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Research Article

Composition Operators from the Hardy Space to the Zygmund-Type Space on the Upper Half-Plane

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Here we introduce the nth weighted space on the upper half-plane $\Pi_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ in the complex plane $\mathbb{C}$. For the case $n = 2$, we call it the Zygmund-type space, and denote it by $\mathcal{Z}(\Pi_+)$. The main result of the paper gives some necessary and sufficient conditions for the boundedness of the composition operator $C_\varphi f(z) = f(\varphi(z))$ from the Hardy space $H^p(\Pi_+)$ on the upper half-plane, to the Zygmund-type space, where φ is an analytic self-map of the upper half-plane.

1. Introduction

Let Π_+ be the upper half-plane, that is, the set $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $H(\Pi_+)$ the space of all analytic functions on Π_+. The Hardy space $H^p(\Pi_+) = H^p$, $p > 0$, consists of all $f \in H(\Pi_+)$ such that

$$\|f\|_{H^p}^p = \sup_{y>0} \int_{-\infty}^{\infty} f\left|(x+iy)\right|^p dx < \infty. \tag{1.1}$$

With this norm $H^p(\Pi_+)$ is a Banach space when $p \geq 1$, while for $p \in (0,1)$ it is a Fréchet space with the translation invariant metric $d(f,g) = \|f-g\|_{H^p}^p$, $f, g \in H^p(\Pi_+)$, [1].

We introduce here the nth *weighted space* on the upper half-plane. The nth weighted space consists of all $f \in H(\Pi_+)$ such that

$$\sup_{z\in\Pi_+} \operatorname{Im} z\left|f^{(n)}(z)\right| < \infty, \tag{1.2}$$

where $n \in \mathbb{N}_0$. For $n = 0$ the space is called the *growth space* and is denoted by $\mathcal{A}_\infty(\Pi_+) = \mathcal{A}_\infty$ and for $n = 1$ it is called the *Bloch space* $\mathcal{B}_\infty(\Pi_+) = \mathcal{B}_\infty$ (for Bloch-type spaces on the unit disk, polydisk, or the unit ball and some operators on them, see, e.g., [2–14] and the references therein).

When $n = 2$, we call the space the Zygmund-type space on the upper half-plane (or simply the Zygmund space) and denote it by $\mathcal{Z}(\Pi_+) = \mathcal{Z}$. Recall that the space consists of all $f \in H(\Pi_+)$ such that

$$b_{\mathcal{Z}}(f) = \sup_{z\in\Pi_+} \operatorname{Im} z \left|f''(z)\right| < \infty. \tag{1.3}$$

The quantity is a seminorm on the Zygmund space or a norm on $\mathcal{Z}/\mathbb{P}_1$, where $\mathbb{P}_1$ is the set of all linear polynomials. A natural norm on the Zygmund space can be introduced as follows:

$$\|f\|_{\mathcal{Z}} = \left|f(i)\right| + \left|f'(i)\right| + b_{\mathcal{Z}}(f). \tag{1.4}$$

With this norm the Zygmund space becomes a Banach space.

To clarify the notation we have just introduced, we have to say that the main reason for this name is found in the fact that for the case of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex palne $\mathbb{C}$, Zygmund (see, e.g., [1, Theorem 5.3]) proved that a holomorphic function on $\mathbb{D}$ continuous on the closed unit disk $\overline{\mathbb{D}}$ satisfies the following condition:

$$\sup_{h>0,\,\theta\in[0,2\pi]} \frac{\left|f\left(e^{i(\theta+h)}\right) + f\left(e^{i(\theta-h)}\right) - 2f\left(e^{i\theta}\right)\right|}{h} < \infty \tag{1.5}$$

if and only if

$$\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left|f''(z)\right| < \infty. \tag{1.6}$$

The family of all analytic functions on $\mathbb{D}$ satisfying condition (1.6) is called the Zygmund class on the unit disk.

With the norm

$$\|f\| = \left|f(0)\right| + \left|f'(0)\right| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left|f''(z)\right|, \tag{1.7}$$

the Zygmund class becomes a Banach space. Zygmund class with this norm is called the Zygmund space and is denoted by $\mathcal{Z}(\mathbb{D})$. For some other information on this space and some operators on it, see, for example, [15–19].

Now note that $1 - |z|$ is the distance from the point $z \in \mathbb{D}$ to the boundary of the unit disc, that is, $\partial\mathbb{D}$, and that $\operatorname{Im} z$ is the distance from the point $z \in \Pi_+$ to the real axis in $\mathbb{C}$ which is the boundary of Π_+.

In two main theorems in [20], the authors proved the following results, which we now incorporate in the next theorem.

