Research Article

The Static Stress Intensity Factor around the Antiplane Crack in an Infinite FGM Strip

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The problem of the static stress intensity factor around antiplane crack in an infinite strip functionally graded material was studied by using the method of integral transform-dual integral equations in this paper. The shear modulus in the two principal directions of the functionally graded material was assumed to vary proportionately as gradient model of double parameters index function. The partial differential equation was first reduced to Euler equation with Fourier cosine transform. By solving dual integral equations that were derived by applying the solution of Euler equation with the method of Copson, stress intensity factor around the crack tip was derived. And the variation curves of the dimensionless stress intensity factor with the strip height, crack length, gradient parameter, and inhomogeneous coefficient are obtained by using the numerical calculation.

1. Introduction

Functionally gradient material is a new type of materials. The advent of functionally gradient material is in order to meet performance requirements of parts in all kinds of special environment. The sudden changes of the material composition and properties in a member often lead to obvious local stress concentration. If a material transition to another kind of material is the step by step, the stress concentration will greatly reduce. A new type of functional gradient materials (FGM) was developed in order to reduce the stress concentration of the material and improve the performance requirements of the material as discussed elsewhere [1–5]. Most previous studies about FGM in the past were focused on isotropic material as discussed elsewhere [6, 7]. However, a few FGMs were isotropic material. At present, the application of FGM has been expanded to space, energy, traffic, optics, chemistry, biomedical engineering, and so forth. However, due to the influence of the process conditions, FGM will produce some defects in manufacturing process. These defects will inevitably form crack due to stress action and crack expansion will affect the whole member. So, the research about crack and the crack expansion of FGM are very important as discussed elsewhere [8–11].

In the fracture mechanics study of FGM, most scholars use the power functions model as a material property parameter model. The model of $\mu(z) = \mu_0(1 + c|z|)$ as discussed by Smith [1], the model of $\mu(z) = \mu_0(1 + \alpha|z|)^2$ as discussed by Smith [12], and the model of $\mu(z) = \mu_0(1 + \alpha|z|)^k$ as discussed by Smith [13]. However, the models of double parameters index function were adopted in this paper.

Usually the method of studies about stress intensity factor of antiplane crack in FGM is integral transforms-dual-integral equations. At present, most scholars put the partial differential equation into the Bessel equation with Fourier cosine transform in a designed material property parameter model. Then dual integral equations were obtained by using the solution of Bessel equation. But the difference between this paper and above references is that the partial differential equation is first reduced to Euler equation with Fourier cosine transform, and then dual integral equations were obtained by using the solution of Euler equation. Finally the local stress field and stress intensity factor around the crack tip were derived. This method is very few in the study of FGM.
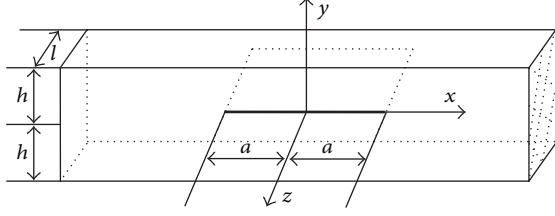
FIGURE 1: The model of crack in infinite strip functionally graded material.

2. Formulation of the Problem

Consider the antiplane problem of an infinite strip functionally graded material containing a crack of length $2a$, and the thickness is $2h$, as shown in Figure 1. The x-axis and y-axis are the elastic principal axis.

Shear moduli μ_x, μ_y are the functions of y, and μ_x, μ_y vary according to the following functions:

$$\mu_x(y) = (\mu_x)_0 \exp(2k\alpha y),$$
$$\mu_y(y) = (\mu_y)_0 \exp(2k\alpha y), \tag{1}$$

where parameter $\alpha > 0$, $k > 0$, the shear modulus $(\mu_x)_0$ and $(\mu_y)_0$, can be determined as $y = 0$.

Suppose that u_x, u_y, and u_z are displacement components of the direction x, y and z. For the antiplane shear deformation, u_x, u_y are 0 everywhere.

