

Research Article

p -Moment Stability of Stochastic Differential Delay Systems with Impulsive Jump and Markovian Switching

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This paper investigates p -moment stability of the stochastic differential delay systems with impulsive jump and Markovian switching. Some stability criteria are obtained based on Lyapunov functional method and stochastic theory. It is shown that, even if all the subsystems governing the continuous dynamics without impulse are not stable, as impulsive and switching signal satisfies a dwell-time upper bound condition, impulses can stabilize the systems in the p -moment stability sense. The opposite situation is also developed for which all the subsystems governing the continuous dynamics are p -moment stable. The results can be easily applied to stochastic systems with arbitrarily large delays. The efficiency of the proposed results is illustrated by two numerical examples.

1. Introduction

Stochastic differential systems which include stochastic delay differential systems have attracted much attention, owing to stochastic modeling having played an important role in many ways such as science and engineering and forecast of the growth of population [1, 2]. Stability analysis of different stochastic systems has been a subject of intense activities in the literature [3–9]. Switched systems are an important class of hybrid dynamical systems which are composed of a family of continuous time or discrete-time dynamical systems and a rule that orchestrates the switching among them [10]. A particular class of switched systems is named as Markovian switched systems, whose system mode is governed by a Markov process. During the past few decades, many issues on Markovian switched systems such as stability and stabilization [11–14], H_∞ control and filtering [15], and adaptive control problem [16, 17] have been well investigated.

Beside stochastic effects and Markovian switching, impulsive effect likewise exists in many evolution processes in which systems states change abruptly at certain moments of time, involving such fields as biology, engineering, and information science [18]. Therefore, the stability investigation of stochastic differential systems with impulsive jump is interesting to many investigators. The p -moment stability

of stochastic differential systems is studied in [19] for the systems with impulsive jump, nonswitched, and no time delay. Thus, the results in [19] cannot be easily applied to the class of impulsive systems with Markovian switching and time-varying delay. Liu and Peng [20] discussed the p -moment stability of stochastic differential delay systems with impulsive jump and Markovian switching. It is shown that when the delayed continuous dynamics are p -moment stable, the stability properties are not destroyed by impulse at discrete instants irrespective of the length of delay. It is also noticed that the conclusions received in [20] are restricted to the case that all the subsystems that govern the continuous dynamics are stable in p -moment sense. There is no attention has been paid to the class of hybrid systems in which all the subsystems that govern the continuous dynamics are not stable. Thus, how to establish a sufficient condition of p -moment stability for the class of stochastic delay hybrid systems is the key problem to be solved in future research.

Based on the above analysis, in this work, the p -moment stability of stochastic differential delay systems with impulsive jump and Markovian switching is investigated. Motivated by the work of [20], we first relax the global Lipschitz condition of impulsive control law $\Delta x(t_k)$. Then, we will propose some conditions of the p -moment stability for two classes of stochastic hybrid delay systems, that is, all the

subsystems that govern the continuous dynamics are stable and not. Compared with [20], the criteria for the stable case in this paper are more general. Further, the results are applied to linear stochastic delay systems with arbitrarily large delays.

2. Preliminaries

Throughout this paper, unless otherwise specified, we let $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions, that is, it is right continuous and F_0 contains all P -null sets. $\omega(t)$ is an m -dimensional Brownian motion defined on $(\Omega, F, \{F_t\}_{t \geq 0}, P)$. Let Z^+ define the set of nonnegative real numbers and R^n the n -dimensional real Euclidean space. $|\cdot|$ denotes the Euclidean norm for vectors or the spectral norm for matrices. For $r > 0$, let $PC([-r, 0], R^n)$ denote the class of functions from $[-r, 0]$ to R^n satisfying the following: (i) it has at most a finite number of jump discontinuities on $(-r, 0]$; (ii) it is continuous from the right at all points in $[-r, 0]$. For simplicity, PC is used for $PC([-r, 0], R^n)$ for the rest of this paper. For function $\phi : [-r, 0] \rightarrow R^n$, a norm is defined as $\|\phi\|_r = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$. Given $x \in PC([-r, \infty], R^n)$ and for each $t \in R^+$, define $x_t, x_{t^-} \in PC$ by $x_t(s) = x(t+s)$ for $-r \leq s \leq 0$ and $x_{t^-}(s) = x(t+s)$ for $-r \leq s < 0$, respectively. Denote by $PC_{F_t}^p$ the family of all F_t -measurable PC-valued random variables $\phi = \{\phi(\theta) : -r \leq \theta \leq 0\}$, satisfying $\sup_{-r \leq \theta \leq 0} E\|\phi(\theta)\|^p < \infty$, where E stands for the mathematical expectation. Let $PC_{F_t}^b(\delta) = \{\phi : \phi \in PC_{F_t}^p([-r, 0], R^n) \text{ and } \sup_{-r \leq \theta \leq 0} E\|\phi(\theta)\|^p < \delta\}$.

The Markov process $\{r(t), t \geq 0\}$ represents the switching between the different modes taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\pi = (\pi_{ij})_{N \times N}$ given by

$$\Pr \{r(t + \Delta) = j \mid r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases} \quad (1)$$

where π_{ij} is the transition rate from mode i to j and satisfies the following relations:

$$\pi_{ij} \geq 0, \quad \pi_{ii} = -\sum_{j \neq i} \pi_{ij}, \quad (2)$$

and $o(\Delta)$ is such that $\lim_{\Delta \rightarrow 0} o(\Delta)/\Delta = 0$. We assume that the Markov chain $r(t)$ is independent of the Brownian motion $\omega(\cdot)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $R^+ = [0, +\infty)$.

