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Research Article

On Some Intermediate Mean Values

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We give a necessary and sufficient mean condition for the quotient of two Jensen functionals and define a new class $\Lambda_{f,g}(a,b)$ of mean values where f, g are continuously differentiable convex functions satisfying the relation $f''(t) = tg''(t)$, $t \in \mathbb{R}^+$. Then we asked for a characterization of f, g such that the inequalities $H(a,b) \le \Lambda_{f,g}(a,b) \le A(a,b)$ or $L(a,b) \le \Lambda_{f,g}(a,b) \le I(a,b)$ hold for each positive a, b, where H, A, L, I are the harmonic, arithmetic, logarithmic, and identric means, respectively. For a subclass of Λ with $g''(t) = t^s$, $s \in \mathbb{R}$, this problem is thoroughly solved.

1. Introduction

It is said that the mean P is intermediate relating to the means M and N, $M \le N$ if the relation

$$M(a,b) \le P(a,b) \le N(a,b) \tag{1}$$

holds for each two positive numbers a, b.

It is also well known that

$$\begin{aligned}\min\{a,b\} &\le H(a,b) \le G(a,b)\\ &\le L(a,b) \le I(a,b) \le A(a,b) \le S(a,b)\\ &\le \max\{a,b\},\end{aligned} \tag{2}$$

where

$$H = H(a,b) := 2\left(\frac{1}{a}+\frac{1}{b}\right)^{-1};$$
$$G = G(a,b) := \sqrt{ab}; \qquad L = L(a,b) := \frac{b-a}{\log b - \log a};$$
$$I = I(a,b) := \frac{\left(b^b/a^a\right)^{1/(b-a)}}{e};$$
$$A = A(a,b) := \frac{a+b}{2}; \qquad S = S(a,b) := a^{a/(a+b)}b^{b/(a+b)} \tag{3}$$

are the harmonic, geometric, logarithmic, identric, arithmetic, and Gini mean, respectively.

An easy task is to construct intermediate means related to two given means M and N with $M \le N$. For instance, for an arbitrary mean P, we have that

$$M(a,b) \le P(M(a,b), N(a,b)) \le N(a,b). \tag{4}$$

The problem is more difficult if we have to decide whether the given mean is intermediate or not. For example, the relation

$$L(a,b) \le S_s(a,b) \le I(a,b) \tag{5}$$

holds for each positive a and b if and only if $0 \le s \le 1$, where the Stolarsky mean S_s is defined by (cf [1])

$$S_s(a,b) := \left(\frac{b^s - a^s}{s(b-a)}\right)^{1/(s-1)}. \tag{6}$$

Also,

$$G(a,b) \le A_s(a,b) \le A(a,b) \tag{7}$$

holds if and only if $0 \le s \le 1$, where the Hölder mean of order s is defined by

$$A_s(a,b) := \left(\frac{a^s + b^s}{2}\right)^{1/s}. \tag{8}$$

An inverse problem is to find best possible approximation of a given mean P by elements of an ordered class of means S. A good example for this topic is comparison between the logarithmic mean and the class A_s of Hölder means of order s. Namely, since $A_0 = \lim_{s\to 0} A_s = G$ and $A_1 = A$, it follows from (2) that

$$A_0 \le L \le A_1. \quad (9)$$

Since A_s is monotone increasing in s, an improving of the above is given by Carlson [2]:

$$A_0 \le L \le A_{1/2}. \quad (10)$$

Finally, Lin showed in [3] that

$$A_0 \le L \le A_{1/3} \quad (11)$$

is the best possible approximation of the logarithmic mean by the means from the class A_s.

Numerous similar results have been obtained recently. For example, an approximation of Seiffert's mean by the class A_s is given in [4, 5].

In this paper we will give best possible approximations for a whole variety of elementary means (2) by the class λ_s defined below (see Theorem 5).

