta + a|T^{-\beta}\Gamma(\alpha-\beta)} \\ &\quad \times \int_0^T (T-s)^{\alpha-\beta-1} \left[d_1 |u_2(s) - u_1(s)| \right. \\ &\quad \left. + d_2 |u_2'(s) - u_1'(s)| \right] ds \end{aligned}$$

$$\begin{aligned}
 & + t \left[\frac{(q-1) M^{(q-2)} \Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \right. \\
 & \quad \times \int_0^T (T-s)^{\alpha-\beta-1} \left[d_1 |u_2(s) - u_1(s)| \right. \\
 & \quad \quad \left. \left. + d_2 |u'_2(s) - u'_1(s)| \right] ds \right] \\
 & \leq \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \right. \\
 & \quad + \frac{a\Gamma(2-\beta)}{|(-1)^\theta + a| T^{-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} ds \\
 & \quad + \frac{a}{|(-1)^\theta + \alpha| \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \\
 & \quad \left. + t \left[\frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} ds \right] \right) \\
 & \quad \times (q-1) M^{(q-2)} (d_1 + d_2) \|u_2 - u_1\|_1 \\
 & \leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha \Gamma(2-\beta)}{|1-a| \Gamma(\alpha-\beta+1)} \right. \\
 & \quad \left. + \frac{aT^\alpha}{|1-a| \Gamma(\alpha+1)} + \frac{T^\alpha \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) \\
 & \quad \times (q-1) M^{(q-2)} (d_1 + d_2) \|u_2 - u_1\|_1, \\
 & |F'u_2(t) - F'u_1(t)| \\
 & \leq \frac{1}{\Gamma(\alpha-1)} \\
 & \quad \times \int_0^t (t-s)^{\alpha-2} \left| \phi_q \left(f(s, u_2(s), u'_2(s)) \right) \right. \\
 & \quad \quad \left. - \phi_q \left(f(s, u_1(s), u'_1(s)) \right) \right| ds \\
 & \quad + \frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \\
 & \quad \times \int_0^T (T-s)^{\alpha-\beta-1} \left| \phi_q \left(f(s, u_2(s), u'_2(s)) \right) \right. \\
 & \quad \quad \left. - \phi_q \left(f(s, u_1(s), u'_1(s)) \right) \right| ds \\
 & \leq \frac{(q-1) M^{(q-2)}}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left| f(s, u_2(s), u'_2(s)) \right. \\
 & \quad \quad \left. - f(s, u_1(s), u'_1(s)) \right| ds \\
 & \quad + \frac{(q-1) M^{(q-2)} \Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)}
 \end{aligned}$$

$$\begin{aligned}
 & \times \int_0^T (T-s)^{\alpha-\beta-1} \left| f(s, u_2(s), u'_2(s)) \right. \\
 & \quad \quad \left. - f(s, u_1(s), u'_1(s)) \right| ds \\
 & \leq \frac{(q-1) M^{(q-2)}}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left[d_1 |u_2(s) - u_1(s)| \right. \\
 & \quad \quad \left. + d_2 |u'_2(s) - u'_1(s)| \right] ds \\
 & \quad + \frac{(q-1) M^{(q-2)} \Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \\
 & \quad \times \int_0^T (T-s)^{\alpha-\beta-1} \left[d_1 |u_2(s) - u_1(s)| \right. \\
 & \quad \quad \left. + d_2 |u'_2(s) - u'_1(s)| \right] ds \\
 & \leq \left(\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \right. \\
 & \quad \left. + \frac{\Gamma(2-\beta)}{T^{1-\beta} \Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} ds \right) \\
 & \quad \times (q-1) M^{(q-2)} (d_1 + d_2) \|u_2 - u_1\|_1 \\
 & \leq \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1} \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) \\
 & \quad \times (q-1) M^{(q-2)} (d_1 + d_2) \|u_2 - u_1\|_1. \tag{40}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & \|Fu_2 - Fu_1\|_1 \\
 & \leq \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha \Gamma(2-\beta)}{|1-a| \Gamma(\alpha-\beta+1)} \right. \\
 & \quad \left. + \frac{aT^\alpha}{|1-a| \Gamma(\alpha+1)} + \frac{T^{\alpha-1} \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) \\
 & \quad \times (q-1) M^{(q-2)} (d_1 + d_2) \|u_2 - u_1\|_1. \tag{41}
 \end{aligned}$$

It follows from (33) that F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. \square

4. Examples

Example 8. Consider the following boundary value problem:

$$\phi_2 \left({}^c D_{0+}^{3/2} u(t) \right) = f(t, u(t), u'(t)), \quad 0 < t < 1,$$

$$\begin{aligned}
 u(0) + (-1)^\theta 2u(1) &= 0, \\
 {}^c D_{0+}^{1/2} u(0) + (-1)^\theta \frac{3\sqrt{\pi}}{2} {}^c D_{0+}^{1/2} u(1) &= \lambda, \quad \theta = 0, 1,
 \end{aligned} \tag{42}$$

where

$$\begin{aligned}
 T = 1, \quad \alpha = \frac{3}{2}, \quad \beta = \frac{1}{2}, \\
 a = 2, \quad b = \frac{3\sqrt{\pi}}{2}.
 \end{aligned} \tag{43}$$