Consider the following stochastic nonlinear delay system with impulsive jump and markovian switching

$$\begin{aligned} dx(t) &= f(x(t), x_t, t, r(t)) dt \\ &\quad + g(x(t), x_t, t, r(t)) d\omega(t), \quad t \neq t_k, \\ \Delta x(t) &= I_k(x(t^-), t), \quad t = t_k, \\ x(t_0 + \theta) &= \phi(\theta), \quad \theta \in [-r, 0], \end{aligned} \quad (3)$$

where $\phi(\theta) \in PC$ is the initial data, $x(t) \in R^n$ is the systems state, $x(t^+)$ and $x(t^-)$ denote the limit from the right and the left at point t , respectively, and $\{t_k : k \in Z^+\} \subset R^+$ a strictly increasing sequence. We assume that for each $i \in S$, given functional $f : R^n \times PC \times S \rightarrow R^n$, $g : R^n \times PC \times S \rightarrow R^{n \times m}$ satisfying $f(0, 0, t, i) \equiv g(0, 0, t, i) \equiv 0$. $I_k : R^n \times R \rightarrow R^n$ with $I_k(0, t) \equiv 0$ is the change of state variable at instant t_k . In this paper, we always assume that there exists a unique stochastic process satisfying systems (3), and all solutions of systems (3) are continuous on the right-hand side and on the left-hand side. Moreover, by $f(0, 0, t, i) \equiv g(0, 0, t, i) \equiv 0$ and $I_k(0, t) \equiv 0$, it is easily obtained that system (3) admits a trivial solution.

Definition 1 (see [20]). The trivial solution of system (3) is

- (1) p -moment stable if for any initial data $x_{t_0} = \phi$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$E\|x(t)\|^p < \varepsilon, \quad t \geq t_0, \quad (4)$$

whenever $\|\phi\|_r^p < \delta$;

- (2) uniformly p -moment stable if the δ in (1) is independent of t_0 .

Definition 2. Let $C^{2,1}(R^n \times [t_0 - r, \infty) \times S; R^+)$ denote the family of all nonnegative functions $V(x, t, i)$ that are continuously twice differentiable in x and once differentiable with respect to t . For each $V(x, t, i) \in C^{2,1}(R^n \times [t_0 - r, \infty) \times S; R^+)$, define an operator LV from $R^n \times [t_0 - r, \infty) \times S$ to R^+ as follows:

$$\begin{aligned} LV(x, t, i) &= V_t(x, t, i) + V_x(x, t, i) f(x, y, t, i) \\ &\quad + \frac{1}{2} \text{trace} \left[g^T(x, y, t, i) V_{xx}(x, t, i) g(x, y, t, i) \right] \\ &\quad + \sum_{j=1}^N q_{ij} V(x, t, j), \end{aligned} \quad (5)$$

where

$$\begin{aligned} V_t(x, t, i) &= \frac{\partial V(x, t, i)}{\partial t}, \\ V_x(x, t, i) &= \left(\frac{\partial V(x, t, i)}{\partial x_1}, \dots, \frac{\partial V(x, t, i)}{\partial x_n} \right), \\ V_{xx}(x, t, i) &= \left(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j} \right)_{n \times n}. \end{aligned} \quad (6)$$

Before giving the results, we need a lemma.

Lemma 3 (see [20]). *If $V \in C^{2,1}(R^n \times [t_0 - r, \infty) \times S; R^+)$, then for any stopping times $0 \leq t_1 \leq t_2 < +\infty$,*

$$\begin{aligned} E(V(x(t_2), t_2, r(t_2))) &= E(V(x(t_1), t_1, r(t_1))) \\ &\quad + E\left(\int_{t_1}^{t_2} LV(x(s), s, r(s)) ds\right), \end{aligned} \quad (7)$$

as long as the integration involved exist and finite.

3. Main Results

In this section, Lyapunov-based sufficient conditions for p -moment stability of system (3) are developed. The first result is concerned with p -moment stability of system (3), in the case when all the subsystems governing the continuous dynamics of (3) are stable. Intuitively, the conditions in the following theorem consist of three aspects: (i) the Lyapunov-Krasovskii functionals satisfy certain positive definite and decrescent conditions; (ii) there exist some negative estimates of the upper right-hand derivatives of the functionals with respect to each stable mode of system (3); (iii) the jumps induced by the impulses and the estimates on the decay rate of continuous dynamics satisfy certain conditions.

Theorem 4. Assume that there exist function $V(x(t), t, i) \in C^{2,1}(R^n \times [t_0 - r, \infty) \times S; R^+)$ and some positive constants p, a, b, e_i , the nonnegative and continuous functions $c_i(t)$ and $d_i(t)$, such that

- (i) $a|x|^p \leq V(x(t), t, i) \leq b|x_i|^p$;
- (ii) $ELV(x(t), t, i) \leq -c_i(t)EV(x(t), t, i) + d_i(t)EV(x_t, t, i)$, $t \in [t_{k-1}, t_k], k = 1, 2, \dots$;
- (iii) $EV(\phi(0) + I_k(\phi, t_k, \bar{i}) \leq e_i V(\phi(0), t_k^-, i)$;
- (iv) $0 < c^* < 1$, where $c^* = \sup\{c_i^* \mid c_i^* = be_i/a, i = 1, 2, \dots\}$;
- (v) $c_i(t) > d_i(t)/c^*, t \in [t_k, t_{k+1})$;

then the trivial solution of system (3) is uniformly p -moment stable.

Proof. For any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\delta < (ac^*/b)\varepsilon$ independent of t_0 . For any $t_0 \geq 0$ and $x_{t_0} = \phi \in PC_{F_i}^b(\delta)$, let $x(t) = x(t, t_0, \phi)$ be the solution of (3).

