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Research Article

Minimax Theorems for Set-Valued Mappings under Cone-Convexities

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The aim of this paper is to study the minimax theorems for set-valued mappings with or without linear structure. We define several kinds of cone-convexities for set-valued mappings, give some examples of such set-valued mappings, and study the relationships among these cone-convexities. By using our minimax theorems, we derive some existence results for saddle points of set-valued mappings. Some examples to illustrate our results are also given.

1. Introduction

The minimax theorems for real-valued functions were introduced by Fan [1, 2] in the early fifties. Since then, these were extended and generalized in many different directions because of their applications in variational analysis, game theory, mathematical economics, fixed-point theory, and so forth (see, for example, [3–11] and the references therein). The minimax theorems for vector-valued functions have been studied in [4, 9, 10] with applications to vector saddle point problems. However, the minimax theorems for set-valued bifunctions have been studied only in few papers, namely, [4–8] and the references therein.

In this paper, we establish some new minimax theorems for set-valued mappings. Section 2 deals with preliminaries which will be used in rest of the paper. Section 3 denotes the cone-convexities of set-valued mappings. In Section 4, we establish some minimax theorems by using separation theorems, Fan-Browder fixed-point theorem. In the last section, we discuss some existence results for different kinds of saddle points for set-valued mappings.

2. Preliminaries

Throughout the paper, unless otherwise specified, we assume that X, Y are two nonempty subsets, and $\mathcal{Z}$ is a real Hausdorff topological vector space, C is a closed convex pointed cone in $\mathcal{Z}$ with $\operatorname{int} C \neq \emptyset$. Let $\mathcal{Z}^*$ be the topological dual space of $\mathcal{Z}$, and let

$$C^* = \{g \in \mathcal{Z}^* : g(c) \geq 0 \, \forall c \in C\}. \tag{2.1}$$

We present some fundamental concepts which will be used in the sequel.

Definition 2.1 (see [3, 4, 8]). Let A be a nonempty subset of $\mathcal{Z}$. A point $z \in A$ is called a

(a) *minimal point* of A if $A \cap (z - C) = \{z\}$; $\operatorname{Min} A$ denotes the set of all minimal points of A;

(b) *maximal point* of A if $A \cap (z + C) = \{z\}$; $\operatorname{Max} A$ denotes the set of all maximal points of A;

(c) *weakly minimal point* of A if $A \cap (z - \operatorname{int} C) = \emptyset$; $\operatorname{Min}_w A$ denotes the set of all weakly minimal points of A;

(d) *weakly maximal point* of A if $A \cap (z + \operatorname{int} C) = \emptyset$; $\operatorname{Max}_w A$ denotes the set of all weakly maximal points of A.

It can be easily seen that $\operatorname{Min} A \subset \operatorname{Min}_w A$ and $\operatorname{Max} A \subset \operatorname{Max}_w A$.

Lemma 2.2 (see [3, 4]). *Let A be a nonempty compact subset of $\mathcal{Z}$. Then,*

(a) $\operatorname{Min} A \neq \emptyset$;

(b) $A \subset \operatorname{Min} A + C$;

(c) $\operatorname{Max} A \neq \emptyset$;

(d) $A \subset \operatorname{Max} A - C$.

Following [6], we denote both Max and Max_w by max (both Min and Min_w by min) in $\mathbb{R}$ since both Max and Max_w (both Min and Min_w) are the same in $\mathbb{R}$.

Definition 2.3. Let $\mathcal{X}$, $\mathcal{Y}$ be Hausdorff topological spaces. A set-valued map $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ with nonempty values is said to be

(a) *upper semicontinuous at* $x_0 \in \mathcal{X}$ if for every $x_0 \in \mathcal{X}$ and for every open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$;

(b) *lower semi-continuous at* $x_0 \in \mathcal{X}$ if for any sequence $\{x_n\} \subset \mathcal{X}$ such that $x_n \to x_0$ and any $y_0 \in F(x_0)$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to y_0$;

(c) *continuous at* $x_0 \in \mathcal{X}$ if F is upper semi-continuous as well as lower semi-continuous at x_0.

We present the following fundamental lemmas which will be used in the sequel.

Lemma 2.4 (see [9, Lemma 3.1]). *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be three topological spaces. Let $\mathcal{Y}$ be compact, $F : \mathcal{X} \times \mathcal{Y} \rightrightarrows \mathcal{Z}$ a set-valued mapping, and the set-valued mapping $T : \mathcal{X} \rightrightarrows \mathcal{Z}$ defined by*

$$T(x) = \bigcup_{y \in \mathcal{Y}} F(x, y), \quad \forall x \in \mathcal{X}. \tag{2.2}$$

(a) *If F is upper semi-continuous on $\mathcal{X} \times \mathcal{Y}$, then T is upper semi-continuous on $\mathcal{X}$.*

(b) *If F is lower semi-continuous on $\mathcal{X}$, so is T.*

Lemma 2.5 (see [9, Lemma 3.2]). *Let $\mathcal{Z}$ be a Hausdorff topological vector space, $F : \mathcal{Z} \rightrightarrows \mathbb{R}$ a set-valued mapping with nonempty compact values, and the functions p, $q : \mathcal{Z} \to \mathbb{R}$ defined by $p(z) = \max F(z)$ and $q(z) = \min F(z)$.*

(a) *If F is upper semi-continuous, so is p.*

(b) *If F is lower semi-continuous, so is p.*

(c) *If F is continuous, so are p and q.*

We shall use the following nonlinear scalarization function to establish our results.

Definition 2.6 (see [6, 10]). Let $k \in \operatorname{int} C$ and $v \in \mathcal{Z}$. The *Gerstewitz function* $\xi_{kv} : \mathcal{Z} \to \mathbb{R}$ is defined by

$$\xi_{kv}(u) = \min\{t \in \mathbb{R} : u \in v + tk - C\}. \tag{2.3}$$

We present some fundamental properties of the scalarization function.

Proposition 2.7 (see [6, 10]). *Let $k \in \operatorname{int} C$ and $v \in \mathcal{Z}$. The Gerstewitz function $\xi_{kv} : \mathcal{Z} \to \mathbb{R}$ has the following properties:*

(a) $\xi_{kv}(u) < r \Leftrightarrow u \in v + rk - \operatorname{int} C$;

(b) $\xi_{kv}(u) \leq r \Leftrightarrow u \in v + rk - C$;

(c) $\xi_{kv}(u) = 0 \Leftrightarrow u \in v - \partial C$, *where ∂C is the topological boundary of C;*

(d) $\xi_{kv}(u) > r \Leftrightarrow u \notin v + rk - C$;

(e) $\xi_{kv}(u) \geq r \Leftrightarrow u \notin v + rk - \operatorname{int} C$;

(f) $\xi_{kv}(\cdot)$ *is a convex function;*

(g) $\xi_{kv}(\cdot)$ *is an increasing function, that is,* $u_2 - u_1 \in \operatorname{int} S \Rightarrow \xi_{kv}(u_1) < \xi_{kv}(u_2)$;

(h) $\xi_{kv}(\cdot)$ *is a continuous function.*

Theorem 2.8 (Fan-Browder fixed-point theorem (see [12])). *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and let $T : X \rightrightarrows X$ be a set-valued mapping with nonempty convex values and open fibers, that is, $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open for all $y \in X$. Then, T has a fixed point.*

3. Cone-Convexities

In this section, we present different kinds of cone-convexities for set-valued mappings and give some relations among them. Some examples of such set-valued mappings are also given.