Let

$$\begin{aligned}
 d = 2, \quad p = 2, \\
 f(t, u(t), u'(t)) = \frac{t}{100} \frac{u}{5 + |u| + |u'|}.
 \end{aligned} \tag{44}$$

By computation, we deduce that

$$\begin{aligned}
 0 \leq \lambda &\leq \frac{d}{2} \left(\frac{a\Gamma(2-\beta)}{|1-a|b\Gamma^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \right)^{-1} \\
 &= \left(\frac{2\Gamma(3/2)}{(3\sqrt{\pi}/2)} + \frac{\Gamma(3/2)}{(3\sqrt{\pi}/2)} + \frac{\Gamma(3/2)}{(3\sqrt{\pi}/2)} \right)^{-1} \\
 &= \left(\frac{2(\sqrt{\pi}/2)}{(3\sqrt{\pi}/2)} + \frac{\sqrt{\pi}/2}{(3\sqrt{\pi}/2)} + \frac{\sqrt{\pi}/2}{(3\sqrt{\pi}/2)} \right)^{-1} = \frac{3}{4}, \\
 \phi_2(\gamma) = \gamma &\leq \frac{1}{2} \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} \right. \\
 &\quad \left. + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} + \frac{aT^\alpha\Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} \right. \\
 &\quad \left. + \frac{T^\alpha\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right)^{-1} \\
 &= \frac{1}{2} \left(\frac{1}{\Gamma(3/2)} + \frac{1}{\Gamma(5/2)} + \frac{2}{\Gamma(5/2)} + \frac{2\Gamma(3/2)}{\Gamma(2)} \right. \\
 &\quad \left. + \frac{\Gamma(3/2)}{\Gamma(2)} + \frac{\Gamma(3/2)}{\Gamma(2)} \right)^{-1} \\
 &= \frac{1}{2} \left(\frac{18+6\pi}{3\sqrt{\pi}} \right)^{-1} = \frac{3\sqrt{\pi}}{36+12\pi} < \frac{1}{5}.
 \end{aligned} \tag{45}$$

Thus, let $\gamma = 1/9$; we have

$$f(t, u(t), u'(t)) = \frac{t}{100} \frac{u}{5 + |u| + |u'|} < \gamma \phi_2(|u|) = \frac{1}{9} |u|. \tag{46}$$

Hence, by Theorem 5, BVP (42) has at least one solution for $0 \leq \lambda \leq 3/4$.

Example 9. Consider the following boundary value problem:

$$\begin{aligned}
 \phi_2({}^c D_{0+}^{3/2} u(t)) &= f(t, u(t), u'(t)), \quad 0 < t < 1, \\
 u(0) + (-1)^\theta 2u(1) &= 0, \\
 {}^c D_{0+}^{1/2} u(0) + (-1)^\theta \sqrt{\pi} {}^c D_{0+}^{1/2} u(1) &= \lambda, \quad \theta = 0, 1,
 \end{aligned} \tag{47}$$

where

$$\begin{aligned}
 T = 1, \quad \alpha = \frac{3}{2}, \quad \beta = \frac{1}{2}, \\
 a = 2, \quad b = \sqrt{\pi}, \quad \lambda \geq 0.
 \end{aligned} \tag{48}$$

Let

$$\begin{aligned}
 p = 2, \\
 f(t, u(t), u'(t)) \\
 = \frac{1}{(t + \cos t + 10)^2} \left(\frac{|u(t) + u'(t)|}{1 + |u(t) + u'(t)|} + 10(1 + \sin t) \right).
 \end{aligned} \tag{49}$$

By computation, we deduce that

$$\begin{aligned}
 |f(t, u(t), u'(t)) - f(t, v(t), v'(t))| \\
 \leq \frac{1}{100} (|u(t) - v(t)| + |u'(t) - v'(t)|).
 \end{aligned} \tag{50}$$

