Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 298531, 11 pages
doi:10.1155/2012/298531

Research Article

Explicit Formulas Involving q-Euler Numbers and Polynomials

Serkan Araci,[1] Mehmet Acikgoz,[1] and Jong Jin Seo[2]

[1] *Department of Mathematics, Faculty of Science and Arts, University of Gaziantep, 27310 Gaziantep, Turkey*
[2] *Department of Applied Mathematics, Pukyong National University, Busan 608-737, Republic of Korea*

Correspondence should be addressed to Jong Jin Seo, seo2011@pknu.ac.kr

Received 4 April 2012; Accepted 4 October 2012

Academic Editor: Gerd Teschke

We deal with q-Euler numbers and q-Bernoulli numbers. We derive some interesting relations for q-Euler numbers and polynomials by using their generating function and derivative operator. Also, we derive relations between the q-Euler numbers and q-Bernoulli numbers via the p-adic q-integral in the p-adic integer ring.

1. Preliminaries

Imagine that p is a fixed odd prime number. Throughout this paper we use the following notations, where $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The p-adic absolute value is defined by

$$|p|_p = \frac{1}{p}. \tag{1.1}$$

In this paper, we will assume that $|q-1|_p < 1$ as an indeterminate.

$[x]_q$ is a q-extension of x, which is defined by

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1.2}$$

We note that $\lim_{q\to 1}[x]_q = x$ (see [1–12]).

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y}, \tag{1.3}$$

has a limit $f'(a)$ as $(x,y) \to (a,a)$ and denote this by $f \in \mathrm{UD}(\mathbb{Z}_p)$.

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \le \xi < p^N} f(\xi) q^{\xi} = \sum_{0 \le \xi < p^N} f(\xi) \mu_q\left(\xi + p^N \mathbb{Z}_p\right), \tag{1.4}$$

which represents p-adic q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ will be defined as the limit $(N \to \infty)$ of these sums, when it exists. The p-adic q-integral of function $f \in \mathrm{UD}(\mathbb{Z}_p)$ is defined by Kim

$$I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_q(\xi) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{\xi=0}^{p^N - 1} f(\xi) q^{\xi}. \tag{1.5}$$

The bosonic integral is considered as a bosonic limit $q \to 1$, $I_1(f) = \lim_{q \to 1} I_q(f)$. Similarly, the fermionic p-adic integral on $\mathbb{Z}_p$ is introduced by Kim as follows:

$$I_{-q}(f) = \lim_{q \to -q} I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi) \tag{1.6}$$

(for more details, see [9–12]).

In [6], the q-Euler polynomials with weight 0 are introduced as

$$\widetilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y). \tag{1.7}$$

From (1.7), we have

$$\widetilde{E}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^l \widetilde{E}_{n-l,q}, \tag{1.8}$$

where $\widetilde{E}_{n,q}(0) = \widetilde{E}_{n,q}$ are called q-Euler numbers with weight 0. Then, q-Euler numbers are defined as

$$q\left(\widetilde{E}_q + 1\right)^n + \widetilde{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases} \tag{1.9}$$

where the usual convention about replacing $(\widetilde{E}_q)^n$ by $\widetilde{E}_{n,q}$ is used.

Similarly, the q-Bernoulli polynomials and numbers with weight 0 are defined, respectively, as

$$\widetilde{B}_{n,q}(x) = \lim_{n\to\infty}\frac{1}{[p^n]_q}\sum_{y=0}^{p^n-1}(x+y)^n q^y = \int_{\mathbb{Z}_p}(x+y)^n d\mu_q(y), \tag{1.10}$$

$$\widetilde{B}_{n,q} = \int_{\mathbb{Z}_p} y^n d\mu_q(y)$$

(for more information, see [4]).

We, by using the Kim et al. method in [2], will investigate some interesting identities on the q-Euler numbers and polynomials arising from their generating function and derivative operator. Consequently, we derive some properties on the q-Euler numbers and polynomials and q-Bernoulli numbers and polynomials by using q-Volkenborn integral and fermionic p-adic q-integral on $\mathbb{Z}_p$.