Definition 3.1. Let X be a nonempty convex subset of a topological vector space $\mathcal{W}$. A set-valued mapping $F : X \rightrightarrows \mathcal{Z}$ is said to be

(a) *above -C-convex* [4] (resp., *above-C-concave* [5]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C, \quad \big(\text{resp., } \lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) - C\big); \tag{3.1}$$

(b) *below-C-convex* [13] (resp., *below-C-concave* [9, 13]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C \quad \big(\text{resp., } F(\lambda x_1 + (1-\lambda)x_2) \subset \lambda F(x_1) + (1-\lambda)F(x_2) + C\big); \tag{3.2}$$

(c) *above-C-quasi-convex* (resp., *below-C-quasiconcave*) [7, Definition 2.3] on X if the set

$$\text{Lev}_{F\le}(z) := \{x \in X : F(x) \subset z - C\} \quad \big(\text{resp., } \text{Lev}_{F\ge}(z) := \{x \in X : F(x) \subset z + C\}\big), \tag{3.3}$$

is convex for all $z \in \mathcal{Z}$;

(d) *above-properly C-quasiconvex* (resp., *above-properly C-quasiconcave* [6]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, either

$$F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C \quad \big(\text{resp., } F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) - C\big) \tag{3.4}$$

or

$$F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) - C \quad \big(\text{resp., } F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) - C\big); \tag{3.5}$$

(e) *below-properly C-quasiconvex* [7] (resp., *below-properly C-quasiconcave*) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, either

$$F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + C \quad \big(\text{resp., } F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) + C\big) \tag{3.6}$$

or

$$F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$$
$$\left(\text{resp., } F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) + C\right); \tag{3.7}$$

(f) *above-naturally C-quasiconvex* [6] on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \operatorname{co}\{F(x_1) \cup F(x_2)\} - C, \tag{3.8}$$

where co A denotes the convex hull of a set A;

(g) *above -C-convex-like* (resp., *above-C-concave-like*) on X (X is not necessarily convex) if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, there is an $x' \in X$ such that

$$F(x') \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C$$
$$\left(\text{resp., } \lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x') - C\right); \tag{3.9}$$

(h) *below -C-convex-like* [13] (resp., *below -C-concave-like*) on X (X is not necessarily convex) if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, there is an $x' \in X$ such that

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x') + C$$
$$\left(\text{resp.,} F(x') \subset \lambda F(x_1) + (1-\lambda)F(x_2) + C\right). \tag{3.10}$$

It is obvious that every above-C-convex set-valued mapping or above-properly C-quasi-convex set-valued mapping is an above-naturally C-quasi-convex set-valued mapping, and every above-C-convex (above-C-concave) set-valued mapping is an above-C-convex-like (above-C-concave-like) set-valued mapping. Similar relations hold for cases below.

Remark 3.2. The definition of above-properly C-quasi-convex (above-properly C-quasi-concave) set-valued mapping is different from the one mentioned in [7, Definition 2.3] or [5, 6]. The following Examples 3.3 and 3.4 illustrate the reason why they are different from the one mentioned in [5–7]. However, if F is a vector-valued mapping or a single-valued mapping, both mappings reduce to the ordinary definition of a properly C-quasi-convex mapping for vector-valued functions [7]. The above-C-convexity in Definition 3.1 is also different from the below-C-convexity used in [5, 9].

Example 3.3. Consider $C = \{(s,t) \in \mathbb{R}^2 : s \geq 0,\ t \geq 0\}$. Let $F : [x_1, x_2] \subset \mathbb{R} \rightrightarrows \mathbb{R}^2$ be a set-valued mapping defined by

$$F(x_1) := \left\{(s,t) \in \mathbb{R}^2 : (s-2)^2 + (t-4)^2 = 1, 2 \leq s \leq 3, 4 \leq t \leq 5\right\} \bigcup \{(s,5) : -1 \leq s \leq 2\},$$
$$F(x_2) := \left\{(s,t) \in \mathbb{R}^2 : (s-6)^2 + (t+2)^2 = 1, 6 \leq s \leq 7, -2 \leq t \leq -1\right\}, \tag{3.11}$$

and for all $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) := \left\{(s,t) \in \mathbb{R}^2 : (s-2)^2 + (t-2)^2 = 4, 0 \leq s \leq 2, 0 \leq t \leq 2\right\}. \tag{3.12}$$

Then F is an above-properly C-quasi-convex set-valued mapping, but it is not below-properly C-quasi-convex.

On the other hand, let $G : [x_1, x_2] \subset \mathbb{R} \rightrightarrows \mathbb{R}^2$ be a set-valued mapping defined by

$$\begin{aligned} G(x_1) &:= \left\{(s,t) \in \mathbb{R}^2 : (s-1)^2 + (t-4)^2 = 1, 1 \leq s \leq 2, 4 \leq t \leq 5\right\}, \\ G(x_2) &:= \left\{(s,t) \in \mathbb{R}^2 : (s-6)^2 + (t+2)^2 = 1, 6 \leq s \leq 7, -2 \leq t \leq -1\right\}, \end{aligned} \tag{3.13}$$

and for all $\lambda \in (0,1)$,

$$\begin{aligned} G(\lambda x_1 + (1-\lambda)x_2) : = & \left\{(s,t) \in \mathbb{R}^2 : (s-2)^2 + (t-2)^2 = 4, 0 \leq s \leq 2, 0 \leq t \leq 2\right\} \\ & \bigcup \{(s,0) : 2 \leq s \leq 3\}. \end{aligned} \tag{3.14}$$

Then, G is a below-properly C-quasi-convex set-valued mapping, but it is not above-properly C-quasi-convex.

Example 3.4. Let $C = \{(s,t) : s \geq 0, t \geq 0\}$. Define $F : [-1,1] \rightrightarrows \mathbb{R}^2$ by

$$F(x) = \left\{(x,t) : 1 - x^2 \leq t \leq 1\right\}, \quad \forall x \in [-1,1]. \tag{3.15}$$

Then F is continuous, above-C-quasi-convex, below-C-quasi-concave, above-properly C-quasi-convex, and above-properly C-quasi-concave, but it is not below-properly C-quasi-conconvex.

Proposition 3.5. *Let X be a nonempty set (not necessarily convex) and for a given set-valued mapping $F : X \rightrightarrows \mathfrak{Z}$ with nonempty compact values, define a set-valued mapping $M : X \rightrightarrows \mathfrak{Z}$ as*

$$M(x) = \mathrm{Max}_w \mathrm{F(x)}, \quad \forall \mathrm{x} \in \mathrm{X}. \tag{3.16}$$

(a) *If Fis above-C-convex-like, then M is so.*

(b) *If X is a topological space and F is a continuous mapping, then M is upper semicontinuous with nonempty compact values on X.*

Proof. (a) Let F be above-C-convex-like, and let $x_1, x_2 \in X$ be arbitrary. Since F is above-C-convex-like, for any $\alpha \in [0,1]$, there exists $x' \in X$ such that

$$F(x') \subset \alpha F(x_1) + (1-\alpha)F(x_2) - C. \tag{3.17}$$

By Lemma 2.2,

$$\mathrm{Max}_w F(x') \subset \alpha \mathrm{Max}_w F(x_1) + (1-\alpha)\mathrm{Max}_w \mathrm{F}(\mathrm{x}_2) - \mathrm{C}. \tag{3.18}$$

Therefore, $x \mapsto \mathrm{Max}_w F(x)$ is above-C-convex-like.

(b) The upper semicontinuity of M was deduced in [4, Lemma 2]. □

Proposition 3.6. *Let X be a nonempty convex set, and let $F : X \rightrightarrows \mathfrak{Z}$ be a set-valued mapping with nonempty compact values. Then, the set-valued mapping $M : X \rightrightarrows \mathfrak{Z}$ defined by*

$$M(x) = \mathrm{Max}_w \mathrm{F}(\mathrm{x}), \quad \forall \mathrm{x} \in \mathrm{X}, \tag{3.19}$$

is above-C-quasiconvex if F is so.

The following result can be easily derived, and therefore, we omit the proof.

Proposition 3.7. *Let X be a nonempty convex set and $F : X \rightrightarrows \mathbb{R}$ be above-$\mathbb{R}_+$-concave. Then the set-valued mapping $x \mapsto \max F(x)$ is above-$\mathbb{R}_+$-concave and below-$\mathbb{R}_+$-quasiconcave. Furthermore, if $F : X \rightrightarrows \mathbb{R}$ is above-properly $\mathbb{R}_+$-quasiconcave, then the set-valued mapping $x \mapsto \max F(x)$ is also above-properly $\mathbb{R}_+$-quasiconcave and below-$\mathbb{R}_+$-quasiconcave.*

Let $\xi \in C^*$ and $F : X \rightrightarrows \mathfrak{Z}$ be a set-valued mapping. Then, the composition mapping $\xi \circ F : X \rightrightarrows \mathbb{R}$ is defined by

$$(\xi \circ F)(x) = \xi(F(x)) = \bigcup_{y \in F(x)} \xi(y), \quad \forall x \in X. \tag{3.20}$$

Clearly, the composition mapping $\xi \circ F : X \rightrightarrows \mathbb{R}$ is also a set-valued mapping.