Assume that $u_z = \omega(x, y)$ and $\omega(x, y)$ is the displacement function. The relationship between the displacement function $\omega(x, y)$ and stress components τ_{xz} and τ_{yz} is written as

$$\tau_{xz} = \mu_x \frac{\partial \omega}{\partial x}, \qquad \tau_{yz} = \mu_y \frac{\partial \omega}{\partial y}. \tag{2}$$

The stress equilibrium equation can be expressed as

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0. \tag{3}$$

We can obtain the governing equation [14–16] of antiplane problem in infinite strip functionally graded materials by substituting stress components (2) into (3):

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\mu_y}{\mu_x} \frac{\partial^2 \omega}{\partial y^2} + \frac{\mu_y'}{\mu_x} \frac{\partial \omega}{\partial y} = 0. \tag{4}$$

The corresponding boundary conditions of the crack surface under antiplane shear loading τ_0 are as follows:

$$\omega(x, 0) = 0, \quad |x| > a, \tag{5}$$

$$\tau_{yz}(x, 0) = -\tau_0, \quad |x| \le a, \tag{6}$$

$$\tau_{xz}(x, \pm h) = 0, \quad |x| < \infty. \tag{7}$$

Therefore, an analysis of antiplane problem in infinite strip functionally graded materials is to solve the boundary value problem of partial differential equation (4)–(7).

3. Dual Integral Equation

Due to the symmetry, we only need to study the part of $x > 0$, $y > 0$. Introducing the following Fourier cosine transform:

$$\tilde{f}(s) = \int_0^\infty f(x) \cos(sx) \, dx,$$
$$f(x) = \frac{2}{\pi} \int_0^\infty \tilde{f}(s) \cos(sx) \, ds, \tag{8}$$

where $\tilde{f}(s)$ is called Fourier cosine transform of $f(x)$, and $f(x)$ is called Fourier cosine inverse transform of $\tilde{f}(s)$.

Define

$$\omega(x, y) = \frac{2}{\pi} \int_0^\infty U(s, y) \cos(sx) \, ds. \tag{9}$$

Among them, $U(s, y)$ is the Fourier integral cosine transformation function of displacement function $\omega(x, y)$.

By using the Fourier integral transformation, (4) can be converted into as follows:

$$\frac{\partial^2 U(s, y)}{\partial y^2} + 2k\alpha \frac{\partial U(s, y)}{\partial y} - S^2 U(s, y) = 0, \tag{10}$$

where $S = sr$, and inhomogenous coefficient $r = \sqrt{(\mu_x)_0 / (\mu_y)_0}$.

Choosing proper variable substitutions as follows:

$$Y = (\exp(\alpha y))^{(k+1)/2} U, \qquad X = S \exp(\alpha y). \tag{11}$$

So

$$\frac{\partial U}{\partial y} = S^{(k+1)/2} X^{-(k+1)/2} \frac{dY}{dy}$$
$$\qquad - \frac{k+1}{2} \alpha S^{(k+3)/2} X^{-(k+3)/2} \exp(\alpha y) Y,$$

$$\frac{\partial^2 U}{\partial y^2} = S^{(k+1)/2} X^{-(k+1)/2} \frac{d^2 Y}{dy^2}$$
$$\qquad - (k+1) \alpha S^{(k+3)/2} X^{-(k+3)/2} \exp(\alpha y) \frac{dY}{dy}$$
$$\qquad + \frac{k+1}{2} \left(\frac{k+3}{2} S^{(k+5)/2} X^{-(k+5)/2} \alpha^2 \exp(2\alpha y) \right.$$
$$\qquad \left. - S^{(k+3)/2} X^{-(k+3)/2} \alpha^2 \exp(\alpha y) \right) Y,$$

$$\frac{dY}{dy} = S\alpha \exp(\alpha y) \frac{dY}{dX},$$

$$\frac{d^2 Y}{dy^2} = S\alpha^2 \exp(\alpha y) \frac{dY}{dX} + S^2 \alpha^2 \exp(2\alpha y) \frac{d^2 Y}{dX^2}. \tag{12}$$