Theorem A. *Assume $p \geq 1$ and φ is a holomorphic self-map of Π_+. Then the following statements true hold.*

(a) *The operator $C_\varphi : H^p(\Pi_+) \to \mathcal{A}_\infty(\Pi_+)$ is bounded if and only if*

$$\sup_{z\in\Pi_+} \frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{1/p}} < \infty. \tag{1.8}$$

(b) *The operator $C_\varphi : H^p(\Pi_+) \to \mathcal{B}_\infty(\Pi_+)$ is bounded if and only if*

$$\sup_{z\in\Pi_+} \frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{1+1/p}} \left|\varphi'(z)\right| < \infty. \tag{1.9}$$

Motivated by Theorem A, here we investigate the boundedness of the operator $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}(\Pi_+)$. Some recent results on composition and weighted composition operators can be found, for example, in [4, 6, 7, 10, 12, 18, 21–27].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. An Auxiliary Result

In this section we prove an auxiliary result which will be used in the proof of the main result of the paper.

Lemma 2.1. *Assume that $p \geq 1$, $n \in \mathbb{N}$, and $w \in \Pi_+$. Then the function*

$$f_{w,n}(z) = \frac{(\operatorname{Im} w)^{n-1/p}}{(z-\overline{w})^n}, \tag{2.1}$$

belongs to $H^p(\Pi_+)$. Moreover

$$\sup_{w\in\Pi_+} \left\|f_{w,n}\right\|_{H^p} \leq \pi^{1/p}. \tag{2.2}$$

Proof. Let $z = x + iy$ and $w = u + iv$. Then, we have

$$
\begin{aligned}
\|f_{w,n}\|_{H^p}^p &= \sup_{y>0}\int_{-\infty}^{\infty} \left|f_{w,n}(x+iy)\right|^p dx \\
&= (\operatorname{Im} w)^{np-1}\sup_{y>0}\int_{-\infty}^{\infty}\frac{dx}{|z-\overline{w}|^{np-2}|z-\overline{w}|^2} \\
&\le v^{np-1}\sup_{y>0}\int_{-\infty}^{\infty}\frac{dx}{\left((y+v)^2\right)^{(np-2)/2}\left((x-u)^2+(y+v)^2\right)} \\
&\le v^{np-1}\sup_{y>0}\frac{1}{(y+v)^{np-1}}\int_{-\infty}^{\infty}\frac{y+v}{(x-u)^2+(y+v)^2}dx \\
&= \sup_{y>0}\frac{v^{np-1}}{(y+v)^{np-1}}\int_{-\infty}^{\infty}\frac{dt}{t^2+1} = \pi,
\end{aligned}
\tag{2.3}
$$

where we have used the change of variables $x = u + t(y+v)$. □

3. Main Result

Here we formulate and prove the main result of the paper.

Theorem 3.1. *Assume $p \ge 1$ and φ is a holomorphic self-map of Π_+. Then $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}(\Pi_+)$ is bounded if and only if*

$$
\sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{2+1/p}}\left|\varphi'(z)\right|^2 < \infty, \tag{3.1}
$$

$$
\sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{1+1/p}}\left|\varphi''(z)\right| < \infty. \tag{3.2}
$$

Moreover, if the operator $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}/\mathbb{P}_1(\Pi_+)$ is bounded, then

$$
\|C_\varphi\|_{H^p(\Pi_+)\to\mathcal{Z}/\mathbb{P}_1(\Pi_+)} \asymp \sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{2+1/p}}\left|\varphi'(z)\right|^2 + \sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{1+1/p}}\left|\varphi''(z)\right|. \tag{3.3}
$$

Proof. First assume that the operator $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}(\Pi_+)$ is bounded.

For $w \in \Pi_+$, set

$$
f_w(z) = \frac{(\operatorname{Im} w)^{2-1/p}}{\pi^{1/p}(z-\overline{w})^2}. \tag{3.4}
$$

By Lemma 2.1 (case $n = 2$) we know that $f_w \in H^p(\Pi_+)$ for every $w \in \Pi_+$. Moreover, we have that

$$\sup_{w\in\Pi_+} \|f_w\|_{H^p(\Pi_+)} \leq 1. \tag{3.5}$$

From (3.5) and since the operator $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}(\Pi_+)$ is bounded, for every $w \in \Pi_+$, we obtain

$$\sup_{z\in\Pi_+} \operatorname{Im} z \left| f_w''(\varphi(z))(\varphi'(z))^2 + f_w'(\varphi(z))\varphi''(z) \right| = \|C_\varphi(f_w)\|_{\mathcal{Z}(\Pi_+)} \leq \|C_\varphi\|_{H^p(\Pi_+)\to\mathcal{Z}(\Pi_+)}. \tag{3.6}$$