Proposition 3.8. *Let X be a nonempty set, $F : X \rightrightarrows \mathfrak{Z}$ a set-valued mapping, and $\xi \in C^*$.*

(a) *If F is above-C-convex-like, then $\xi \circ F$ is above-$\mathbb{R}_+$-convex-like.*

(b) *If F is below-C-concave-like, then $\xi \circ F$ is below-$\mathbb{R}_+$-concave-like.*

(c) *If X is a topological space and F is upper semi-continuous, then so is $\xi \circ F$.*

Proof. (a) By the definition of above-C-convex-like set-valued mapping $F : X \rightrightarrows \mathfrak{Z}$, for any $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, there exists $x' \in X$ such that $F(x') \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C$. For any $y \in F(x')$, there exist $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ such that

$$\lambda y_1 + (1-\lambda) y_2 \in f - C. \tag{3.21}$$

For any $\xi \in \mathrm{C}^*$, we have $\xi(y) \le \lambda\xi(y_1) + (1-\lambda)\xi(y_2)$. Hence, $\xi(F(x')) \subset \lambda\xi(F(x_1)) + (1-\lambda)\xi(F(x_2)) - \mathbb{R}_+$. Thus, $\xi \circ F$ is above-$\mathbb{R}_+$-convex-like.

The proof of (b) and (c) is easy, and therefore, we omit it. □

Proposition 3.9. *Let X be a nonempty convex set and $\xi \in C^*$.*

(a) *If $F : X \rightrightarrows \mathcal{Z}$ is above-C-concave (above-properly C-quasi-concave), then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above-$\mathbb{R}_+$-concave (above-properly $\mathbb{R}_+$-quasi-concave).*

(b) *If $F : X \rightrightarrows \mathcal{Z}$ is above-properly C-quasi-convex, then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above-$\mathbb{R}_+$-quasi-convex and above-properly $\mathbb{R}_+$-quasi-convex.*

(c) *If $F : X \rightrightarrows \mathcal{Z}$ is above-C-convex, then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above-$\mathbb{R}_+$-convex and above-$\mathbb{R}_+$-quasi-convex.*

Lemma 3.10. *Let $\mathcal{Z}$ be a real Hausdorff topological vector space and C a closed convex pointed cone in $\mathcal{Z}$ with $\operatorname{int} C \neq \emptyset$. Let X be a nonempty compact subset of a topological space $\mathcal{X}$, and let $F : X \rightrightarrows \mathcal{Z}$ be an upper semi-continuous set-valued mapping with nonempty compact values. Then, for any $\xi \in C^*$, there exists $y \in \operatorname{Max}_w F(X)$ such that $\xi(y) = \max \bigcup_{x \in X} \xi(F(x))$.*

Proof. For any given $\xi \in C^*$, the mapping $x \rightrightarrows \xi(F(x))$ is upper semi-continuous by Proposition 3.8 (c). By the compactness of X, there exist $x_0 \in X$ and $y_0 \in F(x_0)$ such that $\xi(y_0) = \max \bigcup_{x \in X} \xi(F(x))$. By Lemma 2.2, there exists $y \in \operatorname{Max}_w \bigcup_{x \in X} F(x)$ such that $y_0 - y \in -C$, and hence $\xi(y) \geq \xi(y_0)$. On the other hand, $y \in \operatorname{Max}_w \bigcup_{x \in X} F(x) \subset F(X)$, we know that $\xi(y) \in \xi(F(X))$, and then $\xi(y) \leq \max \bigcup_{x \in X} \xi(F(x)) = \xi(y_0)$. Therefore, the conclusion holds. □

Proposition 3.11. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathcal{Z}$ is above-properly C-quasi-convex, then it is above-C-quasi-convex.*

Proof. For any $z \in \mathcal{Z}$, let $x_1, x_2 \in \operatorname{Lev}_{F\leq}(z)$. Then, $F(x_1)$ and $F(x_2)$ are subsets of $z - C$. Since F is above-properly C-quasi-convex, for any $\lambda \in [0,1]$, $F(\lambda x_1 + (1-\lambda)x_2)$ is contained in either $F(x_1) - C$ or $F(x_2) - C$, and hence, in $z - C$. Thus, the set $\operatorname{Lev}_{F\leq}(z)$ is convex, and therefore, F is above-C-quasi-convex. □

Proposition 3.12. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathcal{Z}$ is above-naturally C-quasi-convex, then it is above-C-quasi-convex.*

Proof. Let z, x_1, and x_2 be the same as given as in Proposition 3.11. Then, $\operatorname{co}\{F(x_1) \cup F(x_2)\} \subset z - C$ since $z - C$ is convex. By the above-naturally C-quasi-convexity, $F(\lambda x_1 + (1-\lambda)x_2)\} \subset z - C$ for all $\lambda \in [0,1]$. Thus, the set $\operatorname{Lev}_{F\leq}(z)$ is convex, and therefore, F is above-C-quasi-convex. □

Proposition 3.13. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathcal{Z}$ is above-naturally C-quasi-convex, then $\xi \circ F$ is above-naturally $\mathbb{R}_+$-quasi-convex for any $\xi \in C^*$.*

Proof. Let $\xi \in C^*$ be given. From the above-naturally C-quasi-convexity of F, for any $x_1, x_2 \in X$ and any $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \operatorname{co}\{F(x_1) \cup F(x_2)\} - C. \tag{3.22}$$

For any $y \in F(\alpha x_1 + (1-\alpha)x_2)$, there is a $w \in \operatorname{co}\{F(x_1) \cup F(x_2)\}$ such that $y \in w - C$. Then there exist $y_i \in F(x_1) \cup F(x_2)$ and $\lambda_i \in [0,1]$, $1 \leq i \leq n$ such that $w = \sum_{i=1}^{n} \lambda_i y_i$. Hence, $\xi(w) = \sum_{i=1}^{n} \lambda_i \xi(y_i)$, and

$$\xi(y) \in \xi(w) - \mathbb{R}_+ = \sum_{i=1}^{n} \lambda_i \xi(y_i) - \mathbb{R}_+ \subset \operatorname{co}\{\xi(F(x_1)) \cup \xi(F(x_2))\} - \mathbb{R}_+. \tag{3.23}$$

Therefore, $\xi \circ F$ is a above-naturally $\mathbb{R}_+$-quasi-convex. □

Proposition 3.14. *Let $F : X \rightrightarrows \mathcal{Z}$ be a set-valued mapping with nonempty compact values. For any $\xi \in C^*$,*

(a) *if $\xi(d) = \min \bigcup_{x\in X} \xi(F(x))$ for some $d \in \mathcal{Z}$, then $d \in \operatorname{Min}_w \bigcup_{x\in X} F(x)$;*

(b) *if $\xi(e) = \max \bigcup_{x\in X} \xi(F(x))$ for some $e \in \mathcal{Z}$, then $e \in \operatorname{Max}_w \bigcup_{x\in X} F(x)$.*

Proof. Let $\xi(d) = \min \bigcup_{x\in X} \xi(F(x))$. Suppose that $d \notin \operatorname{Min}_w \bigcup_{x\in X} F(x)$. Then

$$\left(\bigcup_{x\in X} F(x)\right) \bigcap (d - \operatorname{int} C) \neq \emptyset. \tag{3.24}$$

Then, there exists $w \in \bigcup_{x\in X} F(x)$ and $w \in d - \operatorname{int} C$. Therefore, there exists $s \in X$ such that $w \in F(s)$ and $d - w \in \operatorname{int} C$. Since $\xi \in C^*$, $\xi(d) > \xi(w)$ and $\xi(w) \geq \min \bigcup_{x\in X} \xi(F(x))$. This implies that $\xi(d) > \min \bigcup_{x\in X} \xi(F(x))$, which is a contradiction. This proves (a).

Analogously, we can prove (b), so we omit it. □

Remark 3.15. Propositions 3.8 and 3.9, Lemma 3.10, and Propositions 3.13 and 3.14 are always true except Proposition 3.8 (b) if we replace ξ by any Gerstewitz function.

4. Minimax Theorems for Set-Valued Mappings

In this section, we establish some minimax theorems for set-valued mappings with or without linear structure.