Let f, g be twice continuously differentiable (strictly) convex functions on $\mathbb{R}^+$. By definition (cf [6], page 5),

$$\overline{f}(a,b) := f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) > 0, \quad a \ne b,$$
$$\overline{f}(a,b) = 0, \quad (12)$$

if and only if $a = b$.

It turns out that the expression

$$\Lambda_{f,g}(a,b) := \frac{\overline{f}(a,b)}{\overline{g}(a,b)} = \frac{f(a) + f(b) - 2f((a+b)/2)}{g(a) + g(b) - 2g((a+b)/2)} \quad (13)$$

represents a mean of two positive numbers a, b; that is, the relation

$$\min\{a,b\} \le \Lambda_{f,g}(a,b) \le \max\{a,b\} \quad (14)$$

holds for each $a, b \in \mathbb{R}^+$, if and only if the relation

$$f''(t) = tg''(t) \quad (15)$$

holds for each $t \in \mathbb{R}^+$.

Let $f, g \in C^\infty(0,\infty)$ and denote by Λ the set $\{(f,g)\}$ of convex functions satisfying the relation (15). There is a natural question how to improve the bounds in (14); in this sense we come upon the following intermediate mean problem.

Open Question. Under what additional conditions on $f, g \in \Lambda$, the inequalities

$$H(a,b) \le \Lambda_{f,g}(a,b) \le A(a,b), \quad (16)$$

or, more tightly,

$$L(a,b) \le \Lambda_{f,g}(a,b) \le I(a,b), \quad (17)$$

hold for each $a, b \in \mathbb{R}^+$?

As an illustration, consider the function $f_s(t)$ defined to be

$$f_s(t) = \begin{cases} \dfrac{t^s - st + s - 1}{s(s-1)}, & s(s-1) \ne 0; \\ t - \log t - 1, & s = 0; \\ t\log t - t + 1, & s = 1. \end{cases} \quad (18)$$

Since

$$f_s'(t) = \begin{cases} \dfrac{t^{s-1} - 1}{s-1}, & s(s-1) \ne 0; \\ 1 - \dfrac{1}{t}, & s = 0; \\ \log t, & s = 1, \end{cases} \quad (19)$$

$$f_s''(t) = t^{s-2}, \quad s \in \mathbb{R},\ t > 0,$$

it follows that $f_s(t)$ is a twice continuously differentiable convex function for $s \in \mathbb{R}$, $t \in \mathbb{R}^+$.

Moreover, it is evident that $(f_{s+1}, f_s) \in \Lambda$.

We will give in the sequel a complete answer to the above question concerning the means

$$\frac{\overline{f}_{s+1}(a,b)}{\overline{f}_s(a,b)} := \lambda_s(a,b) \quad (20)$$

defined by

$$\lambda_s(a,b) = \begin{cases} \dfrac{s-1}{s+1}\dfrac{a^{s+1} + b^{s+1} - 2((a+b)/2)^{s+1}}{a^s + b^s - 2((a+b)/2)^s}, & s \in \mathbb{R}/\{-1,0,1\}; \\ \dfrac{2\log((a+b)/2) - \log a - \log b}{1/2a + 1/2b - 2/(a+b)}, & s = -1; \\ \dfrac{a\log a + b\log b - (a+b)\log((a+b)/2)}{2\log((a+b)/2) - \log a - \log b}, & s = 0; \\ \dfrac{(b-a)^2}{4(a\log a + b\log b - (a+b)\log((a+b)/2))}, & s = 1. \end{cases} \quad (21)$$

Those means are obviously symmetric and homogeneous of order one.