In the following, we will show that

$$EV(x(t), t, i) \leq \frac{b}{c^*}\delta, \quad t \in [t_{k-1}, t_k), k \in Z^+. \quad (8)$$

It will be firstly proved that

$$EV(x(t), t, i) \leq \frac{b}{c^*}\delta, \quad t \in [t_0, t_1). \quad (9)$$

For any initial data $x_{t_0} \in PC_{F_i}^b(\delta)$, it follows from assumption (i) that

$$\begin{aligned} EV(x(t), t, r(t)) &\leq bE\|x_{t_0}\|_r^p \\ &\leq b\delta < \frac{b}{c^*}\delta, \quad t \in [t_0 - r, t_0]. \end{aligned} \quad (10)$$

Suppose that (9) is not true, then there exists some $t \in (t_0, t_1)$ such that

$$EV(x(t), t, r(t)) > \frac{b}{c^*}\delta > b\delta \geq EV(x(t_0), t_0, r(t_0)). \quad (11)$$

Set $t^* = \inf\{t \in [t_0, t_1) : EV(x(t), t, r(t)) > (b/c^*)\delta\}$. Observe that $EV(x(t), t, r(t))$ is continuous on $t \in [t_0, t_1)$, then $t^* \in (t_0, t_1)$ and

$$EV(x(t^*), t^*, r(t^*)) = \frac{b}{c^*}\delta, \quad (12)$$

$$EV(x(t), t, r(t)) < \frac{b}{c^*}\delta, \quad t \in [t_0 - r, t^*). \quad (13)$$

In view of (10), define $t^{**} = \sup\{t \in [t_0 - r, t^*] : EV(x(t), t, r(t)) \leq b\delta\}$. Then $t^{**} \in (t_0, t^*)$ and

$$EV(x(t^{**}), t^{**}, r(t^{**})) = b\delta, \quad (14)$$

$$EV(x(t), t, r(t)) > b\delta, \quad t \in (t^{**}, t^*]. \quad (15)$$

Consequently, in view of (10)–(15), for all $t \in [t^*, t^{**}]$ and $\theta \in [-r, 0]$, one has

$$\begin{aligned} EV(x(t + \theta), t + \theta, r(t + \theta)) &\leq \frac{b}{c^*}\delta \leq \frac{1}{c^*}EV(x(t^*), t^*, r(t^*)) \\ &\leq \frac{1}{c^*}EV(x(t), t, r(t)). \end{aligned} \quad (16)$$

Now, combining (16) and condition (ii), we have

$$\begin{aligned} ELV(x(t), t, r(t)) &\leq -c_i(t)EV(x(t), t, r(t)) + d_i(t)EV(x_t, t, r(t)) \\ &\leq \left(-c_i(t) + \frac{d_i(t)}{c^*}\right)EV(x(t), t, r(t)). \end{aligned} \quad (17)$$

Applying Lemma 3, integrating (17) on $[t^{**}, t^*]$, one obtains that

$$\begin{aligned} EV(x(t^*), t^*, r(t^*)) &\leq EV(x(t^{**}), t^{**}, r(t^{**})) \\ &+ \int_{t^{**}}^{t^*} \left(-c_i(t) + \frac{d_i(t)}{c^*}\right)EV(x(s), s, r(s)) ds. \end{aligned} \quad (18)$$

By condition (v) and (14) and the Gronwall inequality, one has

$$\begin{aligned} EV(x(t^*), t^*, r(t^*)) &\leq EV(x(t^{**}), t^{**}, r(t^{**})) e^{(-c_i(t) + (d_i(t)/c^*)) (t^* - t^{**})} \\ &\leq EV(x(t^{**}), t^{**}, r(t^{**})) = b\delta < \frac{b}{c^*}\delta. \end{aligned} \quad (19)$$

Since (19) contradicts with (12), inequality (9) holds and (8) is true for $k = 1$.

Now, assume that

$$EV(x(t), t, i) \leq \frac{b}{c^*}\delta, \quad t \in [t_{k-1}, t_k), \quad (20)$$

for all $k \leq m$, where $k, m \in \mathbb{Z}^+$. We proceed to show that

$$EV(x(t), t, i) \leq \frac{b}{c^*} \delta, \quad t \in [t_m, t_{m+1}). \quad (21)$$

Suppose that (21) is not true, set $\bar{t} = \inf\{t \in [t_m, t_{m+1}) : EV(x(t), t, r(t)) > (b/c^*)\delta\}$. From condition (i), (iv), and (21), we know that

$$\begin{aligned} & EV(x(t_m), t_m, r(t_m)) \\ & \leq e_{i_m} EV(x(t_m^-), t_m^-, r(t_m^-)) \\ & \leq be_{i_m} \|x_{t_m^-}\|_r^p \\ & \leq \frac{be_{i_m}}{a} \sup_{-r \leq \theta \leq 0} EV(x(t_m^- + \theta), t_m^- + \theta, r(t_m^- + \theta)) \\ & \leq \frac{be_{i_m}}{a} \frac{b}{c^*} \delta \leq b\delta < \frac{b}{c^*} \delta. \end{aligned} \quad (22)$$

Owing to $EV(x(t), t, r(t))$ is continuous on $t \in [t_m, t_{m+1})$, then $\bar{t} \in (t_m, t_{m+1})$ and

$$EV(x(\bar{t}), \bar{t}, r(\bar{t})) = \frac{b}{c^*} \delta, \quad (23)$$

$$EV(x(t), t, r(t)) < \frac{b}{c^*} \delta, \quad t \in [t_m, \bar{t}). \quad (24)$$

Define $\underline{t} = \sup\{t \in [t_0 - r, \bar{t}] : EV(x(t), t, r(t)) \leq b\delta\}$, then $\underline{t} \in (t_m, \bar{t})$ and

$$EV(x(\underline{t}), \underline{t}, r(\underline{t})) = b\delta, \quad (25)$$

$$EV(x(t), t, r(t)) > b\delta, \quad t \in (\underline{t}, \bar{t}]. \quad (26)$$

Fix any $t \in [\underline{t}, \bar{t}]$, when $t + \theta \geq t_m$ for all $\theta \in [-r, 0]$, then from (22)–(26), one has

$$\begin{aligned} & EV(x(t + \theta), t + \theta, r(t + \theta)) \\ & \leq \frac{b}{c^*} \delta \leq \frac{1}{c^*} EV(x(\bar{t}), \bar{t}, r(\bar{t})) \\ & \leq \frac{1}{c^*} EV(x(t), t, r(t)). \end{aligned} \quad (27)$$