Theorem 4.1. *Let X, Y be two nonempty compact subsets (not necessarily convex) of real Hausdorff topological spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ be lower semi-continuous on X and upper semi-continuous on Y such that for all $(x,y) \in X \times Y$, $F(x,y)$ is nonempty compact and satisfies the following conditions:*

(i) *for each $x \in X$, $y \mapsto F(x,y)$ is below-$\mathbb{R}_+$-concave-like on Y;*

(ii) *for each $y \in Y$, $x \mapsto F(x,y)$ is above-$\mathbb{R}_+$-convex-like on X.*

Then,

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) = \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.1}$$

Proof. Since

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \leq \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y), \tag{4.2}$$

it is sufficient to prove that

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \geq \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.3}$$

Choose any $\alpha \in \mathbb{R}$ such that $\alpha < \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y)$. For any $y \in Y$, let

$$\operatorname{Lev}_{F\leq}(y;\alpha) = \{x \in X : F(x,y) \subset \alpha - \mathbb{R}_+\}. \tag{4.4}$$

Then, by the lower semi-continuity of the set-valued mapping $x \mapsto F(x,y)$, the set $\operatorname{Lev}_{F\leq}(y;\alpha)$ is closed, hence it is compact for all $y \in Y$. By the choice of α, we have

$$\bigcap_{y\in Y} \operatorname{Lev}_{F\leq}(y;\alpha) = \emptyset. \tag{4.5}$$

Since X is compact and the collection $\{X \setminus \operatorname{Lev}_{F\leq}(y;\alpha) : y \in Y\}$ covers X, there exist finite number of points $y_1, y_2, \ldots, y_m$ in Y such that

$$X \subset \bigcup_{i=1}^{m} (X \setminus \operatorname{Lev}_{F\leq}(y_i;\alpha)) \tag{4.6}$$

or

$$\bigcap_{i=1}^{m} \operatorname{Lev}_{F\leq}(y_i;\alpha) = \emptyset. \tag{4.7}$$

This implies that

$$\max \bigcup_{i=1}^{m} F(x,y_i) > \alpha, \quad \forall x \in X, \tag{4.8}$$

and therefore,

$$\min \bigcup_{x\in X} \max \bigcup_{i=1}^{m} F(x,y_i) > \alpha. \tag{4.9}$$

Following the idea of Borwein and Zhuang [14], let

$$\mathfrak{E} := \left\{(\mathbf{z},r) \in \mathbb{R}^{m+1} : \text{there is } x \in X, F(x,y_i) \subset r + z_i - \mathbb{R}_+, i = 1,2,\ldots,m\right\}, \tag{4.10}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_m)$. Then the set $\mathfrak{C}$ is convex, so is $\operatorname{int}\mathfrak{C}$. We note that the interior $\operatorname{int}\mathfrak{C}$ of $\mathfrak{C}$ is nonempty since

$$\left(\mathbf{0}, 1 + \max \bigcup_{i=1}^{m} F(x, y_i)\right) \in \operatorname{int}\mathfrak{C}, \quad \forall x \in X. \tag{4.11}$$

Since $(\mathbf{0}, \alpha) \notin \mathfrak{C}$, by separation hyperplane theorem [15, Theorem 14.2], there is a $(\Xi, \epsilon) \neq \mathbf{0} \times \{0\}$ such that

$$\langle (\Xi, \varepsilon), (\mathbf{z}, r) \rangle \geq \langle (\Xi, \varepsilon), (\mathbf{0}, \alpha) \rangle, \quad \forall (\mathbf{z}, r) \in \mathfrak{C}, \tag{4.12}$$

where $\Xi = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, that is,

$$\Xi\mathbf{z} + \varepsilon \mathrm{r} \geq \varepsilon\alpha, \quad \forall (\mathbf{z}, r) \in \mathfrak{C}. \tag{4.13}$$

By (4.11), (4.13), and the choice of α, we have that $\varepsilon > 0$. Furthermore, from the fact

$$\prod_{i=1}^{m} (F(x, y_i) + r) \times \{-r\} \subset \mathfrak{C}, \tag{4.14}$$

we have

$$(\eta_{x,1} + r, \eta_{x,2} + r, \ldots, \eta_{x,m} + r, -r) \in \mathfrak{C}, \quad \forall \eta_{x,i} \in F(x, y_i). \tag{4.15}$$

Hence, by (4.13), we have

$$\sum_{i=1}^{m} \lambda_i (\eta_{x,i} + r) + \varepsilon(-r) \geq \varepsilon\alpha \tag{4.16}$$

or

$$\sum_{i=1}^{m} \left(\frac{\lambda_i}{\varepsilon}\right) \eta_{x,i} + \left(\frac{\sum_{i=1}^{m} \lambda_i}{\varepsilon} - 1\right) r \geq \alpha, \quad \forall x \in X,\ r \in \mathbb{R}. \tag{4.17}$$

Thus, we have $\sum_{i=1}^{m} (\lambda_{\mathrm{i}}/\varepsilon) = 1$. Hence, by (4.17), we have

$$\sum_{i=1}^{m} \left(\frac{\lambda_i}{\varepsilon}\right) F(x, y_i) \subset \alpha + \mathbb{R}_+. \tag{4.18}$$

Since $F(x, y)$ is below-$\mathbb{R}_+$-concave-like in y, there is $y' \in Y$ such that

$$F(x, y') \subset \sum_{i=1}^{m} \left(\frac{\lambda_i}{\varepsilon}\right) F(x, y_i) + \mathbb{R}_+, \quad \forall x \in X. \tag{4.19}$$

Therefore,

$$\bigcup_{x\in X} F(x,y') \subset \alpha + \mathbb{R}_+, \tag{4.20}$$

and hence,

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \geq \alpha. \tag{4.21}$$

This completes the proof. □

Remark 4.2. Theorem 4.1 is a modification of [14, Theorem A]. If F is a real-valued function, then Theorem 4.1 reduces to the well-known minimax theorem due to Fan [2].

We next establish a minimax theorem for set-valued mappings defined on the sets with linear structure.

Theorem 4.3. *Let X, Y be two nonempty compact convex subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ be lower semi-continuous on X and upper semi-continuous on Y such that for all $(x,y) \in X \times Y$, $F(x,y)$ is nonempty compact, and satisfies the following conditions:*

(i) *for each $y \in Y$, $x \mapsto F(x,y)$ is above-$\mathbb{R}_+$-quasi-convex on X;*

(ii) *for each $x \in X$, $y \mapsto F(x,y)$ is above-$\mathbb{R}_+$-concave, or above-properly $\mathbb{R}_+$-quasi-concave on Y;*

(iii) *for each $y \in Y$, there is a $x_y \in Y$ such that*

$$\max F(x_y,y) \leq \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y). \tag{4.22}$$

Then,

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) = \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y). \tag{4.23}$$

Proof. We only need to prove that

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) < \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) \tag{4.24}$$

is impossible, since it is always true that

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \leq \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.25}$$

Suppose that there is an $\alpha \in \mathbb{R}$ such that

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) < \alpha < \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.26}$$

Define $G : X \times Y \rightrightarrows X \times Y$ by

$$G(x,y) = \{s \in X : \max F(s,y) < \alpha\} \times \{t \in Y : \max F(x,t) > \alpha\}. \tag{4.27}$$

For each $x \in X$, $\max \bigcup_{y\in Y} F(x,y) \geq \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) > \alpha$. Since Y is compact and the set-valued mapping $y \mapsto \max F(x,y)$ is upper semi-continuous, there is a $t \in Y$ such that $\max F(x,t) = \max \bigcup_{y\in Y} F(x,y) > \alpha$.

On the other hand, from the condition (iii), for each $y \in Y$, there is a $x_y \in Y$ such that $\max F(x_y, y) < \alpha$. Hence, for each $(x,y) \in X \times Y$, $G(x,y) \neq \emptyset$. By (i) and Proposition 3.6, the mapping $x \rightarrow \max F(x,y)$ is above-$\mathbb{R}_+$-quasi-convex on X. By (ii) and Proposition 3.7, the mapping $y \rightarrow \max F(x,y)$ is below-$\mathbb{R}_+$-quasi-concave on y. Hence, for each $(x,y) \in X \times Y$, the set $G(x,y)$ is convex. From the lower semi-continuities on X and upper semi-continuity on Y of F, the set

$$G^{-1}(s,t) = \{x \in X : \max F(x,t) > \alpha\} \times \{y \in Y : \max F(s,y) < \alpha\} \tag{4.28}$$

is open in $X \times Y$. By Fan-Browder fixed-point Theorem 2.8, there exists $(\overline{x}, \overline{y}) \in X \times Y$ such that

$$(\overline{x}, \overline{y}) \in G(\overline{x}, \overline{y}), \tag{4.29}$$

that is,

$$\max F(\overline{x}, \overline{y}) > \alpha > \max F(\overline{x}, \overline{y}), \tag{4.30}$$

which is a contradiction. This completes the proof. □

Remark 4.4. [5, Propositions 2.7 and 2.1] can be deduced from Theorem 4.3. Indeed, in [5, Proposition 2.1], the above-naturally C-quasi-convexity is used. By Proposition 3.12, the condition (i) of Theorem 4.3 holds. Hence the conclusion of Proposition 2.1 in [5] holds. We also note that, in Theorem 4.3, the mapping F need not be continuous on $X \times Y$. Hence Theorem 4.3 is a slight generalization of [7, Theorem 3.1].