By substituting (12) into (10), partial differential equation (10) can be translated into Euler equation as follows:

$$X^2 \frac{d^2 Y}{dX^2} + kX \frac{dY}{dX} + \beta^2 Y = 0, \tag{13}$$

where

$$\beta = \sqrt{\frac{(1+k)(1-3k)}{4} - \frac{S^2}{\alpha^2}}. \tag{14}$$

Introducing transformations of the independent variable

$$X = \exp(t), \quad t = \ln(X). \tag{15}$$

So, (13) can be transformed into the following formula:

$$\frac{d^2Y}{dt^2} + (k-1)\frac{dY}{dt} + \beta^2 Y = 0. \tag{16}$$

The general solution of Euler equation (16) can be written as

$$Y = C_1 \exp\left(\frac{(1-k) + \sqrt{(k-1)^2 - 4\beta^2}}{2} \cdot t\right)$$

$$+ C_2 \exp\left(\frac{(1-k) - \sqrt{(k-1)^2 - 4\beta^2}}{2} \cdot t\right) \tag{17}$$

$$= C_1 X^{((1-k)+\sqrt{(k-1)^2 - 4\beta^2})/2} + C_2 X^{((1-k)-\sqrt{(k-1)^2 - 4\beta^2})/2},$$

where C_1 and C_2 are the coefficients of solution.

Substituting (17) into (11), we can obtain the formula of solution as follows:

$$U(s, y) = A_1(s) \exp\left[\alpha\left(k_1 - \frac{k+1}{2}\right)y\right]$$

$$+ A_2(s) \exp\left[\alpha\left(k_2 - \frac{k+1}{2}\right)y\right], \tag{18}$$

where

$$A_1(s) = C_1 S^{((1-k)+\sqrt{(k-1)^2 - 4\beta^2})/2},$$

$$A_2(s) = C_2 S^{((1-k)-\sqrt{(k-1)^2 - 4\beta^2})/2},$$

$$k_1 = \frac{(1-k) + \sqrt{(k-1)^2 - 4\beta^2}}{2}, \tag{19}$$

$$k_2 = \frac{(1-k) - \sqrt{(k-1)^2 - 4\beta^2}}{2}.$$

From (9) and (18), the integral expression of displacement function was obtained as follow:

$$\omega(x, y)$$

$$= \frac{2}{\pi} \int_0^\infty \left\{ A_1(s) \exp\left[\alpha\left(k_1 - \frac{k+1}{2}\right)y\right] \right.$$

$$\left. + A_2(s) \exp\left[\alpha\left(k_2 - \frac{k+1}{2}\right)y\right] \right\} \cos(sx)\, ds. \tag{20}$$

The shear stresses τ_{xz} and τ_{yz} can be obtained from (20) and the relationship of stress-displacement (2). The stress formulas are listed as follows:

$$\tau_{yz}(x, y)$$

$$= \mu_y \frac{\partial \omega}{\partial y} = \frac{2}{\pi}\mu_y$$

$$\cdot \int_0^\infty \left\{ \alpha\left(k_1 - \frac{k+1}{2}\right) A_1(s) \right.$$

$$\cdot \exp\left[\alpha\left(k_1 - \frac{k+1}{2}\right)y\right]$$

$$+ \alpha\left(k_2 - \frac{k+1}{2}\right) \cdot A_2(s)$$

$$\left. \cdot \exp\left[\alpha\left(k_2 - \frac{k+1}{2}\right)y\right] \right\} \cos(sx)\, ds, \tag{21}$$

$$\tau_{xz}(x, y)$$

$$= \mu_x \frac{\partial \omega}{\partial x} = -\frac{2}{\pi}\mu_x$$

$$\cdot \int_0^\infty s \left\{ A_1(s) \exp\left[\alpha\left(k_1 - \frac{k+1}{2}\right)y\right] \right. \tag{22}$$