As a consequence we obtain some new intermediate mean values; for instance, we show that the inequalities

$$H(a,b) \le \lambda_{-1}(a,b) \le G(a,b) \le \lambda_0(a,b) \le L(a,b)$$
$$\le \lambda_1(a,b) \le I(a,b) \quad (22)$$

hold for arbitrary $a, b \in \mathbb{R}^+$. Note that

$$\lambda_{-1} = \frac{2G^2\log(A/G)}{A-H}; \qquad \lambda_0 = A\frac{\log(S/A)}{\log(A/G)};$$
$$\lambda_1 = \frac{1}{2}\frac{A-H}{\log(S/A)}. \quad (23)$$

2. Results

We prove firstly the following

Theorem 1. *Let $f, g \in C^2(I)$ with $g'' > 0$. The expression $\Lambda_{f,g}(a,b)$ represents a mean of arbitrary numbers $a, b \in I$ if and only if the relation* (15) *holds for $t \in I$.*

Remark 2. In the same way, for arbitrary $p, q > 0,\ p + q = 1$, it can be deduced that the quotient

$$\Lambda_{f,g}(p, q; a, b) := \frac{pf(a) + qf(b) - f(pa + qb)}{pg(a) + qg(b) - g(pa + qb)} \quad (24)$$

represents a mean value of numbers a, b if and only if (15) holds.

A generalization of the above assertion is the next.

Theorem 3. *Let $f, g : I \to \mathbb{R}$ be twice continuously differentiable functions with $g'' > 0$ on I and let $p = \{p_i\}$, $i = 1, 2, \ldots, \sum p_i = 1$ be an arbitrary positive weight sequence. Then the quotient of two Jensen functionals*

$$\Lambda_{f,g}(p, x) := \frac{\sum_1^n p_i f(x_i) - f(\sum_1^n p_i x_i)}{\sum_1^n p_i g(x_i) - g(\sum_1^n p_i x_i)}, \quad n \geq 2, \quad (25)$$

represents a mean of an arbitrary set of real numbers $x_1, x_2, \ldots, x_n \in I$ if and only if the relation

$$f''(t) = tg''(t) \quad (26)$$

holds for each $t \in I$.

Remark 4. It should be noted that the relation $f''(t) = tg''(t)$ determines f in terms of g in an easy way. Precisely,

$$f(t) = tg(t) - 2G(t) + ct + d, \quad (27)$$

where $G(t) := \int_1^t g(u)du$ and c and d are constants.

Our results concerning the means $\lambda_s(a,b)$, $s \in \mathbb{R}$ are included in the following.

Theorem 5. *For the class of means $\lambda_s(a,b)$ defined above, the following assertions hold for each $a, b \in \mathbb{R}^+$.*

(1) *The means $\lambda_s(a,b)$ are monotone increasing in s;*
(2) *$\lambda_s(a,b) \leq H(a,b)$ for each $s \leq -4$;*
(3) *$H(a,b) \leq \lambda_s(a,b) \leq G(a,b)$ for $-3 \leq s \leq -1$;*
(4) *$G(a,b) \leq \lambda_s(a,b) \leq L(a,b)$ for $-1/2 \leq s \leq 0$;*
(5) *there is a number $s_0 \in (1/12, 1/11)$ such that $L(a,b) \leq \lambda_s(a,b) \leq I(a,b)$ for $s_0 \leq s \leq 1$;*
(6) *there is a number $s_1 \in (1.03, 1.04)$ such that $I(a,b) \leq \lambda_s(a,b) \leq A(a,b)$ for $s_1 \leq s \leq 2$;*
(7) *$A(a,b) \leq \lambda_s(a,b) \leq S(a,b)$ for each $2 \leq s \leq 5$;*
(8) *there is no finite s such that the inequality $S(a,b) \leq \lambda_s(a,b)$ holds for each $a, b \in \mathbb{R}^+$.*

The above estimations are best possible.