2. On the q-Euler Numbers and Polynomials

Let us consider Kim's q-Euler polynomials as follows:

$$F_x^q = F_x^q(t) = \frac{[2]_q}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty}\widetilde{E}_{n,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

Here x is a fixed parameter. Thus, by expression of (2.1), we can readily see the following:

$$qe^t F_x^q + F_x^q = [2]_q e^{xt}. \tag{2.2}$$

Last from equality, taking derivative operator D as $D = d/dt$ on the both sides of (2.2). Then, we easily see that

$$qe^t(D+I)^k F_x^q + D^k F_x^q = [2]_q x^k e^{xt}, \tag{2.3}$$

where $k \in \mathbb{N}^*$ and I is identity operator. By multiplying e^{-t} on both sides of (2.3), we get

$$q(D+I)^k F_x^q + e^{-t}D^k F_x^q = [2]_q x^k e^{(x-1)t}. \tag{2.4}$$

Let us take derivative operator $D^m (m \in \mathbb{N})$ on both sides of (2.4). Then we get

$$qe^t D^m(D+I)^k F_x^q + D^k(D-I)^m F_x^q = [2]_q x^k (x-1)^m e^{xt}. \tag{2.5}$$

Let $G[0]$ (not $G(0)$) be the constant term in a Laurent series of $G(t)$. Then, from (2.5), we get

$$\sum_{j=0}^{k}\binom{k}{j}\left(qe^{t}D^{k+m-j}F_{x}^{q}(t)\right)[0]+\sum_{j=0}^{m}\binom{m}{j}(-1)^{j}\left(D^{k+m-j}F_{x}^{q}(t)\right)[0]=[2]_{q}x^{k}(x-1)^{m}. \tag{2.6}$$

By (2.1), we see

$$\left(D^{N}F_{x}^{q}(t)\right)[0]=\widetilde{E}_{N,q}(x),\qquad \left(e^{t}D^{N}F_{x}^{q}(t)\right)[0]=\widetilde{E}_{N,q}(x). \tag{2.7}$$

By expressions of (2.6) and (2.7), we see that

$$\sum_{j=0}^{\max\{k,m\}}\left[q\binom{k}{j}+(-1)^{j}\binom{m}{j}\right]\widetilde{E}_{k+m-j,q}(x)=[2]_{q}x^{k}(x-1)^{m}. \tag{2.8}$$

From (2.1), we note that

$$\frac{d}{dx}\left(\widetilde{E}_{n,q}(x)\right)=n\sum_{l=0}^{n-1}\binom{n-1}{l}\widetilde{E}_{l,q}x^{n-1-l}=n\widetilde{E}_{n-1,q}(x). \tag{2.9}$$

By (2.9), we easily see

$$\int_{0}^{1}\widetilde{E}_{n,q}(x)dx=\frac{\widetilde{E}_{n+1,q}(1)-\widetilde{E}_{n+1,q}}{n+1}=-\frac{[2]_{q^{-1}}}{n+1}\widetilde{E}_{n+1,q}. \tag{2.10}$$

Now, let us consider definition of integral from 0 to 1 in (2.8), then we have

$$\begin{aligned}
&-[2]_{q^{-1}}\sum_{j=0}^{\max\{k,m\}}\left[q\binom{k}{j}+(-1)^{j}\binom{m}{j}\right]\frac{\widetilde{E}_{k+m-j+1,q}}{k+m-j+1}\\
&\quad=[2]_{q}(-1)^{m}B(k+1,m+1)\\
&\quad=[2]_{q}(-1)^{m}\frac{\Gamma(k+1)\Gamma(m+1)}{\Gamma(k+m+2)},
\end{aligned} \tag{2.11}$$

where $B(m,n)$ is beta function which is defined by

$$\begin{aligned}
B(m,n)&=\int_{0}^{1}x^{m-1}(1-x)^{n-1}dx\\
&=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\quad m>0,\ n>0.
\end{aligned} \tag{2.12}$$