$$\left. + A_2(s) \exp\left[\alpha\left(k_2 - \frac{k+1}{2}\right)y\right] \right\}$$

$$\times \sin(sx)\, ds.$$

We can obtain the relation between $A_1(s)$ and $A_2(s)$ by substituting (22) into the boundary conditions (7) as follows:

$$A_1(s) = -A_2(s) \exp\left[-2\alpha h \sqrt{k^2 + \frac{s^2 r^2}{\alpha^2}}\right]. \tag{23}$$

Let

$$A_1(s) = -\frac{\exp\left[-2\alpha h \sqrt{k^2 + s^2 r^2/\alpha^2}\right]}{1 - \exp\left[-2\alpha h \sqrt{k^2 + s^2 r^2/\alpha^2}\right]} A(s),$$

$$A_2(s) = \frac{1}{1 - \exp\left[-2\alpha h \sqrt{k^2 + s^2 r^2/\alpha^2}\right]} A(s). \tag{24}$$

From (20), (21), (24), and boundary conditions (5) and (6), we can obtain the dual integral equations:

$$\int_0^\infty A(s) \cos(sx)\, ds = 0, \quad x > a, \tag{25a}$$

$$\int_0^\infty s A(s) G(s) \cos(sx)\, ds = \frac{\pi \tau_0}{2r(\mu_y)_0}, \quad 0 < x \le a, \tag{25b}$$

in which,

$$G(s) = \left(-\alpha \left(k_2 - \frac{k+1}{2} \right) + \alpha \left(k_1 - \frac{k+1}{2} \right) \right.$$

$$\times \exp \left[-2\alpha h \sqrt{k^2 + \frac{s^2 r^2}{\alpha^2}} \right] \right)$$ (26)

$$\times \left(sr \left(1 - \exp \left[-2\alpha h \sqrt{k^2 + \frac{s^2 r^2}{\alpha^2}} \right] \right) \right)^{-1}.$$

Solving the dual integral equations (25a) and (25b) by using Copson-Sih method [17], the solution can be expressed as

$$A(s) = \frac{\pi \tau_0 a^2}{2r(\mu_y)_0} \int_0^1 \sqrt{\xi} \Phi(\xi) J_0(sa\xi) d\xi,$$ (27)

where $J_0(\cdot)$ is the so-called zero Bessel function of the first kind. The function $\psi(\xi)$ can be governed by the Fredholm integral equation of the second kind:

$$\Phi(\xi) + \int_0^1 \Phi(\eta) M(\xi, \eta) d\eta = \sqrt{\xi},$$ (28)

where kernel function $M(\xi, \eta)$ is

$$M(\xi, \eta) = \sqrt{\xi \eta} \int_0^\infty s \left[G\left(\frac{s}{a} \right) - 1 \right] J_0(s\xi) J_0(s\eta) ds.$$ (29)

By using numerical integral method, the numerical solution of $\Phi(1)$ can be given from (28) and (29).

4. The Stress Intensity Factor

By using of the formula of integration by parts, formula (27) can be written as

$$A(s) = \frac{\pi \tau_0}{2r(\mu_y)_0} \frac{a}{s}$$

$$\times \left[\Phi(1) J_1(sa) - \int_0^1 \xi J_1(sa\xi) \frac{d\left(\xi^{-1/2}\Phi(\xi)\right)}{d\xi} d\xi \right].$$ (30)

We can obtain the stress principle by substituting (30) into (21)–(24) as follows:

$$\tau_{yz}(x, 0) = \frac{\tau_0 a \Phi(1)}{r} \int_0^\infty H(s) J_1(sa) \cos(sx) ds,$$ (31)

$$\tau_{xz}(x, 0) = -r\tau_0 a \Phi(1) \int_0^\infty J_1(sa) \sin(sx) ds.$$ (32)