When $t + \theta < t_m$ for some $\theta \in [-r, 0]$, without loss of generality, we assume that $t + \theta \in [t_{l-1}, t_l)$ for some $l \in \mathbb{Z}^+$, $l < m$, then from (20) and (25), we obtain that

$$\begin{aligned} & EV(x(t + \theta), t + \theta, r(t + \theta)) \\ & \leq \frac{b}{c^*} \delta \leq \frac{1}{c^*} EV(x(\bar{t}), \bar{t}, r(\bar{t})) \\ & \leq \frac{1}{c^*} EV(x(t), t, r(t)). \end{aligned} \quad (28)$$

Therefore, from condition (ii), (27), and (28), one gets

$$\begin{aligned} & ELV(x(t + \theta), t + \theta, r(t + \theta)) \\ & \leq \left(-c_{i_m}(t) + \frac{d_{i_m}(t)}{c^*} \right) EV(x(t), t, r(t)). \end{aligned} \quad (29)$$

Similar to the argument on $[t^{**}, t^*]$, an application of Ito's formula on $[t, \bar{t}]$ will lead to $EV(x(\bar{t}), \bar{t}, r(\bar{t})) < (b/c^*)\delta$, which would contradict with (23). So the inequality (21) is true. Therefore, by mathematical induction, one can obtain that (8) holds for all $k \in \mathbb{Z}^+$. Then from condition (i) and the definition of δ , we have

$$E\|x(t)\|^p \leq \frac{b}{ac^*} \delta < \varepsilon, \quad t \geq t_0. \quad (30)$$

According to Definition 1, it is concluded that the trivial solution of system (3) is uniformly p -moment stable. The proof is complete. \square

Corollary 5. Assume that there exist function $V(x(t), t, i) \in C^{2,1}(\mathbb{R}^n \times [t_0 - r, \infty) \times S; \mathbb{R}^+)$ and some positive constants p, a, b, e_i, c_i , and d_i such that

- (i) $a|x|^p \leq V(x(t), t, i) \leq b|x|_r^p$;
- (ii) $ELV(x(t), t, i) \leq -c_i EV(x(t), t, i) + d_i EV(x_t, t, i)$, $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots$;
- (iii) $EV(\phi(0) + I_k(\phi, t_k, \bar{i})) \leq e_i V(\phi(0), t_k^-, \bar{i})$;
- (iv) $0 < c^* < 1$, where $c^* = \sup\{c_i^* \mid c_i^* = (be_i/a), i = 1, 2, \dots\}$;
- (v) $c_i > d_i/c^*$, $t \in [t_k, t_{k+1})$;

then the trivial solution of system (3) is uniformly p -moment stable.

Proof. Replacing $c_i(t)$ and $d_i(t)$ in Theorem 4 with c_i and d_i , respectively, we find that it is a direct conclusion of Theorem 4. \square

Remark 6. From condition (ii) in Theorem 4, it is seen that each of the continuous dynamics is stable. On the other hand, from condition (iv), we see that each of the discrete dynamics is also stabilizing ($0 < c^* < 1$ implies $e_i < 1$). It implies that the stability properties of a time-delay impulsive markovian switched system with stable continuous dynamics can be preserved under stabilizing impulsive perturbations irrespective of the times of impulses and switching. Hence, the bound of dwell-time is not necessary.

Remark 7. If condition (iii) is omitted in Corollary 5, the result is consistent with that of Theorem 1 in [20]. Thus, the results in this work are an extension of that in [20].

We proceed to consider in next the p -moment stability of systems (3). It is supposed that all the subsystems governing the continuous dynamics of (3) can be unstable while the impulses are stabilizing. Intuitively, the conditions in the following theorem consist of four aspects: (i) the Lyapunov functionals satisfy certain positive definite and decrescent conditions; (ii) there exist some positive estimates of the upper right-hand derivatives of the functionals with respect to each unstable mode of system (3); (iii) the dwell time of each mode of system (3) satisfies some upper bounds; (iv) the jumps induced by the stabilizing impulses satisfy certain diminishing conditions.

Theorem 8. Assume that there exist function $V(x(t), t, i) \in C^{2,1}(R^n \times [t_0 - r, \infty] \times S; R^+)$ and some positive scalars $p, a, b, e_i, \rho < 1, \alpha$, and β such that

- (i) $a|x|^p \leq V(x(t), t, i) \leq b|x_t|_r^p$;
- (ii) $EV(\phi(0) + I_k(\phi, t_k, \tilde{i}) \leq \rho e_i V(\phi(0), t_k^-, i)$;
- (iii) $0 < c^* < 1$, where $c^* = \sup\{c_i^* \mid c_i^* = (b/a)\rho e_i, i = 1, 2, \dots\}$;
- (iv) there exist nonnegative and piecewise continuous functions $c_i(t) : [t_0, \infty) \rightarrow R^n$ satisfying $\int_t^{t+\alpha} c_i(s) ds \leq \alpha\beta$ for all $t \geq t_0$, such that

$$ELV(x(t), t, i) \leq c_i(t) EV(x(t), t, i), \tag{31}$$

$$t \in [t_{k-1}, t_k), \quad k = 1, 2, \dots,$$

whenever $t \geq t_0$ and $\phi \in PC_{F_t}^b$ are such that $c^* ELV(\phi(s), t + s, \tilde{i}) \leq EV(\phi(0), t, i)$;

- (v) $\sup_{k \in Z^+} \{t_k - t_{k-1}\} = \alpha < -(1/\beta) \ln((b/a)\rho e_i), k \in Z^+$;
- then the trivial solution of system (3) is uniformly p -moment stable.

Proof. For any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\delta < (ac^*/b)\varepsilon$ independent of t_0 . For any $t_0 \geq 0$ and $x_{t_0} = \phi \in PC_{F_t}^b(\delta)$, let $x(t) = x(t, t_0, \phi)$ be the solution of (3).