As a result, we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$, one has*

$$\sum_{j=1}^{\max\{k,m\}} \left[q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \frac{\widetilde{E}_{k+m-j+1,q}}{k+m-j+1} = q\frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} - [2]_q \frac{\widetilde{E}_{k+m+1,q}}{k+m+1}. \tag{2.13}$$

Substituting $m = k+1$ into Theorem 2.1, we readily get

$$\sum_{j=1}^{k+1} \left[q\binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \frac{\widetilde{E}_{2k+2-j,q}}{2k+2-j} = q\frac{(-1)^{k}}{(2k+2)\binom{2k+1}{k}} - [2]_q \frac{\widetilde{E}_{2k+2,q}}{2k+2}. \tag{2.14}$$

By (2.1), it follows that

$$\sum_{j=0}^{\max\{k,m\}} (k+m-j) \left[q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \widetilde{E}_{k+m-j-1,q}(x) = [2]_q x^{k-1}(x-1)^{m-1}((k+m)x-k). \tag{2.15}$$

Let $m = k$ in (2.1), we see that

$$\sum_{j=0}^{k} \left[q\binom{k}{j} + (-1)^j \binom{k}{j} \right] \widetilde{E}_{2k-j,q}(x) = [2]_q x^k (x-1)^k. \tag{2.16}$$

Last from equality, we discover the following:

$$[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \widetilde{E}_{2k-2j,q}(x) + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \widetilde{E}_{2k-2j-1,q}(x) = [2]_q x^k (x-1)^k. \tag{2.17}$$

Here $[\cdot]$ is Gauss' symbol. Then, taking integral from 0 to 1 in both sides of last equality, we get

$$\begin{aligned} &-[2]_{q^{-1}}[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{\widetilde{E}_{2k-2j+1,q}}{2k-2j+1} + [2]_{q^{-1}}(1-q) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \frac{\widetilde{E}_{2k-2j,q}}{2k-2j} \\ &= [2]_q (-1)^k B(k+1,k+1) \\ &= \frac{[2]_q (-1)^k}{(2k+1)\binom{2k}{k}}. \end{aligned} \tag{2.18}$$

Consequently, we derive the following theorem.

Theorem 2.2. *The following identity*

$$[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{\widetilde{E}_{2k-2j+1,q}}{2k-2j+1} + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \frac{\widetilde{E}_{2k-2j,q}}{2k-2j} = \frac{q(-1)^{k+1}}{(2k+1)\binom{2k}{k}} \tag{2.19}$$

is true.

In view of (2.1) and (2.17), we discover the following applications:

$$\begin{aligned}
&= \sum_{j=0}^{k+1} \left[q\binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \widetilde{E}_{2k+1-j,q}(x) \\
&= [2]_q \widetilde{E}_{2k+1,q}(x) + \sum_{j=1}^{[(k+1)/2]} \left[q\binom{k}{2j} + \binom{k}{2j} + \binom{k}{2j-1} \right] \widetilde{E}_{2k+1-2j,q}(x) \\
&\quad + \sum_{j=0}^{[(k+1)/2]} \left[q\binom{k}{2j+1} - \binom{k}{2j+1} - \binom{k}{2j} \right] \widetilde{E}_{2k-2j,q}(x) \\
&= -\left[\sum_{j=0}^{[k/2]} \binom{k}{2j} \widetilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \widetilde{E}_{2k-2j+1}(x) \right] \\
&\quad + [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \widetilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \widetilde{E}_{2k+1-2j,q}(x) \\
&\quad + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \widetilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \widetilde{E}_{2k-2j+1}(x).
\end{aligned} \tag{2.20}$$

By expressions (2.17) and (2.20), we have the following theorem.