In formula (31),

$$H(s) = \frac{-\alpha \left(k_1 - (k+1)/2 \right) \exp \left(-2\alpha h \sqrt{k^2 + s^2 r^2/\alpha^2} \right)}{s \left[1 - \exp \left(-2\alpha h \sqrt{k^2 + s^2 r^2/\alpha^2} \right) \right]}$$

$$+ \frac{\alpha \left(k_2 - (k+1)/2 \right)}{s \left[1 - \exp \left(-2\alpha h \sqrt{k^2 + s^2 r^2/\alpha^2} \right) \right]}.$$ (33)

It is noticed that

$$H(s) = -r, \quad \text{as } s \longrightarrow \infty.$$ (34)

Considering the stress singularity around crack tip as $s \to \infty$, the stress principle can be written as

$$\tau_{yz}(x, 0) = -\tau_0 a \Phi(1) \int_0^\infty J_1(sa) \cos(sx) ds,$$ (35)

$$\tau_{xz}(x, 0) = -r\tau_0 a \Phi(1) \int_0^\infty J_1(sa) \sin(sx) ds.$$ (36)

Introducing the complex transformations as follows:

$$z = x + iy = re^{i\theta}, \quad z - a = r_1 e^{i\theta_1}, \quad z + a = r_2 e^{i\theta_2}.$$ (37)

For

$$\int_0^\infty J_1(sa) \cos(sx) ds = -\frac{x}{a\sqrt{(x-a)(x+a)}}.$$ (38)

The formula (35) can be simplified to

$$\tau_{yz}(x, 0) = \frac{\tau_0 \Phi(1) x}{\sqrt{(x-a)(x+a)}}.$$ (39)

Defining the stress intensity factor as follows:

$$K_{\mathrm{III}} = \lim_{x \to a^+} \sqrt{2\pi(x-a)} \tau_{yz}(x, 0).$$ (40)

So, we can obtain

$$K_{\mathrm{III}} = \tau_0 \sqrt{\pi a} \Phi(1).$$ (41)

Formula (41) can be written as

$$\Phi(1) = \frac{K_{\mathrm{III}}}{\tau_0 \sqrt{\pi a}},$$ (42)

where $\Phi(1)$ is called dimensionless stress intensity factor.

The variation curves of the dimensionless stress intensity factor $\Phi(1)$ with the strip height h, crack length a, gradient parameter are k and α, and inhomogenous coefficient r are obtained by using the numerical calculation as shown in Figures 2–4.

Figure 2 shows the variation curves of the dimensionless stress intensity factor with strip height and crack length under the same condition of inhomogeneous coefficient and gradient parameter. It is noticed that the longer crack and the thinner strip lead to the greater the stress intensity factor.
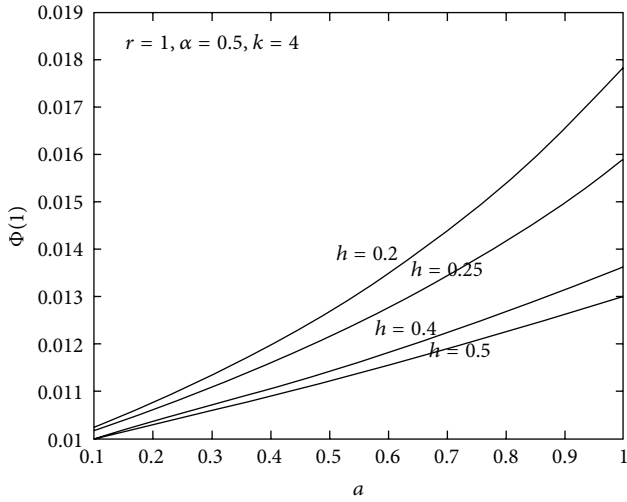
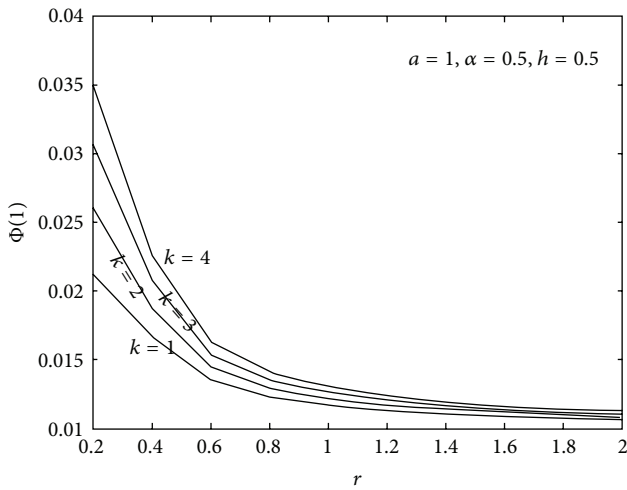FIGURE 2: The effect of strip height h and crack length a on the stress intensity factor $\Phi(1)$.