By generalized Ito's formula (Lemma 3), when $t \in [t_k, t_{k+1})$, we get

$$EV(x(t), t, r(t)) = EV(x(t_k), t_k, r(t_k)) + E\left(\int_{t_k}^t LV(x(s), s, r(s)) ds\right) + E\left(\int_{t_k}^t \frac{\partial V(x(s), s, r(s))}{\partial x(s)} g(x(s), s, r(s)) d\omega(s)\right) = EV(x(t_k), t_k, r(t_k)) + E\left(\int_{t_k}^t LV(x(s), s, r(s)) ds\right). \tag{32}$$

Given small enough $\Delta t > 0$ such that $t + \Delta t \in [t_k, t_{k+1})$, one has

$$EV(x(t + \Delta t), t + \Delta t, r(t + \Delta t)) = EV(x(t_k), t_k, r(t_k)) + E\left(\int_{t_k}^{t+\Delta t} LV(x(s), s, r(s)) ds\right), \tag{33}$$

$$t \in [t_k, t_{k+1}).$$

In view of condition (iv), (32), and (33), it follows that

$$EV(x(t + \Delta t), t + \Delta t, r(t + \Delta t)) - EV(x(t), t, r(t)) = \int_t^{t+\Delta t} ELV(x(s), s, r(s)) ds \leq \int_t^{t+\Delta t} c_i(t) EV(x(s), s, r(s)) ds, \tag{34}$$

$$t \in [t_k, t_{k+1}).$$

From (34), one gets

$$D^+ EV(x(t), t, r(t)) \leq c_i(t) EV(x(t), t, r(t)), \tag{35}$$

$$t \in [t_k, t_{k+1}),$$

where D^+ is the Dini-derivative defined as

$$D^+ V(t) = \lim_{h \rightarrow 0^+} \sup \frac{V(t+h) - V(t)}{h}, \tag{36}$$

in which $V(t)$ is continuous function.

In the following, we will show that

$$EV(x(t), t, i) \leq \frac{b}{c^*} \delta, \quad t \in [t_{k-1}, t_k), \quad k \in Z^+. \tag{37}$$

For this purpose, we first prove that

$$EV(x(t), t, i) \leq \frac{b}{c^*} \delta, \quad t \in [t_0, t_1). \tag{38}$$

From condition (i) and $x_{t_0} \in PC_{F_t}^b(\delta)$, it follows that

$$EV(x(t), t, r(t)) \leq bE\|x_{t_0}\|_r^p \leq b\delta < \frac{b}{c^*} \delta, \quad t \in [t_0 - r, t_0]. \tag{39}$$

If (38) is not true, there must exist some $t \in [t_0, t_1)$ such that

$$EV(x(t), t, r(t)) > \frac{b}{c^*} \delta > b\delta \geq EV(x(t_0), t_0, r(t_0)). \tag{40}$$

Let $t^* = \inf\{t \in [t_0, t_1) : EV(x(t), t, r(t)) > (b/c^*)\delta\}$. From (39), one has $t^* \in (t_0, t_1)$ and

$$EV(x(t^*), t^*, r(t^*)) = \frac{b}{c^*} \delta,$$

$$EV(x(t), t, r(t)) < \frac{b}{c^*} \delta, \quad t \in [t_0 - r, t^*), \tag{41}$$

$$D^+ EV(x(t^*), t^*, r(t^*)) \geq 0.$$

Owing to $(b/c^*)\delta > b\delta$ and $EV(x(t), t, r(t)) \leq b\delta$ for $t \in [t_0 - r, t_0]$, then there exists a $t^{**} \in [t_0, t^*)$ such that

$$EV(x(t^{**}), t^{**}, r(t^{**})) = b\delta,$$

$$EV(x(t), t, r(t)) > b\delta, \quad t \in (t^{**}, t^*], \tag{42}$$

$$D^+ EV(x(t^*), t^*, r(t^*)) \geq 0.$$

Consequently, by (41) and (42), for $t \in [t^*, t^{**}]$, we have

$$EV(x(t+s), t+s, r(t+s)) \leq \frac{b\delta}{c^*} \leq \frac{1}{c^*} EV(x(t), t, r(t)). \tag{43}$$

So, for $t \in [t^*, t^{**}]$, it has

$$D^+ EV(x(t), t, r(t)) \leq c_i(t) EV(x(t), t, r(t)). \tag{44}$$

Integrating (44) from t^{**} to t^*

$$\int_{t^{**}}^{t^*} \frac{D^+ EV(x(s), s, r(s))}{EV(x(s), s, r(s))} ds \leq \int_{t_0}^{t_1} c_i(t) ds \leq \alpha\beta. \tag{45}$$

Meanwhile, by condition (v), one has

$$\begin{aligned} & \int_{t^{**}}^{t^*} \frac{D^+ EV(x(s), s, r(s))}{EV(x(s), s, r(s))} ds \\ &= \int_{EV(x(t^{**}), t^{**}, r(t^{**}))}^{EV(x(t^*), t^*, r(t^*))} \frac{1}{u} du \\ &= \int_{b\delta}^{(b/c^*)\delta} \frac{1}{u} du = -\ln c^* > c_i(t_1 - t_0), \end{aligned} \tag{46}$$

which is contradictory with (45), so (38) is true.