3. Proofs

3.1. Proof of Theorem 1. We prove firstly the necessity of the condition (15).

Since $\Lambda_{f,g}(a,b)$ is a mean value for arbitrary $a, b \in I$; $a \neq b$, we have

$$\min\{a,b\} \leq \Lambda_{f,g}(a,b) \leq \max\{a,b\}. \quad (28)$$

Hence

$$\lim_{b \to a} \Lambda_{f,g}(a,b) = a. \quad (29)$$

From the other hand, due to l'Hospital's rule we obtain

$$\begin{aligned}\lim_{b \to a} \Lambda_{f,g}(a,b) &= \lim_{b \to a}\left(\frac{f'(b) - f'((a+b)/2)}{g'(b) - g'((a+b)/2)}\right) \\ &= \lim_{b \to a}\left(\frac{2f''(b) - f''((a+b)/2)}{2g''(b) - g''((a+b)/2)}\right) \\ &= \frac{f''(a)}{g''(a)}.\end{aligned} \quad (30)$$

Comparing (29) and (30) the desired result follows.

Suppose now that (15) holds and let $a < b$. Since $g''(t) > 0$ $\ t \in [a,b]$ by the Cauchy mean value theorem there exists $\xi \in ((a+t)/2, t)$ such that

$$\frac{f'(t) - f'((a+t)/2)}{g'(t) - g'((a+t)/2)} = \frac{f''(\xi)}{g''(\xi)} = \xi. \quad (31)$$

But,

$$a \leq \frac{a+t}{2} < \xi < t \leq b, \quad (32)$$

and, since g' is strictly increasing, $g'(t) - g'((a+t)/2) > 0,\ t \in [a,b]$.

Therefore, by (31) we get

$$\begin{aligned}a\left(g'(t) - g'\left(\frac{a+t}{2}\right)\right) &\leq f'(t) - f'\left(\frac{a+t}{2}\right) \\ &\leq b\left(g'(t) - g'\left(\frac{a+t}{2}\right)\right).\end{aligned} \quad (33)$$

Finally, integrating (33) over $t \in [a,b]$ we obtain the assertion from Theorem 1.

3.2. Proof of Theorem 3. We will give a proof of this assertion by induction on n.

By Remark 2, it holds for $n = 2$.

Next, it is not difficult to check the identity

$$\begin{aligned}&\sum_1^n p_i f(x_i) - f\left(\sum_1^n p_i x_i\right) \\ &= (1 - p_n)\left(\sum_1^{n-1} p_i' f(x_i) - f\left(\sum_1^{n-1} p_i' x_i\right)\right) \\ &\quad + [(1 - p_n) f(T) + p_n f(x_n) - f((1 - p_n)T + p_n x_n)],\end{aligned} \quad (34)$$

where

$$T := \sum_1^{n-1} p_i' x_i; \quad p_i' := \frac{p_i}{(1-p_n)}, \quad i = 1, 2, \ldots, n-1;$$
$$\sum_1^{n-1} p_i' = 1. \tag{35}$$

Therefore, by induction hypothesis and Remark 2, we get

$$\begin{aligned}
&\sum_1^n p_i f(x_i) - f\left(\sum_1^n p_i x_i\right) \\
&\quad \le \max\{x_1, x_2, \ldots x_{n-1}\}(1-p_n) \\
&\quad \times \left(\sum_1^{n-1} p_i' g(x_i) - g\left(\sum_1^{n-1} p_i' x_i\right)\right) \\
&\quad + \max\{T, x_n\}\left[(1-p_n) g(T) + p_n g(x_n) \right.\\
&\qquad\qquad \left. - g((1-p_n)T + p_n x_n)\right] \\
&\quad \le \max\{x_1, x_2, \ldots, x_n\} \\
&\quad \times \left((1-p_n)\left(\sum_1^{n-1} p_i' g(x_i) - g\left(\sum_1^{n-1} p_i' x_i\right)\right)\right. \\
&\quad \left. + \left[(1-p_n) g(T) + p_n g(x_n) - g((1-p_n)T + p_n x_n)\right]\right) \\
&\quad = \max\{x_1, x_2, \ldots, x_n\}\left(\sum_1^n p_i g(x_i) - g\left(\sum_1^n p_i x_i\right)\right).
\end{aligned} \tag{36}$$

The inequality

$$\min\{x_1, x_2, \ldots, x_n\} \le \Lambda_{f,g}(p, x) \tag{37}$$

can be proved analogously.