Let

$$l_1 = l_2 = \frac{1}{100}. \tag{51}$$

Thus,

$$\begin{aligned}
 &\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha\Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} \right. \\
 &\quad \left. + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right. \\
 &\quad \left. + \frac{T^\alpha\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) (l_1 + l_2) \\
 &= \left(\frac{1}{\Gamma(3/2)} + \frac{1}{\Gamma(5/2)} + \frac{2\Gamma(3/2)}{\Gamma(2)} + \frac{2}{\Gamma(5/2)} \right. \\
 &\quad \left. + \frac{\Gamma(3/2)}{\Gamma(2)} + \frac{\Gamma(3/2)}{\Gamma(2)} \right) \left(\frac{1}{100} + \frac{1}{100} \right) \\
 &= \left(\frac{1}{(\sqrt{\pi}/2)} + \frac{1}{(3\sqrt{\pi}/4)} + \frac{2(\sqrt{\pi}/2)}{1} + \frac{2}{(3\sqrt{\pi}/4)} \right. \\
 &\quad \left. + \frac{(\sqrt{\pi}/2)}{1} + \frac{(\sqrt{\pi}/2)}{1} \right) \left(\frac{1}{100} + \frac{1}{100} \right) \\
 &= \frac{3+\pi}{25\sqrt{\pi}} < 1.
 \end{aligned} \tag{52}$$

Hence, by Theorem 6, BVP (47) has a unique solution.

Example 10. Consider the following boundary value problem:

$$\begin{aligned} \phi_{4/3}({}^c D_{0+}^{3/2} u(t)) &= f(t, u(t), u'(t)), \quad 0 < t < 1, \\ u(0) + (-1)^\theta 2u(1) &= 0, \end{aligned} \tag{53}$$

$${}^c D_{0+}^{1/2} u(0) + (-1)^\theta \sqrt{\pi} {}^c D_{0+}^{1/2} u(1) = \lambda, \quad \theta = 0, 1,$$

where

$$\begin{aligned} p &= \frac{4}{3}, & T &= 1, & \alpha &= \frac{3}{2}, \\ \beta &= \frac{1}{2}, & a &= 2, & b &= \sqrt{\pi}. \end{aligned} \tag{54}$$

Let

$$\begin{aligned} f(t, u(t), u'(t)) &= \frac{1}{(t + \cos t + 10)^2} \left(\frac{|u(t) + u'(t)|}{1 + |u(t) + u'(t)|} + 10(1 + \sin t) \right). \end{aligned} \tag{55}$$

Thus,

$$\begin{aligned} q &= 4, \\ |f(t, u(t), u'(t))| &\leq \frac{1}{100} (1 + 12) = \frac{13}{100} = M, \\ (q - 1)M^{(q-2)} &= (4 - 1)M^{4-2} = 3\left(\frac{13}{100}\right)^2 < 1. \end{aligned} \tag{56}$$

By Example 9, we know that

$$\begin{aligned} |f(t, u(t), u'(t)) - f(t, v(t), v'(t))| &\leq \frac{1}{100} (|u(t) - v(t)| + |u'(t) - v'(t)|). \end{aligned} \tag{57}$$

Let

$$d_1 = d_2 = \frac{1}{100}. \tag{58}$$

It follows from Example 9 that

$$\begin{aligned} \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha \Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} \right. \\ \left. + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right. \\ \left. + \frac{T^\alpha \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) (d_1 + d_2) < 1. \end{aligned} \tag{59}$$

On the other hand, we have

$$\begin{aligned} \left(\frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \right)^{-1} \\ = \left(\frac{2\Gamma(3/2)}{\sqrt{\pi}} + \frac{\Gamma(3/2)}{\sqrt{\pi}} + \frac{\Gamma(3/2)}{\sqrt{\pi}} \right)^{-1} = \frac{1}{2}. \end{aligned} \tag{60}$$

Let

$$d = 2 \times \frac{3 + \pi}{25\sqrt{\pi}}. \tag{61}$$

Thus,

$$\begin{aligned} \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{aT^\alpha \Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} \right. \\ \left. + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right. \\ \left. + \frac{T^\alpha \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) (q-1)M^{(q-2)}(d_1 + d_2) < 1, \end{aligned}$$

$0 \leq \lambda$

$$\begin{aligned} \leq \left(\frac{a\Gamma(2-\beta)}{|1-a|bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{-\beta}} + \frac{\Gamma(2-\beta)}{bT^{1-\beta}} \right)^{-1} \\ \times \left[d - \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} \right. \right. \\ \left. \left. + \frac{aT^\alpha}{|1-a|\Gamma(\alpha+1)} + \frac{aT^\alpha \Gamma(2-\beta)}{|1-a|\Gamma(\alpha-\beta+1)} \right. \right. \\ \left. \left. + \frac{T^\alpha \Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right) \phi_q(M) \right] \\ = \frac{1}{2} \left[2 \times \frac{3 + \pi}{25\sqrt{\pi}} - \frac{3 + \pi}{25\sqrt{\pi}} \times \left(\frac{13}{100} \right)^3 \right] \\ = \frac{3 + \pi}{50\sqrt{\pi}} \left[2 - \left(\frac{13}{100} \right)^3 \right]. \end{aligned} \tag{62}$$

Hence, by Theorem 7, BVP (53) has a unique solution for $0 \leq \lambda \leq ((3 + \pi)/50\sqrt{\pi})[2 - (13/100)^3]$.

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