By condition (i) and (38), we have

$$\begin{aligned} & EV(x(t_1), t_1, r(t_1)) \\ & \leq \rho e_{i_1} EV(x(t_1^-), t_1^-, r(t_1^-)) \\ & \leq b e_{i_1} \|x_{t_1^-}\|_r^p \\ & \leq \frac{b}{a} \rho e_{i_1} \sup_{-r \leq \theta \leq 0} EV(x(t_1^- + \theta), t_1^- + \theta, r(t_1^- + \theta)) \\ & \leq \frac{b}{a} \frac{b}{c^*} \rho e_{i_1} \delta \leq b\delta < \frac{b}{c^*} \delta. \end{aligned} \tag{47}$$

Now, we assume that (37) holds for $k = 1, 2, \dots, m$, that is,

$$EV(x(t), t, r(t)) \leq \frac{b}{c^*} \delta, \quad t \in [t_{k-1}, t_k]. \tag{48}$$

We will prove that, for $k = m + 1$,

$$EV(x(t), t, r(t)) \leq \frac{b}{c^*} \delta, \quad t \in [t_m, t_{m+1}). \tag{49}$$

Suppose that (49) does not hold, by the same procedure as in $[t_0, t_1)$, we can get a contradiction and (49) follows. Finally, we get

$$E\|x(t)\|^p \leq \frac{b}{ac^*} \delta < \varepsilon, \quad t \geq t_0. \tag{50}$$

From Definition 1, it is concluded that the trivial solution of system (3) is uniformly p -moment stable. The proof is complete. \square

Corollary 9. Assume that there exist positive scalars $p, a, b, \rho < 1, e_i, c_i (i \in S), \alpha$, and β such that conditions (iv) in Theorem 8 are replaced by the following:

(iv)* there exists positive number c_i such that

$$\begin{aligned} & ELV(x(t), t, i) \leq c_i EV(x(t), t, i), \\ & t \in [t_{k-1}, t_k), \quad k = 1, 2, \dots, \end{aligned} \tag{51}$$

whenever $t \geq t_0$ and $\varphi \in PC_{F_i}^b([-r, 0]; R^n)$ are such that $c^* ELV(\varphi(s), t+s, \tilde{i}) \leq EV(\varphi(0), t, i)$; and all other conditions remain the same, then the trivial solution of system (3) is uniformly p -moment stable.

Proof. Similar to the procedure of Theorem 8, correspondingly, the item $c_i(t)$ in the proof is replaced by c_i . \square

Remark 10. It is clear that Theorem 8 implies that each of the continuous dynamics can be unstable, since the item $c_i(t)$ in condition (iv), which characterizes the changing rate of Lyapunov function $V(x(t), t, r(t))$ at t , is assumed to be nonnegative. Theorem 8 shows that an unstable stochastic delay system can be successfully stabilized by impulse.

Remark 11. Constant e_i in condition (ii) characterizes certain perturbations in the overall impulsive stabilization process, that is, it is not strictly required by Theorem 8 that each impulse contributes to stabilize the system, as long as the overall contribution of the impulses is stabilizing. When $e_i \equiv 1$, it is easily obtained that each impulse must be stabilizing ($\rho < 1$), which is more restrictive.

Now consider the linear stochastic system with impulses and markovian switching

$$\begin{aligned} dx(t) &= [A(r(t))x(t) + B(r(t))x(t-r)]dt \\ &+ [C(r(t))x(t) + D(r(t))x(t-r)]d\omega(t), \\ & t \neq t_k, \\ \Delta x(t) &= E_k(x(t^-), r(t)), \quad t = t_k, \\ x(t_0 + \theta) &= \phi(\theta), \quad \theta \in [-r, 0], \end{aligned} \tag{52}$$

where A_i, B_i, C_i, D_i , and E_k are all $n \times n$ matrices and ω is a one-dimensional standard wiener process. $r(t)$ is a Markov chain taking values in $S = \{1, 2, \dots, N\}$ in which the transition rate is π_{ij} .

Theorem 12. When $r(t) = i \in S$, define $A(r(t)) = A_i, B(r(t)) = B_i, C(r(t)) = C_i, D(r(t)) = D_i$.

(i) If there exist constants $p_i, c_i, d_i, \varepsilon_1$, and ε_2 such that

$$p_i \lambda_{\max}(A_i^T + A_i + \varepsilon_1 I) + p_i(1 + \varepsilon_2) \|D_i\|^2 + \sum_{j=1}^N \pi_{ij} p_j$$

$$\leq -c_i, p_i \{ \varepsilon_1^{-1} \|B_i\|^2 + (1 + \varepsilon_2^{-1}) \|G_i\|^2 \} \leq d_i;$$

$$p_i c_i \|I + E_k\|^2 > d_i,$$
(53)

then system (52) is uniformly p -moment stable.

(ii) If there exist constants c_i , ε_1 , and ε_2 such that

$$p_i \lambda_{\max} (A_i^T + A_i + \varepsilon_1 I) + p_i (1 + \varepsilon_2) \|D_i\|^2$$

$$+ \sum_{j=1}^N \pi_{ij} p_j + 2p_i \ln \|I + E_k\|$$

$$\times [\varepsilon_1^{-1} \|B_i\|^2 + (1 + \varepsilon_2^{-1}) \|G_i\|^2] \leq c_i;$$

$$\ln (p_i \|I + E_k\|) \leq -\frac{1}{2} c_i (t_{k+1} - t_k),$$
(54)

then system (52) is uniformly p -moment stable.

Proof. It is a direct consequence of Corollaries 5 and 9 with $V(x(t), r(t)) = p_i \|x(t)\|^2$. \square

4. Numerical Examples

The applicability of the results derived in the preceding section is illustrated by the following two examples.