For the proof of necessity, put $x_2 = x_3 = \cdots = x_n$ and proceed as in Theorem 1.

Remark 6. It is evident from (15) that if $I \subseteq \mathbb{R}^+$ then f has to be also convex on I. Otherwise, it shouldn't be the case. For example, the conditions of Theorem 3 are satisfied with $f(t) = t^3/3$, $g(t) = t^2$, $t \in \mathbb{R}$. Hence, for an arbitrary sequence $\{x_i\}_1^n$ of real numbers, we obtain

$$\begin{aligned}
\min\{x_1, x_2, \ldots, x_n\} &\le \frac{\sum_1^n p_i x_i^3 - \left(\sum_1^n p_i x_i\right)^3}{3\left(\sum_1^n p_i x_i^2 - \left(\sum_1^n p_i x_i\right)^2\right)} \\
&\le \max\{x_1, x_2, \ldots, x_n\}.
\end{aligned} \tag{38}$$

Because the above inequality does not depend on n, a probabilistic interpretation of the above result is contained in the following.

Theorem 7. *For an arbitrary probability law F of random variable X with support on $(-\infty, +\infty)$, one has*

$$(EX)^3 + 3(\min X)\,\sigma_X^2 \le EX^3 \le (EX)^3 + 3(\max X)\,\sigma_X^2. \tag{39}$$

3.3. Proof of Theorem 5, Part (1). We will prove a general assertion of this type. Namely, for an arbitrary positive sequence $\mathbf{x} = \{x_i\}$ and an associated weight sequence $\mathbf{p} = \{p_i\}$, $i = 1, 2, \ldots$, denote

$$\chi_s(\mathbf{p}, \mathbf{x}) := \begin{cases} \dfrac{\sum p_i x_i^s - \left(\sum p_i x_i\right)^s}{s(s-1)}, & s \in \mathbb{R}/\{0,1\}; \\ \log\left(\sum p_i x_i\right) - \sum p_i \log x_i, & s = 0; \\ \sum p_i x_i \log x_i - \left(\sum p_i x_i\right)\log\left(\sum p_i x_i\right), & s = 1. \end{cases} \tag{40}$$

For $s \in \mathbb{R}$, $r > 0$ we have

$$\chi_s(\mathbf{p}, \mathbf{x})\,\chi_{s+r+1}(\mathbf{p}, \mathbf{x}) \ge \chi_{s+1}(\mathbf{p}, \mathbf{x})\,\chi_{s+r}(\mathbf{p}, \mathbf{x}), \tag{41}$$

which is equivalent to

Theorem 8. *The sequence $\{\chi_{s+1}(\mathbf{p}, \mathbf{x})/\chi_s(\mathbf{p}, \mathbf{x})\}$ is monotone increasing in s, $s \in \mathbb{R}$.*

This assertion follows applying the result from [7, Theorem 2] which states the following.

Lemma 9. *For $-\infty < a < b < c < +\infty$, the inequality*

$$\left(\chi_b(\mathbf{p}, \mathbf{x})\right)^{c-a} \le \left(\chi_a(\mathbf{p}, \mathbf{x})\right)^{c-b}\left(\chi_c(\mathbf{p}, \mathbf{x})\right)^{b-a} \tag{42}$$

holds for arbitrary sequences $\mathbf{p}$, $\mathbf{x}$.