Theorem 4.5. *Let X and Y be nonempty compact (not necessarily convex) subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the mapping $F : X \times Y \rightrightarrows \mathcal{Z}$ be upper semi-continuous with nonempty compact values and lower semi-continuous on X such that*

(i) *for each $x \in X$, $y \rightarrow F(x,y)$ is below-C-concave-like on Y;*

(ii) *for each $y \in Y$, $x \rightarrow F(x,y)$ is above-C-convex-like on X;*

(iii) *for every* $y \in Y$,

$$\operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w \bigcup_{x \in X} F(x, y) \subset \operatorname{Min}_w \bigcup_{x \in X} F(x, y) + C. \tag{4.31}$$

Then for any

$$z_1 \in \operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w \bigcup_{x \in X} F(x, y), \tag{4.32}$$

there is a

$$z_2 \in \operatorname{Min}\left(\operatorname{co}\left\{ \bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) \tag{4.33}$$

such that

$$z_1 \in z_2 + C, \tag{4.34}$$

that is,

$$\operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w \bigcup_{x \in X} F(x, y) \subset \operatorname{Min}\left(\operatorname{co}\left\{ \bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) + C. \tag{4.35}$$

Proof. Let $\Gamma(x) := \operatorname{Max}_w \bigcup_{y \in Y} F(x, y)$ for all $x \in X$. From Lemma 2.4 and Proposition 3.5, the set-valued mapping $x \mapsto \Gamma(x)$ is upper semi-continuous with nonempty compact values on X. Hence the set $\Gamma(X)$ is compact, and so is $\operatorname{co}\{\Gamma(X)\}$. Then $\operatorname{co}\{\Gamma(X)\}+C$ is a closed convex set with nonempty interior. Suppose that $v \notin \operatorname{co}\{\Gamma(X)\} + C$. By separation hyperplane theorem [15, Theorem 14.2], there exist $k \in \mathbb{R}$, $\varepsilon > 0$ and a nonzero continuous linear functional $\xi : Z \to \mathbb{R}$ such that

$$\xi(v) \leq k - \varepsilon < k \leq \xi(u + c), \quad \text{for every } u \in \operatorname{co}\{\Gamma(X)\},\ c \in C. \tag{4.36}$$

Therefore,

$$\xi(c) > \xi(v - u), \quad \text{for every } u \in \operatorname{co}\{\Gamma(X)\},\ c \in C. \tag{4.37}$$

This implies that $\xi \in C^*$ and $\xi(v) < \xi(u)$ for all $u \in \operatorname{co}\{\Gamma(X)\}$.

Let $g := \xi F : X \times Y \rightrightarrows \mathbb{R}$. From Lemma 3.10, for each fixed $x \in X$, there exist $y_x^* \in Y$ and $f(x, y_x^*) \in F(x, y_x^*)$ with $f(x, y_x^*) \in \Gamma(x)$ such that $\xi(f(x, y_x^*)) = \max \bigcup_{y \in Y} \xi(F(x, y))$. Choosing $c = 0$ and $u = f(x, y_x^*)$ in (4.36), we have

$$\max \bigcup_{y \in X} \xi(F(x, y)) = \xi f(x, y_x^*) \geq k > k - \varepsilon \geq \xi(v), \quad \forall x \in X. \tag{4.38}$$

Therefore,

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} \xi(F(x,y)) > \xi(v). \tag{4.39}$$

By the conditions (i), (ii) and Proposition 3.8, the set-valued mapping $y \mapsto \xi(F(x,y))$ is below-$\mathbb{R}_+$-concave-like on Y for all $x \in X$, and the set-valued mapping $x \mapsto \xi(F(x,y))$ is above-$\mathbb{R}_+$-convex-like on X for all $y \in Y$. From Theorem 4.1, we have

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} \xi(F(x,y)) > \xi(v). \tag{4.40}$$

Since Y is compact, there is an $y' \in Y$ such that $\min \bigcup_{x\in X} \xi(F(x,y')) > \xi(v)$. For any $x \in X$ and all $g(x,y') \in F(x,y')$, we have

$$\xi(g(x,y')) > \xi(v), \tag{4.41}$$

that is,

$$\xi(g(x,y') - v) > 0, \quad \forall x \in X,\ g(x,y') \in F(x,y'). \tag{4.42}$$

Thus, $v \notin \bigcup_{x\in X} F(x,y') + C$, and hence,

$$v \notin \mathrm{Min}_w \bigcup_{x\in X} F(x,y') + C. \tag{4.43}$$

If $v \in \mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y)$, by the condition (iii), $v \in \mathrm{Min}_w \bigcup_{x\in X} F(x,y') + C$ which contradicts (4.43). Hence, for every $v \in \mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y)$,

$$v \in \mathrm{co}\left\{ \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) \right\} + C, \tag{4.44}$$

that is,

$$\mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) \subset \mathrm{co}\left\{ \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) \right\} + C \tag{4.45}$$

or

$$\mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) \subset \mathrm{Min}\left(\mathrm{co}\left\{ \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) \right\} \right) + C. \tag{4.46}$$

□

The following examples illustrate Theorem 4.5.

Example 4.6. Let $X = Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$, $C = \mathbb{R}^2_+$ and

$$F(x, y) = \left\{(s, t) \in \mathbb{R}^2 : s = x^2, t = 1 - y^2\right\}, \quad \forall (x, y) \in X \times Y. \tag{4.47}$$

It is obviously that F is below-$\mathbb{R}^2_+$-concave-like on Y and above-$\mathbb{R}^2_+$-convex-like on X. We now verify the condition (iii) of Theorem 4.5. Indeed, for any $y \in Y$,

$$\begin{gathered}\bigcup_{x \in X} F(x, y) = \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left\{1 - y^2\right\}, \\ \mathrm{Min}_w \bigcup_{x \in X} F(x, y) = \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left\{1 - y^2\right\}.\end{gathered} \tag{4.48}$$

Then,

$$\begin{gathered}\bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F(x, y) = \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left(\{1\} \cup \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}\right), \\ \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F(x, y) = \{(1, 1)\}.\end{gathered} \tag{4.49}$$

Thus, for every $y \in Y$,

$$\begin{aligned}\mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F(x, y) &\subset \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left\{1 - y^2\right\} + C \\ &= \mathrm{Min}_w \bigcup_{x \in X} F(x, y) + C,\end{aligned} \tag{4.50}$$

and the condition (iii) of Theorem 4.5 holds.

Furthermore, for any $x \in X$,

$$\begin{gathered}\bigcup_{y \in Y} F(x, y) = \left\{x^2\right\} \times \left(\{1\} \cup \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}\right), \\ \mathrm{Max}_w \bigcup_{y \in Y} F(x, y) = \left\{x^2\right\} \times \left(\{1\} \cup \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}\right).\end{gathered} \tag{4.51}$$

Then,

$$\begin{gathered}\bigcup_{x \in X} \mathrm{Max}_w \bigcup_{y \in Y} F(x, y) = \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left(\{1\} \cup \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}\right), \\ \mathrm{co}\left\{\bigcup_{x \in X} \mathrm{Max}_w \bigcup_{y \in Y} F(x, y)\right\} = [0, 1] \times [0, 1].\end{gathered} \tag{4.52}$$

Thus,

$$\begin{aligned}
&\operatorname{Min}\left(\operatorname{co}\left\{\bigcup_{x\in X}\operatorname{Max}_w\bigcup_{y\in Y}F(x,y)\right\}\right)=\{(0,0)\},\\
&\operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=\{(1,1)\}\subset\operatorname{Min}\left(\operatorname{co}\left\{\bigcup_{x\in X}\operatorname{Max}_w\bigcup_{y\in Y}F(x,y)\right\}\right)+C.
\end{aligned}\tag{4.53}$$

Hence, the conclusion of Theorem 4.5 holds.