Example 1. Consider the following linear impulsive switching delay system:

$$dx(t) = [A(r(t))x(t) + B(r(t))x(t-1)] dt$$

$$+ [C(r(t))x(t) + D(r(t))x(t-1)] d\omega(t),$$

$$t \neq t_k, \quad (55)$$

$$\Delta x(t) = E_k(x(t^-), r(t)), \quad t = t_k,$$

where

$$A_1 = \begin{bmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & 0.15 & 0.3 \\ 0 & 0.24 & 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.24 & 0.04 & 0 \\ 0.24 & -0.4 & 0.1 \\ 0 & 0.24 & -0.2 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ -0.24 & 0.32 & 0.3 \\ 0.6 & 0.26 & -0.5 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.14 & 0 & -0.21 \\ 0.2 & 0.1 & 0.16 \\ 0.1 & 0.15 & 0.17 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2 & 0.3 & -0.1 \\ 0.3 & 0.1 & 0.3 \\ 0 & 0.2 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.2 & 0.04 & 0 \\ 0.2 & -0.4 & 0.1 \\ 0 & 0.2 & -0.2 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ -0.2 & 0.3 & 0.3 \\ 0.6 & 0.2 & -0.5 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.24 & 0 & -0.21 \\ 0.2 & 0.1 & 0.16 \\ 0.1 & 0.1 & 0.17 \end{bmatrix},$$

$$E_k = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & -0.3 \end{bmatrix}, \quad \pi = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$
(56)

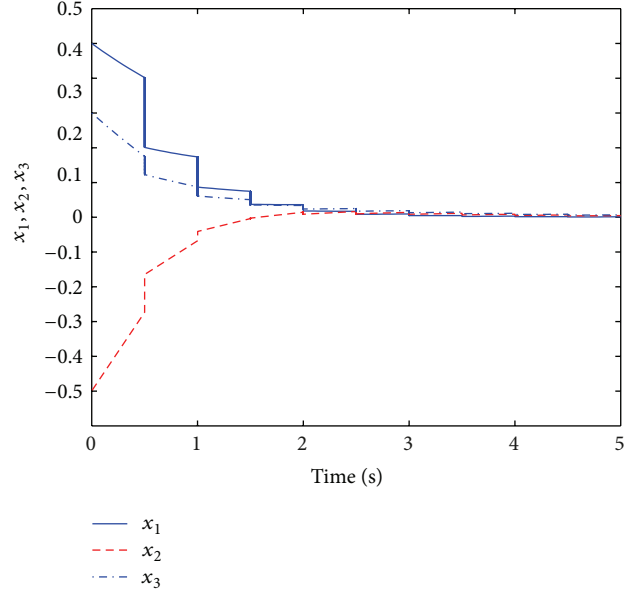


FIGURE 1: State response of system (55) under impulsive perturbations.

Set $\varepsilon_1 = \varepsilon_2 = 0.5$, $p_i = 1$. It is easy to see that $\lambda_{\max}(A_1^T + A_1 + 0.5I) = 1.3819$, $\lambda_{\max}(A_2^T + A_2 + 0.5I) = 1.5020$, $\|B_1\|^2 = 0.3141$, $\|B_2\|^2 = 0.2764$, $\|D_1\|^2 = 0.9175$, $\|D_2\|^2 = 0.8737$, $\|G_1\|^2 = 0.1360$, $\|G_2\|^2 = 0.1219$, and $\|I + E_k\|^2 = 4.9$. Taking $c_1 = 3.1$, $c_2 = 3.2$, $t_k - t_{k-1} = 0.22$, we can verify that all the conditions (ii) of Theorem 12 are satisfied:

$$\lambda_{\max} (A_1^T + A_1 + \varepsilon_1 I) + (1 + \varepsilon_2) \|D_1\|^2$$

$$+ \sum_{j=1}^N \pi_{1j} + 2 \ln \|I + E_k\|$$

$$\times [\varepsilon_1^{-1} \|B_1\|^2 + (1 + \varepsilon_2^{-1}) \|G_1\|^2]$$

$$= 3.0190 \leq c_1 = 3.1;$$

$$\lambda_{\max} (A_2^T + A_2 + \varepsilon_1 I) + (1 + \varepsilon_2) \|D_2\|^2$$

$$+ \sum_{j=1}^N \pi_{2j} + 2 \ln \|I + E_k\|$$

$$\times [\varepsilon_1^{-1} \|B_2\|^2 + (1 + \varepsilon_2^{-1}) \|G_2\|^2]$$

$$= 3.1574 \leq c_2 = 3.2;$$

$$\ln \|I + E_k\|^2 = -0.7133 \leq -c_i (t_{k+1} - t_k)$$

$$= -3.2 \times 0.22 = -0.7040.$$
(57)

So, system (55) is p -moment stable. Numerical simulations for this example are shown in Figures 1 and 2. It is clearly demonstrated that impulses can successfully stabilize an otherwise unstable stochastic delay system.