Putting there $a = s$, $b = s + 1$, $c = s + r + 1$ and $a = s$, $b = s + r$, $c = s + r + 1$, we successively obtain

$$\begin{aligned}
\left(\chi_{s+1}(\mathbf{p}, \mathbf{x})\right)^{r+1} &\le \left(\chi_s(\mathbf{p}, \mathbf{x})\right)^r \chi_{s+r+1}(\mathbf{p}, \mathbf{x}), \\
\left(\chi_{s+r}(\mathbf{p}, \mathbf{x})\right)^{r+1} &\le \chi_s(\mathbf{p}, \mathbf{x})\left(\chi_{s+r+1}(\mathbf{p}, \mathbf{x})\right)^r.
\end{aligned} \tag{43}$$

Since $r > 0$, multiplying those inequalities we get the relation (41), that is, the proof of Theorem 8.

The part (1) of Theorem 5 follows for $p_1 = p_2 = 1/2$.

A general way to prove the rest of Theorem 5 is to use an easy-checkable identity

$$\frac{\lambda_s(a, b)}{A(a, b)} = \lambda_s(1 + t,\ 1 - t), \tag{44}$$

with $t := (b - a)/(b + a)$.

Since $0 < a < b$, we get $0 < t < 1$. Also,

$$\frac{H(a,b)}{A(a,b)} = 1 - t^2; \qquad \frac{G(a,b)}{A(a,b)} = \sqrt{1-t^2};$$
$$\frac{L(a,b)}{A(a,b)} = \frac{2t}{\log(1+t) - \log(1-t)};$$
$$\frac{I(a,b)}{A(a,b)} = \exp\left(\frac{(1+t)\log(1+t) - (1-t)\log(1-t)}{2t} - 1\right);$$
$$\frac{S(a,b)}{A(a,b)} = \exp\left(\frac{1}{2}\left((1+t)\log(1+t) + (1-t)\log(1-t)\right)\right). \tag{45}$$

Therefore, we have to compare some one-variable inequalities and to check their validness for each $t \in (0,1)$.

For example, we will prove that the inequality

$$\lambda_s(a,b) \le L(a,b) \tag{46}$$

holds for each positive a, b if and only if $s \le 0$.

Since $\lambda_s(a,b)$ is monotone increasing in s, it is enough to prove that

$$\frac{\lambda_0(a,b)}{L(a,b)} \le 1. \tag{47}$$

By the above formulae, this is equivalent to the assertion that the inequality

$$\phi(t) \le 0 \tag{48}$$

holds for each $t \in (0,1)$, with

$$\begin{aligned}\phi(t) := {} & \frac{\log(1+t) - \log(1-t)}{2t} \\ & \times \left((1+t)\log(1+t) + (1-t)\log(1-t)\right) \\ & + \log(1+t) + \log(1-t).\end{aligned} \tag{49}$$

We will prove that the power series expansion of $\phi(t)$ have non-positive coefficients. Thus the relation (48) will be proved.

Since

$$\begin{aligned}&\frac{\log(1+t) - \log(1-t)}{2t} = \sum_0^\infty \frac{t^{2k}}{2k+1};\\ &\log(1+t) + \log(1-t) = -t^2 \sum_0^\infty \frac{t^{2k}}{k+1};\\ &(1+t)\log(1+t) + (1-t)\log(1-t) = t^2 \sum_0^\infty \frac{t^{2k}}{(k+1)(2k+1)},\end{aligned} \tag{50}$$

we get

$$\frac{\phi(t)}{t^2} = \sum_{n=0}^\infty \left(-\frac{1}{n+1} + \sum_{k=0}^n \frac{1}{(2n-2k+1)(k+1)(2k+1)}\right) t^{2n} = \sum_0^\infty c_n t^{2n}. \tag{51}$$

Hence,

$$c_0 = c_1 = 0; \qquad c_2 = -\frac{1}{90}, \tag{52}$$

and, after some calculation, we get

$$c_n = \frac{2}{(n+1)(2n+3)}\left((n+2)\sum_1^n \frac{1}{2k+1} - (n+1)\sum_1^n \frac{1}{2k}\right), \quad n > 1. \tag{53}$$