Example 4.7. Let $X=[0,1]$, $Y=[-1,0]$, $C=\mathbb{R}_+^2$, and $G:Y\rightrightarrows Y$ be defined by

$$G(y)=\begin{cases}[-1,0], & y=0,\\ \{0\}, & y\neq 0.\end{cases}\tag{4.54}$$

Let $F(x,y)=\{x^2\}\times G(y)$ for all $(x,y)\in X\times Y$. Then G is upper semi-continuous, but not lower semi-continuous on $\mathbb{R}$, and F is not continuous but is upper semi-continuous on $X\times Y$. Moreover, F has nonempty compact values and is lower semi-continuous on X. It is easy to see that F is below-C-concave-like on Y and is above-C-convex-like on X. We verify the condition (iii) of Theorem 4.5. Indeed, for all $y\in Y$, $\bigcup_{x\in X}F(x,y)=[0,1]\times G(y)$.

$$\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=\begin{cases}[0,1]\times\{0\}, & y\neq 0,\\ (\{0\}\times[-1,0])\cup([0,1]\times\{-1\}), & y=0.\end{cases}\tag{4.55}$$

Then,

$$\begin{aligned}
&\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=(\{0\}\times[-1,0])\cup([0,1]\times\{-1\})\cup([0,1]\times\{0\}),\\
&\operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=\{(1,0)\}\subset\operatorname{Min}_w\bigcup_{x\in X}F(x,y)+C.
\end{aligned}\tag{4.56}$$

Therefore, the condition (iii) of Theorem 4.5 holds.

Since

$$F(x,y)=\begin{cases}\{x^2\}\times[-1,0], & y=0,\\ \{x^2\}\times\{0\}, & y\neq 0,\end{cases}\tag{4.57}$$

for all $(x,y)\in X\times Y$, and $\operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=\{(1,0)\}$, for each $y\in Y$, we can choose $x_y=0\in X$ such that

$$\operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)\subset F(x_y,y)+C.\tag{4.58}$$

Furthermore,

$$\begin{aligned}\bigcup_{y\in Y} F(x,y) &= \left\{x^2\right\} \times \left(\bigcup_{y\in Y} G(y)\right)\\ &= \left\{x^2\right\} \times ([-1,0] \cup \{0\})\\ &= \left\{x^2\right\} \times [-1,0],\\ \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) &= [0,1] \times [-1,0].\end{aligned} \tag{4.59}$$

Therefore,

$$\begin{aligned}\mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) &= \{(1,0)\} \subset \{(0,-1)\} + C\\ &= \mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) + C.\end{aligned} \tag{4.60}$$

Hence, the conclusion of Theorem 4.5 holds.

Remark 4.8. Theorem 3.1 in [5] Theorem 3.1 in [6], or Theorem 4.2 in [7] cannot be applied to Examples 4.6 and 4.7 because of the following reasons:

(i) the two sets X and Y are not convex in Example 4.6;

(ii) F is not continuous on $X \times Y$ in Examples 4.6 and 4.7.

Theorem 4.9. *Let X, Y be two nonempty compact convex subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathfrak{Z}$ has nonempty compact values, and it is continuous on Y and lower semi-continuous on X such that*

(i) *for each $y \in Y$, $x \mapsto F(x,y)$ is above-naturally C-quasi-convex on X;*

(ii) *for each $x \in X$, $y \mapsto F(x,y)$ is above-C-concave or above-properly C-quasi-concave on Y;*

(iii) *for every $y \in Y$,*

$$\mathrm{Max} \bigcup_{y\in Y} \mathit{Min}_w \bigcup_{x\in X} F(x,y) \subset \mathit{Min}_w \bigcup_{x\in X} F(x,y) + C; \tag{4.61}$$

(iv) for any continuous increasing function h and for each $y \in Y$, there exists $x_y \in X$ such that

$$\max h(F(x_y,y)) \le \max \bigcup_{y\in Y} \min \bigcup_{x\in X} h(F(x,y)). \tag{4.62}$$

Then, for any $z_1 \in \operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w \bigcup_{x \in X} F(x, y)$*, there is a*

$$z_2 \in \operatorname{Min} \bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in Y} \mathrm{F}(\mathrm{x}, \mathrm{y}) \tag{4.63}$$

such that $z_1 \in z_2 + C$*, that is,*

$$\operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w \bigcup_{x \in X} \mathrm{F}(x, y) \subset \operatorname{Min} \bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in Y} \mathrm{F}(x, y) + C. \tag{4.64}$$

Proof. Let $\Gamma(x)$ be defined as the same as in the proof of Theorem 4.5. Following the same perspective as in the proof of Theorem 4.5, suppose that $v \notin \bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in Y} F(x, y) + C$. For any $k \in \operatorname{int} C$ and Gerstewitz function $\xi_{\mathrm{kv}} : \mathfrak{Z} \rightrightarrows \mathbb{R}$. By Proposition 2.7(d), we have

$$\xi_{kv}(u) > 0, \quad \text{for every } u \in \Gamma(\mathrm{X}). \tag{4.65}$$

Let $g := \xi_{kv} \circ F : X \times Y \rightrightarrows \mathbb{R}$. From Lemma 3.10, for the mapping ξ_{kv} and Remark 3.15, for each $x \in \mathrm{X}$, there exist $y_x^* \in Y$and $f(x, y_x^*) \in F(x, y_x^*)$ with $f(x, y_x^*) \in \operatorname{Max}_w \bigcup_{y \in Y} F(x, y)$ such that $\xi_{kv} f(x, y_x^*) = \max \bigcup_{y \in Y} \xi_{kv}(F(x, y))$. Choosing $u = f(x, y_x^*)$ in (4.65), we have

$$\max \bigcup_{y \in Y} \xi_{kv}(F(x, y)) > 0, \quad \forall x \in \mathrm{X}. \tag{4.66}$$

Therefore,

$$\min \bigcup_{x \in X} \max \bigcup_{y \in Y} \xi_{kv}(F(x, y)) > 0. \tag{4.67}$$

By conditions (i), (ii) and Remark 3.15, the set-valued mapping $y \mapsto \xi_{kv}(F(x, y))$ is upper semi-continuous, and either above-$\mathbb{R}_+$-concave or above-properly $\mathbb{R}_+$-quasi-concave on Y, and the set-valued mapping $x \mapsto \xi_{kv}(F(x, y))$ is lower semi-continuous and above-$\mathbb{R}_+$-quasi-convex on X. From Theorem 4.3, we have

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} \xi_{kv}(F(x, y)) > 0. \tag{4.68}$$

Since the set-valued mapping $y \mapsto F(x, y)$ is lower semi-continuous on Y, by Lemma 2.4 (b) and Lemma 2.5 (b), the set-valued mapping $y \mapsto \min \bigcup_{x \in X} \xi_{kv}(F(x, y))$ is upper semi-continuous on Y. By the compactness of Y, there exists $y' \in Y$ such that $\min \bigcup_{x \in X} \xi_{kv}(F(x, y')) > 0$. For all $x \in X$ and all $g(x, y') \in F(x, y')$, we have $\xi_{kv}(g(x, y')) > 0$. Thus, $v \notin \bigcup_{x \in X} F(x, y') + C$, and hence,

$$v \notin \operatorname{Min}_w \bigcup_{x \in X} F(x, y') + \mathrm{C}. \tag{4.69}$$

If $v \in \mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F(x,y)$, by the condition (iii), $v \in \mathrm{Min}_w\bigcup_{x\in X}F(x,y') + C$ which contradicts (4.69). Hence, for every $v \in \mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F(x,y)$,

$$v \in \mathrm{Min}\bigcup_{x\in X}\mathrm{Max}_w\bigcup_{y\in Y}F(x,y) + C, \tag{4.70}$$

that is,

$$\mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F(x,y) \subset \mathrm{Min}\bigcup_{x\in X}\mathrm{Max}_w\bigcup_{y\in Y}F(x,y) + C. \tag{4.71}$$

This completes the proof. □

The following example illustrates Theorem 4.9.

Example 4.10. Let $X = Y = [0,1]$, $C = \mathbb{R}^2_+$ and $G : X \rightrightarrows Y$ be a set-valued mapping defined as

$$G(x) = \begin{cases} [0,1], & x \neq 0, \\ \{0\}, & x = 0. \end{cases} \tag{4.72}$$

Let $F(x,y) = G(x) \times \{-y^2\}$ for all $(x,y) \in X \times Y$. Then G is lower semi-continuous, but not upper semi-continuous on $\mathbb{R}$, and F is continuous on Y, and F has nonempty compact values and is lower semi-continuous on X. It is easy to see that F is above-C-concave or above-properly C-quasi-concave on Y and is above-naturally C-quasi-convex on X.