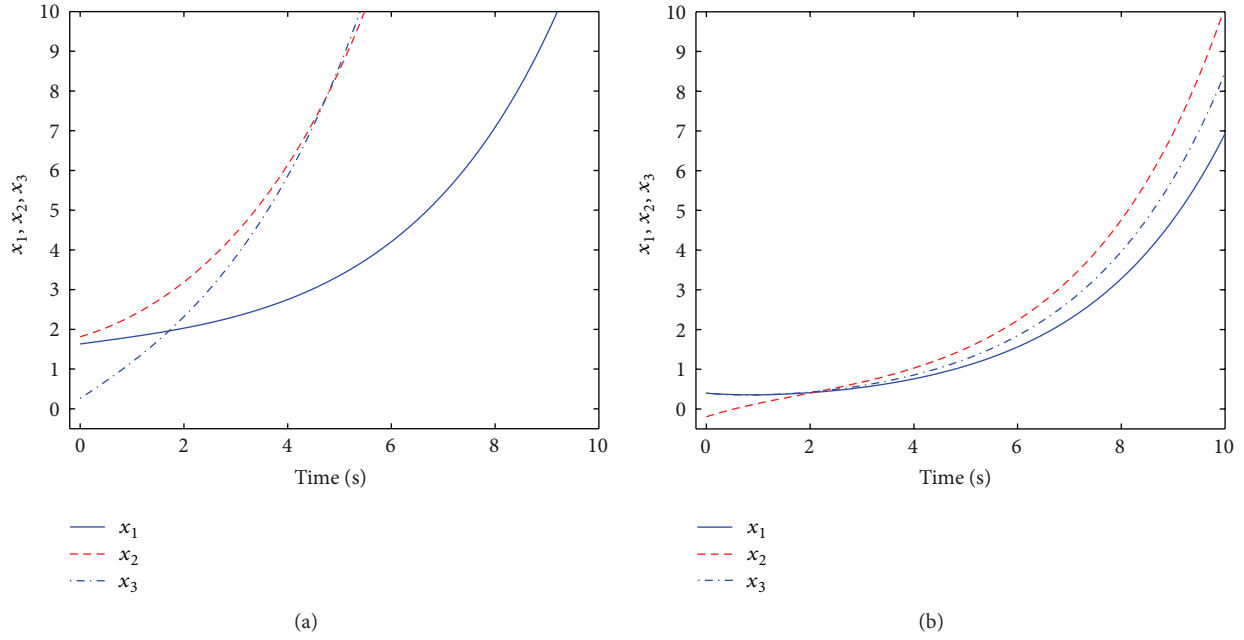


FIGURE 2: State response of system (55) without impulsive perturbations.

Example 2. Consider the following linear impulsive switching delay system:

$$\begin{aligned} dx(t) = & [A(r(t))x(t) + B(r(t))x(t-1)] dt \\ & + [C(r(t))x(t) + D(r(t))x(t-1)] d\omega(t), \\ & t \neq t_k, \end{aligned} \quad (58)$$

$$\Delta x(t) = E_k(x(t^-), r(t)), \quad t = t_k,$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0.2 & 0.3 \\ 0.5 & 0.1 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.2 \end{bmatrix}, & G_1 &= \begin{bmatrix} 0.8 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \\ E_k &= \begin{bmatrix} -0.6 & 0.5 \\ 0.3 & -0.5 \end{bmatrix}, \end{aligned} \quad (59)$$

$$\begin{aligned} A_2 &= \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0.3 \end{bmatrix}, \\ D_2 &= \begin{bmatrix} 0.5 & 0.1 \\ 0.4 & 0.1 \end{bmatrix}, & G_2 &= \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}, \\ P &= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}. \end{aligned} \quad (60)$$

Choosing $\varepsilon_1 = \varepsilon_2 = 2$, $p_i = 1$. It is easy to see that $\lambda_{\max}(A_1^T + A_1 + 2I) = -6$, $\lambda_{\max}(A_2^T + A_2 + 2I) = -4$, $\|B_1\|^2 = 0.3403$, $\|B_2\|^2 = 0.5687$, $\|D_1\|^2 = 0.5009$, $\|D_2\|^2 = 0.4298$, $\|G_1\|^2 = 0.6712$, $\|G_2\|^2 = 0.2347$, $\|I + E_k\|^2 = 0.7467$. Taking $c_1 = 3.3$,

$c_2 = 1.6$, $d_1 = 1.2$, $d_2 = 0.65$, we can verify that all the conditions (i) of Theorem 12 are satisfied:

$$\begin{aligned} \lambda_{\max}(A_i^T + A_i + \varepsilon_1 I) + (1 + \varepsilon_2) \|D_i\|^2 \\ + \sum_{j=1}^N \pi_{ij} = \begin{cases} -3.4973 < -3.3 = c_1; \\ -1.7166 < -1.6 = c_2; \end{cases} \end{aligned} \quad (61)$$

$$\varepsilon_1^{-1} \|B_i\|^2 + (1 + \varepsilon_2^{-1}) \|G_i\|^2 = \begin{cases} 1.1770 < 1.2 = d_1; \\ 0.6364 < 0.65 = d_2; \end{cases}$$

$$\|I + E_k\|^2 c_i = 0.7467 \times \begin{cases} 3.3 = 2.4641 > 1.2 = d_1; \\ 1.6 = 1.1947 > 0.65 = d_2. \end{cases} \quad (62)$$

Theorem 12 guarantees that the trivial solution of system (58) is p -moment stable. Numerical simulations are shown in Figure 3. It is clearly demonstrated that the p -moment stability properties can be preserved irrespective of the length of the delay.

5. Conclusions

In this paper, a method of multiple Lyapunov functionals has been applied to deal with the effects of impulses on p -moment stability of stochastic differential delay systems with impulsive jump and markovian switching. Some stability criteria are obtained based on Lyapunov functional method and stochastic theory. It is shown that, even if all the subsystems governing the continuous dynamics without impulse are not stable, as impulsive and switching signal satisfies a dwell-time upper bound condition, impulses can stabilize the systems in the p -moment stability sense. The opposite situation is also developed for which all the subsystems governing the

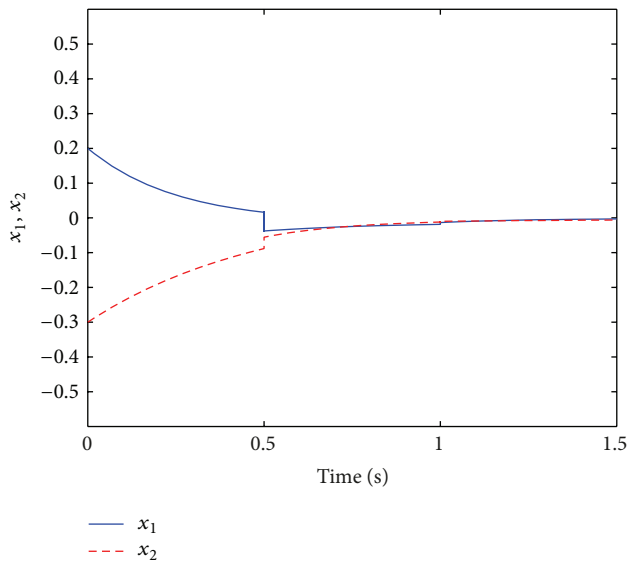


FIGURE 3: State response of system (58) under impulsive perturbations

continuous dynamics without impulse are p -moment stable. Applying the derived results to a class of stochastic systems with arbitrarily large delays, the deduced new stability criteria can relax some restrictions on impulses imposed by the existing results. Two illustrative examples have been provided to demonstrate the main theoretical analysis.

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