FIGURE 4: The effect of gradient parameter α and crack length a on the stress intensity factor $\Phi(1)$.

FIGURE 3: The effect of gradient parameter k and inhomogenous coefficient r on the stress intensity factor $\Phi(1)$.

5. Conclusions

(1) There is $r^{-1/2}$ singularity in crack tip of the orthotropic functionally graded material. It is the same as the orthotropic homogeneous material. Thus, we can study the complex fracture problem of the orthotropic functionally graded material in using the conventional method fracture mechanics.

(2) Figure 2 shows the stress intensity factor $\Phi(1)$ increase with the increasing of crack length a in the condition of fixed strip height h. The stress intensity factor $\Phi(1)$ decreases with the increasing of strip height h in the condition of fixed crack length a.

(3) Figure 3 shows the stress intensity factor $\Phi(1)$ decrease with the increasing of inhomogenous coefficient r in the condition of fixed gradient parameter k, The stress intensity factor $\Phi(1)$ increases with the increasing of gradient parameter k in the condition of fixed inhomogenous coefficient r.

(4) Figure 4 shows the stress intensity factor $\Phi(1)$ increase with the increasing of gradient parameter α in the condition of fixed crack length a. The stress intensity factor $\Phi(1)$ increases with the increasing of crack length a in the condition of fixed gradient parameter α.

Figure 3 shows the variation curves of the dimensionless stress intensity factor with inhomogeneous coefficient and gradient parameter k under the same condition of strip height, crack length, and gradient parameter α. It is noticed that the greater gradient parameter k and the smaller inhomogenous coefficient lead to the greater stress intensity factor.

Figure 4 shows the variation curves of the dimensionless stress intensity factor with crack length and gradient parameter α under the same condition of strip height, inhomogenous coefficient, and gradient parameter k. It is noticed that the greater gradient parameter α and the longer crack lead to the greater of the stress intensity factor.
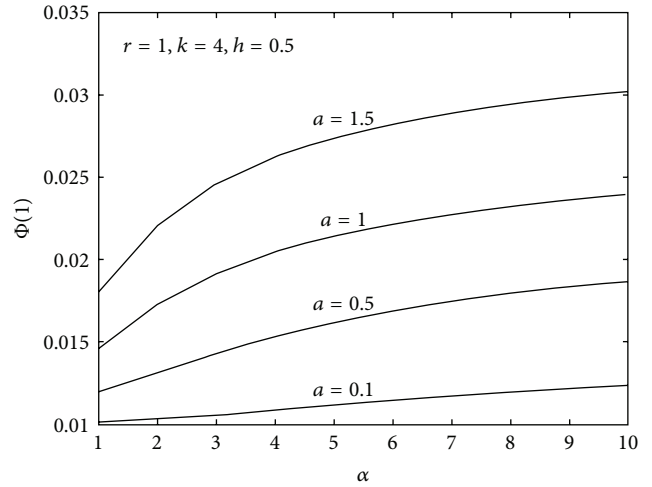
Acknowledgments

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