We verify the condition (iii) of Theorem 4.9. Indeed, for all $y \in Y$, $\bigcup_{x\in X}F(x,y) = [0,1] \times \{-y^2\}$ and $\mathrm{Min}_w\bigcup_{x\in X}F(x,y) = [0,1] \times \{-y^2\}$. Hence,

$$\begin{gathered}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F(x,y) = [0,1] \times [-1,0], \\ \mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F(x,y) = \{(1,0)\} \subset \mathrm{Min}_w\bigcup_{x\in X}F(x,y) + C.\end{gathered} \tag{4.73}$$

Therefore, the condition (iii) of Theorem 4.9 holds.

Since $\mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F(x,y) = \{(1,0)\}$ for any $y \in Y$, we can choose $x_y = 0 \in X$ such that

$$F(x_y,y) = \left\{\left(0,-y^2\right)\right\} \subset \mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F(x,y) - C. \tag{4.74}$$

For any continuous increasing function h, the condition (iv) of Theorem 4.9 holds.

Furthermore, since for each $x \in X$,

$$\bigcup_{y\in Y} F(x,y) = G(x) \times [-1,0],$$
$$\mathrm{Max}_w \bigcup_{y\in Y} F(x,y) = \begin{cases} \{0\} \times [-1,0], & x = 0, \\ (\{1\} \times [-1,0]) \bigcup ([0,1] \times \{0\}), & x \neq 0, \end{cases} \tag{4.75}$$

we have

$$\bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) = (\{0\} \times [-1,0]) \bigcup ([0,1] \times \{0\}) \bigcup (\{1\} \times [-1,0]),$$
$$\mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) = \{(0,-1)\}. \tag{4.76}$$

Thus,

$$\begin{aligned} \mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) &= \{(1,0)\} \subset \{(0,-1)\} + C \\ &= \mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) + C. \end{aligned} \tag{4.77}$$

Therefore, the conclusion of Theorem 4.9 holds.

Remark 4.11. Theorem 3.1 in [5], Theorem 3.1 in [6], or Theorem 4.2 in [7] cannot be applied to Example 4.10 as F is not continuous on $X \times Y$.

If we choose $Z = \mathbb{R}$ and $C = \mathbb{R}_+$ in Theorems 4.5 and 4.9, we always have $C^* = \mathbb{R}_+$ and for every $y \in Y$,

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \geq \min \bigcup_{x\in X} F(x,y). \tag{4.78}$$

Hence, the condition (iii) of Theorem 4.5 holds. Thus, we have the following corollaries.

Corollary 4.12. *Let X, Y be nonempty compact (not necessarily convex) subsets of real Hausdorff topological vector space $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ has nonempty compact values such that it is lower semi-continuous on X and is upper semi-continuous on $X \times Y$. Assume that the following conditions hold:*

(i) *for each $x \in X$, $y \mapsto F(x,y)$ is below-$\mathbb{R}_+$-concave-like on Y;*

(ii) *for each $y \in Y$, $x \mapsto F(x,y)$ is above-$\mathbb{R}_+$-convex-like on X;*

(iii) *for every $y \in Y$,*

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \geq \min \bigcup_{x\in X} F(x,y). \tag{4.79}$$

Then, for any

$$z_1 \in \max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y), \tag{4.80}$$

there is a

$$z_2 \in \min\left(\operatorname{co}\left\{ \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \right\}\right) \tag{4.81}$$

such that

$$z_1 \geq z_2, \tag{4.82}$$

that is,

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) \geq \min\left(\operatorname{co}\left\{ \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \right\}\right). \tag{4.83}$$

Corollary 4.13. *Under the framework of Corollary 4.12, in addition, let X, Y be two convex subsets, and let F be upper semi-continuous on $X \times Y$. Then,*

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) = \min \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y). \tag{4.84}$$

Proof. By Corollary 4.12, we have

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) \geq \min\left(\operatorname{co}\left\{ \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \right\}\right). \tag{4.85}$$

Since the set-valued mapping F is upper semi-continuous on $X \times Y$ and Y is compact, by Lemmas 2.4 and 2.5, the set-valued mapping $x \mapsto \max \bigcup_{y \in Y} F(x, y)$ is upper semi-continuous on X. Since X is convex, it is connected. By [16, Theorem 3.1],

$$\bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \tag{4.86}$$

is connected in $\mathbb{R}$, and hence, it is convex. From (4.85),

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) \geq \min\left(\bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \right). \tag{4.87}$$

This completes the proof. □

When $Z = \mathbb{R}$ and $C = \mathbb{R}_+$, from Theorem 4.9, we deduce the following corollary.

Corollary 4.14. *Let X, Y be two nonempty compact convex subsets in real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ has nonempty compact values such that it is continuous on Y and is lower semi-continuous on X. Assume that the following conditions hold:*

(i) *for each $y \in Y$, $x \to F(x,y)$ is above-naturally $\mathbb{R}_+$-quasi-convex on X;*

(ii) *for each $x \in X$, $y \to F(x,y)$ is above-$\mathbb{R}_+$-concave or above-properly $\mathbb{R}_+$-quasi-concave on Y;*

(iii) *for each $y \in Y$, there exists $x_y \in X$ such that*

$$\max F(x_y, y) \leq \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y). \tag{4.88}$$

Then,

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) = \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.89}$$

Remark 4.15. Corollary 4.14 includes Proposition 2.1 in [5].

5. Saddle Points for Set-Valued Mappings

In this section, we discuss the existence of several kinds of saddle points for set-valued mappings including the C-loose saddle points, weak C-saddle points, $\mathbb{R}_+$-loose saddle points, and R_+-saddle points of F on $X \times Y$.

Definition 5.1. Let $\mathrm{F} : \mathrm{X} \times \mathrm{Y} \rightrightarrows \mathcal{Z}$ be a set-valued mapping. A point $(\overline{x}, \overline{y}) \in X \times Y$ is said to be a

(a) C-loose saddle point [7] of F on $X \times Y$ if

$$\begin{gathered} F(\overline{x},\overline{y}) \bigcap \left(\operatorname{Max} \bigcup_{y\in Y} F(\overline{x},y) \right) \neq \emptyset, \\ F(\overline{x},\overline{y}) \bigcap \left(\operatorname{Min} \bigcup_{x\in X} F(x,\overline{y}) \right) \neq \emptyset; \end{gathered} \tag{5.1}$$

(b) weak C-saddle point [7] of F on $X \times Y$ if

$$F(\overline{x},\overline{y}) \bigcap \left(\operatorname{Max}_w \bigcup_{y\in Y} F(\overline{x},y) \right) \bigcap \left(\operatorname{Min}_w \bigcup_{x\in X} F(x,\overline{y}) \right) \neq \emptyset; \tag{5.2}$$

(c) $\mathbb{R}_+$-*loose saddle point* of F on $X \times Y$ if $Z = \mathbb{R}$ and

$$F(\overline{x},\overline{y}) = \left[\min \bigcup_{x\in X} F(x,\overline{y}), \max \bigcup_{y\in Y} F(\overline{x},y)\right]; \tag{5.3}$$

(d) $\mathbb{R}_+$-*saddle point* of F on $X \times Y$ if $Z = \mathbb{R}$ and

$$\max \bigcup_{y\in Y} F(\overline{x},y) = \min \bigcup_{x\in X} F(x,\overline{y}) = F(\overline{x},\overline{y}). \tag{5.4}$$

It is obvious that every weak C-saddle point is a C-loose saddle point and every $\mathbb{R}_+$-saddle point is a $\mathbb{R}_+$-loose saddle point.