Theorem 2.3. *For $k \in \mathbb{N}$, one has*

$$\begin{aligned}
&[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \widetilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \widetilde{E}_{2k+1-2j,q}(x) \\
&\quad + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\widetilde{E}_{2k-2j,q}(x) + \frac{1}{1+q} \widetilde{E}_{2k-2j+1}(x) \right] \\
&= x^k (x-1)^k \left([2]_q x - q \right).
\end{aligned} \tag{2.21}$$

3. p-adic Integral on $\mathbb{Z}_p$ Associated with Kim's q-Euler Polynomials

In this section, we consider Kim's q-Euler polynomials by means of p-adic q-integral on $\mathbb{Z}_p$. Now we start with the following assertion.

Let $m, k \in \mathbb{N}$. Then by (2.8),

$$\begin{aligned} I_1 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \widetilde{E}_{l+k,q}. \end{aligned} \tag{3.1}$$

On the other hand, in right hand side of (2.8),

$$\begin{aligned} I_2 &= \sum_{j=0}^{\max\{k,m\}} \left[q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \widetilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ &= \sum_{j=0}^{\max\{k,m\}} \left[q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \widetilde{E}_{k+m-j-l,q} \widetilde{E}_{l,q}. \end{aligned} \tag{3.2}$$

Equating I_1 and I_2, we get the following theorem.

Theorem 3.1. *For $m, k \in \mathbb{N}$, one has*

$$\begin{aligned} &\sum_{j=0}^{\max\{k,m\}} \left[q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \widetilde{E}_{k+m-j-l,q} \widetilde{E}_{l,q} \\ &\quad = [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \widetilde{E}_{l+k,q}. \end{aligned} \tag{3.3}$$

Let us take fermionic p-adic q-integral on $\mathbb{Z}_p$ in left hand side of (2.21), we get

$$\begin{aligned} I_3 &= \int_{\mathbb{Z}_p} x^k (x-1)^k \left([2]_q x - q \right) d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_{-q}(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \widetilde{E}_{k+l+1,q} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \widetilde{E}_{k+l,q}. \end{aligned} \tag{3.4}$$

In other words, we consider right hand side of (2.21) as follows:

$$\begin{aligned}
I_4 &= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\begin{array}{c} (q-1) \displaystyle\sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \widetilde{E}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ + \dfrac{q-1}{1+q} \displaystyle\sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \end{array} \right] \\
&= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \widetilde{E}_{l,q} \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \widetilde{E}_{l,q} \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\begin{array}{c} (q-1) \displaystyle\sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \widetilde{E}_{2k-2j-l,q} \widetilde{E}_{l,q} \\ + \dfrac{q-1}{1+q} \displaystyle\sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k-2j-l+1} \widetilde{E}_{l,q} \end{array} \right].
\end{aligned} \tag{3.5}$$

Equating I_3 and I_4, we get the following theorem.

Theorem 3.2. *For $k \in \mathbb{N}$, one has*

$$\begin{aligned}
&\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\{ [2]_q \widetilde{E}_{k+l+1,q} - q \widetilde{E}_{k+l,q} \right\} \\
&\quad = [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \widetilde{E}_{l,q} \\
&\qquad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \widetilde{E}_{l,q} \\
&\qquad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{array}{c} (q-1) \displaystyle\sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \widetilde{E}_{2k-2j-l,q} \widetilde{E}_{l,q} \\ + \dfrac{q-1}{1+q} \displaystyle\sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k-2j-l+1} \widetilde{E}_{l,q} \end{array} \right\}.
\end{aligned} \tag{3.6}$$

Now, we consider (2.8) and (2.1) by means of q-Volkenborn integral. Then, by (2.8), we see

$$\begin{aligned}
&[2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_q(x) \\
&\quad = [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_q(x) \\
&\quad = [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \widetilde{B}_{l+k,q}.
\end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned}
&\sum_{j=0}^{\max\{k,m\}} \left[q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \widetilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
&\quad = \sum_{j=0}^{\max\{k,m\}} \left[q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \widetilde{E}_{k+m-j-l,q} \widetilde{B}_{l,q}.
\end{aligned} \tag{3.8}$$

Therefore, we get the following theorem.