Now, one can easily prove (by induction, e.g.) that

$$d_n := (n+2)\sum_1^n \frac{1}{2k+1} - (n+1)\sum_1^n \frac{1}{2k} \tag{54}$$

is a negative real number for $n \ge 2$. Therefore $c_n \le 0$, and the proof of the first part is done. For $0 < s < 1$ we have

$$\begin{aligned}\frac{\lambda_s(a,b)}{L(a,b)} - 1 &= \frac{(1-s)\left((1+t)^{s+1} + (1-t)^{s+1} - 2\right)\log((1+t)/(1-t))}{2t(1+s)\left(2 - (1+t)^s - (1-t)^s\right)} - 1 \\ &= \frac{1}{6} s t^2 + O\left(t^4\right) \quad (t \longrightarrow 0).\end{aligned} \tag{55}$$

Therefore, $\lambda_s(a,b) > L(a,b)$ for $s > 0$ and sufficiently small $t := (b-a)/(b+a)$.

Similarly, we will prove that the inequality

$$\lambda_s(a,b) \le I(a,b) \tag{56}$$

holds for each a, b; $0 < a < b$ if and only if $s \le 1$.

As before, it is enough to consider the expression

$$\frac{I(a,b)}{\lambda_1(a,b)} = e^{\mu(t)} \nu(t) := \psi(t), \tag{57}$$

with

$$\begin{aligned}\mu(t) &= \frac{(1+t)\log(1+t) - (1-t)\log(1-t)}{2t} - 1;\\ \nu(t) &= \frac{(1+t)\log(1+t) + (1-t)\log(1-t)}{t^2}.\end{aligned} \tag{58}$$

It is not difficult to check the identity

$$\psi'(t) = -\frac{e^{\mu(t)}\phi(t)}{t^3}. \tag{59}$$

Hence by (48), we get $\psi'(t) > 0$, that is, $\psi(t)$ is monotone increasing for $t \in (0,1)$.

Therefore

$$\frac{I(a,b)}{\lambda_1(a,b)} \geq \lim_{t\to 0^+}\psi(t) = 1. \tag{60}$$

By monotonicity it follows that $\lambda_s(a,b) \leq I(a,b)$ for $s \leq 1$.

For $s > 1$, $(b-a)/(b+a) = t$, we have

$$\lambda_s(a,b) - I(a,b) = \left(\frac{1}{6}(s-1)t^2 + O\left(t^4\right)\right)A(a,b) \quad \left(t \longrightarrow 0^+\right). \tag{61}$$

Hence, $\lambda_s(a,b) > I(a,b)$ for $s > 1$ and t sufficiently small.

From the other hand,

$$\lim_{t\to 1^-}\left[\frac{\lambda_s(a,b)}{I(a,b)} - 1\right] = \frac{e(s-1)\left(2^{s+1}-2\right)}{2(s+1)(2^s-2)} - 1 := \tau(s). \tag{62}$$

Examining the function $\tau(s)$, we find out that it has the only real zero at $s_0 \approx 1.0376$ and is negative for $s \in (1, s_0)$.

Remark 10. Since $\psi(t)$ is monotone increasing, we also get

$$\frac{I(a,b)}{\lambda_1(a,b)} \leq \lim_{t\to 1^-}\psi(t) = \frac{4\log 2}{e}. \tag{63}$$

Hence

$$1 \leq \frac{I(a,b)}{\lambda_1(a,b)} \leq \frac{4\log 2}{e}. \tag{64}$$

A calculation gives $4\log 2/e \approx 1.0200$.

Note also that

$$\lambda_2(a,b) \equiv A(a,b). \tag{65}$$

Therefore, applying the assertion from the part 1, we get

$$\begin{aligned} &\lambda_s(a,b) \leq A(a,b), \quad s \leq 2; \\ &\lambda_s(a,b) \geq A(a,b), \quad s \geq 2. \end{aligned} \tag{66}$$

Finally, we give a detailed proof of the part 7.