Theorem 5.2. *Under the framework of Theorem 4.1, F has $\mathbb{R}_+$-saddle point if the set-valued mapping $y \mapsto F(x,y)$ is continuous.*

Proof. By Lemmas 2.4 and 2.5, we attained the max and min in Theorem 4.1. By the compactness of X and Y and the lower semi-continuity of F on X and Y, respectively, there exists $(\overline{x},\overline{y}) \in X \times Y$ such that

$$\begin{aligned} \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) &= \min \bigcup_{x\in X} F(x,\overline{y}), \\ \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) &= \max \bigcup_{y\in Y} F(\overline{x},y). \end{aligned} \tag{5.5}$$

Combining this with Theorem 4.1, we have

$$\max \bigcup_{y\in Y} F(\overline{x},y) = \min \bigcup_{x\in X} F(x,\overline{y}) = F(\overline{x},\overline{y}), \tag{5.6}$$

and hence, F has $\mathbb{R}_+$-saddle point. □

Theorem 5.3. *Under the framework of Theorem 4.3, F has $\mathbb{R}_+$-saddle point if the set-valued mapping $y \mapsto F(x,y)$ is continuous.*

Theorem 5.4. *Under the framework of Theorem 4.5 or Theorem 4.9, F has weak C-saddle point if the set-valued mapping $y \mapsto F(x,y)$ is continuous.*

Proof. For any $\xi \in C^*$, the set-valued mapping $\xi \circ F$ satisfies all the conditions of Theorem 5.2 or Theorem 5.3. Hence, $\xi \circ F$ has $\mathbb{R}_+$-saddle point, that is, there exists $(\overline{x},\overline{y}) \in X \times Y$ such that

$$\max \bigcup_{y\in Y} \xi(F(\overline{x},y)) = \min \bigcup_{x\in X} \xi(F(x,\overline{y})) = \xi(F(\overline{x},\overline{y})). \tag{5.7}$$

Then, for any $z \in F(\overline{x}, \overline{y})$,

$$\xi(z) \in \min \bigcup_{x \in X} \xi\left(F\left(x, \overline{y}\right)\right),$$
$$\xi(z) \in \max \bigcup_{y \in Y} \xi\left(F\left(\overline{x}, y\right)\right). \tag{5.8}$$

Thus, by Proposition 3.14,

$$z \in \mathrm{Min}_w \bigcup_{x \in X} F\left(x, \overline{y}\right) \bigcap \mathrm{Max}_w \bigcup_{y \in Y} F\left(\overline{x}, y\right), \tag{5.9}$$

and $(\overline{x}, \overline{y})$ is a weak C-saddle point of F. □

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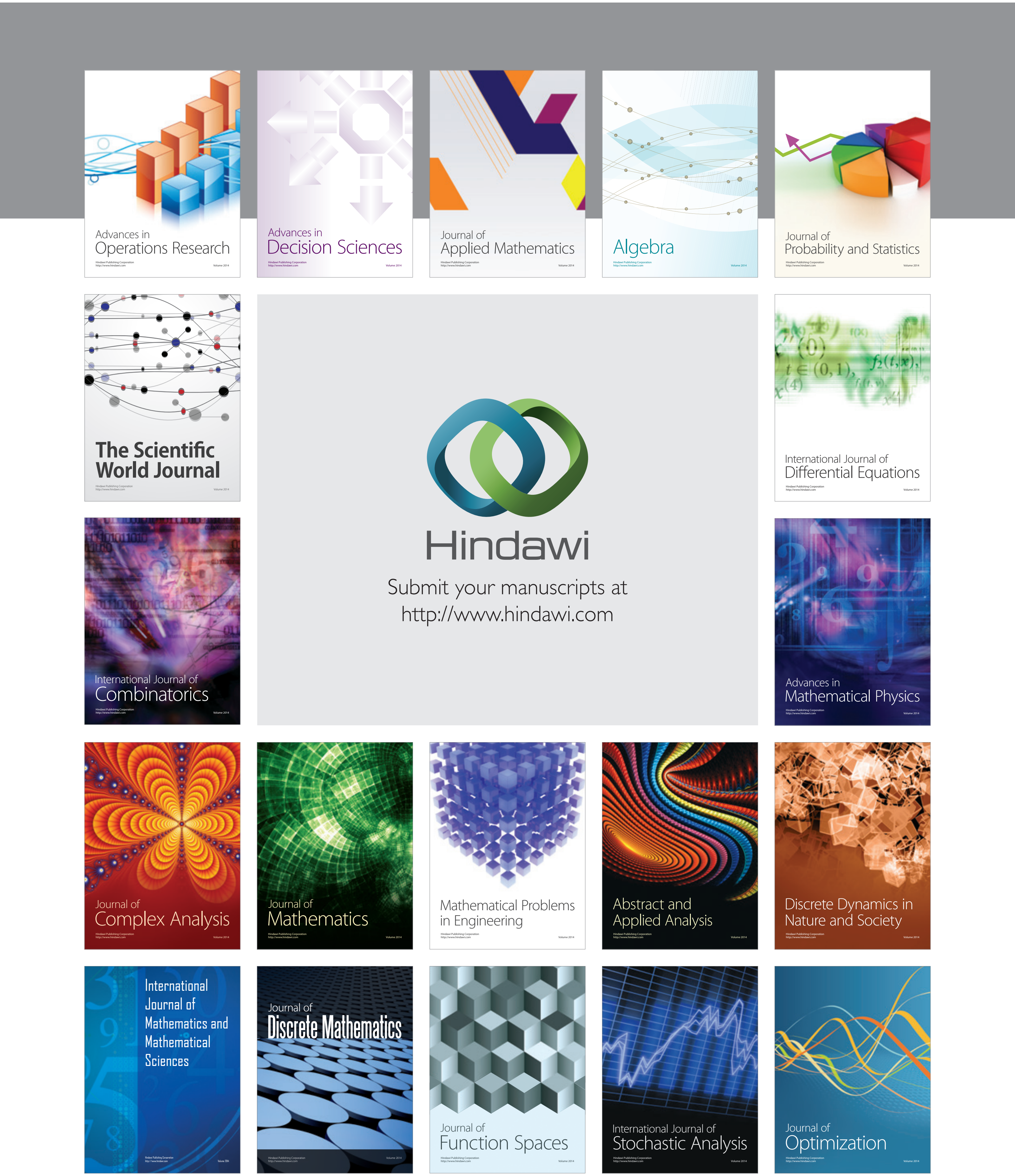
References

[1] Ky. Fan, "Fixed-point and minimax theorems in locally convex topological linear spaces," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 38, pp. 121–126, 1952.
[2] K. Fan, "Minimax theorems," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 39, pp. 42–47, 1953.
[3] F. Ferro, "A minimax theorem for vector-valued functions," *Journal of Optimization Theory and Applications*, vol. 60, no. 1, pp. 19–31, 1989.
[4] G. Y. Chen, "Improvement for Ferro's minimax theorem," *Systems Science and Mathematical Sciences*, vol. 7, no. 1, pp. 1–4, 1994.
[5] S. J. Li, G. Y. Chen, and G. M. Lee, "Minimax theorems for set-valued mappings," *Journal of Optimization Theory and Applications*, vol. 106, no. 1, pp. 183–199, 2000.
[6] S. J. Li, G. Y. Chen, K. L. Teo, and X. Q. Yang, "Generalized minimax inequalities for set-valued mappings," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 2, pp. 707–723, 2003.
[7] Y. Zhang, S. J. Li, and S. K. Zhu, "Minimax problems for set-valued mappings," *Numerical Functional Analysis and Optimization*, vol. 33, no. 2, pp. 239–253, 2012.
[8] Y.-C. Lin, "On generalized vector equilibrium problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 2, pp. 1040–1048, 2009.
[9] K. K. Tan, J. Yu, and X. Z. Yuan, "Existence theorems for saddle points of vector-valued maps," *Journal of Optimization Theory and Applications*, vol. 89, no. 3, pp. 731–747, 1996.
[10] C. Gerth and P. Weidner, "Nonconvex separation theorems and some applications in vector optimization," *Journal of Optimization Theory and Applications*, vol. 67, no. 2, pp. 297–320, 1990.
[11] G. X.-Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker Inc., New York, NY, USA, 1999.
[12] F. E. Browder, "The fixed point theory of multi-valued mappings in topological vector spaces," *Mathematische Annalen*, vol. 177, pp. 283–301, 1968.
[13] D. Kuroiwa, "Convexity for set-valued maps," *Applied Mathematics Letters*, vol. 9, no. 2, pp. 97–101, 1996.
[14] J. M. Borwein and D. Zhuang, "On Fan's minimax theorem," *Mathematical Programming*, vol. 34, no. 2, pp. 232–234, 1986.

[15] J. L. Kelley and I. Namioka, *Linear Topological Spaces*, D. Van Nostrand Co., Inc., Princeton, NJ, USA, 1963.

[16] A. R. Warburton, "Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives," *Journal of Optimization Theory and Applications*, vol. 40, no. 4, pp. 537–557, 1983.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