Theorem 3.3. *For $m, k \in \mathbb{N}$, one has*

$$\begin{aligned}
&[2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \widetilde{B}_{l+k,q} \\
&\quad = \sum_{j=0}^{\max\{k,m\}} \left[q\binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \widetilde{E}_{k+m-j-l,q} \widetilde{B}_{l,q}.
\end{aligned} \tag{3.9}$$

By using fermionic p-adic q-integral on $\mathbb{Z}_p$ in left hand side of (2.21), we get

$$\begin{aligned}
I_5 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^k \big([2]x - q \big) d\mu_q(x) \\
&= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) \\
&= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \widetilde{B}_{k+l+1,q} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \widetilde{B}_{k+l,q}.
\end{aligned} \tag{3.10}$$

Also, we consider right hand side of (2.21) as follows:

$$\begin{aligned}
I_6 &= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\begin{array}{c} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \widetilde{E}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ + \dfrac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \end{array} \right] \\
&= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \widetilde{B}_{l,q} \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \widetilde{B}_{l,q} \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\begin{array}{c} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \widetilde{E}_{2k-2j-l,q} \widetilde{B}_{l,q} \\ + \dfrac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k-2j-l+1} \widetilde{B}_{l,q} \end{array} \right].
\end{aligned} \tag{3.11}$$

Equating I_5 and I_6, we get the following corollary.

Corollary 3.4. *For $k \in \mathbb{N}$, one gets*

$$\begin{aligned}
&\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\{ [2]_q \widetilde{B}_{k+l+1,q} - q\widetilde{B}_{k+l,q} \right\} \\
&\quad = [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \widetilde{B}_{l,q} \\
&\qquad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k+1-2j-l,q} \widetilde{B}_{l,q} \\
&\qquad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{array}{c} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \widetilde{E}_{2k-2j-l,q} \widetilde{B}_{l,q} \\ + \dfrac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{E}_{2k-2j-l+1} \widetilde{B}_{l,q} \end{array} \right\}.
\end{aligned} \tag{3.12}$$

Acknowledgment

The authors would like to thank the referee for his/her valuable comments on this work.

References

[1] S. Araci, D. Erdal, and J.-J. Seo, "A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials," *Abstract and Applied Analysis*, vol. 2011, Article ID 649248, 10 pages, 2011.
[2] T. Kim, B. Lee, S. H. Lee, and S. H. Rim, "Identities for the Bernoulli and Euler numbers and polynomials," *Ars Combinatoria*. In press.
[3] D. Kim, T. Kim, S.-H. Lee, D.-V. Dolgy, and S.-H. Rim, "Some new identities on the Bernoulli numbers and polynomials," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 856132, 11 pages, 2011.
[4] T. Kim, J. Choi, and Y. H. Kim, "Some identities on the q-Bernoulli numbers and polynomials with weight 0," *Abstract and Applied Analysis*, vol. 2011, Article ID 361484, 8 pages, 2011.
[5] T. Kim, "On a q-analogue of the p-adic log gamma functions and related integrals," *Journal of Number Theory*, vol. 76, no. 2, pp. 320–329, 1999.
[6] T. Kim and J. Choi, "On the q-Euler numbers and polynomials with weight 0," *Abstract and Applied Analysis*, vol. 2012, Article ID 795304, 7 pages, 2012.
[7] T. Kim, "On the q-extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.
[8] T. Kim, "On the weighted q-Bernoulli numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, no. 2, pp. 207–215, 2011.
[9] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
[10] T. Kim, "q-Euler numbers and polynomials associated with p-adic q-integrals," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 15–27, 2007.
[11] T. Kim, "New approach to q-Euler polynomials of higher order," *Russian Journal of Mathematical Physics*, vol. 17, no. 2, pp. 218–225, 2010.
[12] T. Kim, "Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.