We have to prove that $\lambda_s(a,b) \leq S(a,b)$ for $s \leq 5$. Since $\lambda_s(a,b)$ is monotone increasing in s, it is sufficient to prove that the inequality

$$\lambda_5(a,b) \leq S(a,b) \tag{67}$$

holds for each $a, b \in \mathbb{R}^+$.

Therefore, by the transformation given above, we get

$$\begin{aligned} \log\frac{\lambda_5}{A} &= \log\left[\frac{2}{3}\frac{(1+t)^6 + (1-t)^6 - 2}{(1+t)^5 + (1-t)^5 - 2}\right] \\ &= \log\left[\frac{2}{15}\frac{15+15t^2+t^4}{2+t^2}\right] \\ &\leq \log\left[\frac{1+t^2+t^4/4}{1+t^2/2}\right] = \log\left(1+\frac{t^2}{2}\right) \\ &= \frac{t^2}{2} - \frac{t^4}{8} + \frac{t^6}{24} - \cdots \\ &\leq \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^6}{30} + \cdots \\ &= \frac{1}{2}\left((1+t)\log(1+t) + (1-t)\log(1-t)\right) \\ &= \log\frac{S}{A}, \end{aligned} \tag{68}$$

and the proof is done.

Further, we have to show that $\lambda_s(a,b) > S(a,b)$ for some positive a, b whenever $s > 5$.

Indeed, since

$$(1+t)^s + (1-t)^s - 2 = \binom{s}{2}t^2 + \binom{s}{4}t^4 + O\left(t^6\right), \tag{69}$$

for $s > 5$ and sufficiently small t, we get

$$\begin{aligned} \frac{\lambda_s}{A} &= \frac{s-1}{s+1}\frac{\binom{s+1}{2}t^2 + \binom{s+1}{4}t^4 + O\left(t^6\right)}{\binom{s}{2}t^2 + \binom{s}{4}t^4 + O\left(t^6\right)} \\ &= \frac{1+(s-1)(s-2)t^2/12 + O\left(t^4\right)}{1+(s-2)(s-3)t^2/12 + O\left(t^4\right)} \\ &= 1 + \left(\frac{s}{6} - \frac{1}{3}\right)t^2 + O\left(t^4\right). \end{aligned} \tag{70}$$

Similarly,

$$\begin{aligned} \frac{S}{A} &= \exp\left(\frac{1}{2}\left((1+t)\log(1+t) + (1-t)\log(1-t)\right)\right) \\ &= \exp\left(\frac{t^2}{2} + O\left(t^4\right)\right) = 1 + \frac{t^2}{2} + O\left(t^4\right). \end{aligned} \tag{71}$$

Hence,

$$\frac{1}{A}(\lambda_s - S) = \frac{1}{6}(s-5)t^2 + O\left(t^4\right), \tag{72}$$

and this expression is positive for $s > 5$ and t sufficiently small, that is, a sufficiently close to b.

As for the part 8, applying the above transformation we obtain

$$\begin{aligned} &\frac{\lambda_s(a,b)}{S(a,b)} \\ &\quad = \frac{s-1}{s+1}\frac{(1+t)^{s+1}+(1-t)^{s+1}-2}{(1+t)^s+(1-t)^s-2} \\ &\qquad \times \exp\left(-\frac{1}{2}\left((1+t)\log(1+t)+(1-t)\log(1-t)\right)\right), \end{aligned} \tag{73}$$

where $0 < a < b$, $t = (b-a)/(b+a)$.

Since for $s > 5$,

$$\lim_{t\to 1^-}\frac{\lambda_s}{S} = \frac{s-1}{s+1}\frac{2^s-1}{2^s-2}, \tag{74}$$

and the last expression is less than one, it follows that the inequality $S(a,b) < \lambda_s(a,b)$ cannot hold whenever b/a is sufficiently large.

The rest of the proof is straightforward.

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