We also have that

$$f_w'(z) = -2\frac{(\operatorname{Im} w)^{2-1/p}}{\pi^{1/p}(z-\overline{w})^3}, \qquad f_w''(z) = 6\frac{(\operatorname{Im} w)^{2-1/p}}{\pi^{1/p}(z-\overline{w})^4}. \tag{3.7}$$

Replacing (3.7) in (3.6) and taking $w = \varphi(z)$, we obtain

$$\operatorname{Im} z \left| \frac{3}{8}\frac{(\varphi'(z))^2}{(\operatorname{Im}\varphi(z))^{2+1/p}} - \frac{i}{4}\frac{\varphi''(z)}{(\operatorname{Im}\varphi(z))^{1+1/p}} \right| \leq \pi^{1/p}\|C_\varphi\|_{H^p(\Pi_+)\to\mathcal{Z}(\Pi_+)}, \tag{3.8}$$

and consequently

$$\frac{1}{4}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{1+1/p}}|\varphi''(z)| \leq \pi^{1/p}\|C_\varphi\|_{H^p(\Pi_+)\to\mathcal{Z}(\Pi_+)} + \frac{3}{8}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{2+1/p}}|\varphi'(z)|^2. \tag{3.9}$$

Hence if we show that (3.1) holds then from the last inequality, condition (3.2) will follow.

For $w \in \Pi_+$, set

$$g_w(z) = \frac{3}{i}\frac{(\operatorname{Im} w)^{2-1/p}}{\pi^{1/p}(z-\overline{w})^2} - 4\frac{(\operatorname{Im} w)^{3-1/p}}{\pi^{1/p}(z-\overline{w})^3}. \tag{3.10}$$

Then it is easy to see that

$$g_w'(w) = 0, \qquad g_w''(w) = \frac{C}{w^{2+1/p}}, \tag{3.11}$$

and by Lemma 2.1 (cases $n = 2$ and $n = 3$) it is easy to see that

$$\sup_{w\in\Pi_+} \|g_w\|_{H^p} < \infty. \tag{3.12}$$

From this, since $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}(\Pi_+)$ is bounded and by taking $w = \varphi(z)$, it follows that

$$C\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{2+1/p}}\left|\varphi'(z)\right|^2 \le \left\|C_\varphi(g_w)\right\|_{\mathcal{Z}(\Pi_+)} \le C\left\|C_\varphi\right\|_{H^p(\Pi_+)\to\mathcal{Z}(\Pi_+)}, \tag{3.13}$$

from which (3.1) follows, as desired.

Moreover, from (3.9) and (3.13) it follows that

$$\sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{2+1/p}}\left|\varphi'(z)\right|^2 + \sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{1+1/p}}\left|\varphi''(z)\right| \le C\left\|C_\varphi\right\|_{H^p(\Pi_+)\to\mathcal{Z}(\Pi_+)}. \tag{3.14}$$

Now assume that conditions (3.1) and (3.2) hold. By the Cauchy integral formula in Π_+ for $H^p(\Pi_+)$ functions (note that $p \ge 1$), we have

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt, \quad z \in \Pi_+. \tag{3.15}$$

By differentiating formula (3.15), we obtain

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{(t-z)^{n+1}}dt, \quad z \in \Pi_+, \tag{3.16}$$

for each $n \in \mathbb{N}$, from which it follows that

$$\left|f^{(n)}(z)\right| \le \frac{n!}{2\pi}\int_{-\infty}^{\infty}\frac{\left|f(t)\right|}{\left[(t-x)^2+y^2\right]^{(n+1)/2}}dt, \quad z \in \Pi_+. \tag{3.17}$$

By using the change $t - x = sy$, we have that

$$\int_{-\infty}^{\infty}\frac{y^n}{\left[(t-x)^2+y^2\right]^{(n+1)/2}}dt = \int_{-\infty}^{\infty}\frac{ds}{(s^2+1)^{(n+1)/2}} =: c_n < \infty, \quad n \in \mathbb{N}. \tag{3.18}$$

From this, applying Jensen's inequality on (3.17) and an elementary inequality, we obtain

$$\begin{aligned}\left|f^{(n)}(z)\right|^p &\le d_n\int_{-\infty}^{\infty}\frac{\left|f(t)\right|^p}{y^{np}}\frac{y^n}{\left[(t-x)^2+y^2\right]^{(n+1)/2}}dt\\ &\le d_n\int_{-\infty}^{\infty}\frac{\left|f(t)\right|^p}{y^{np+1}}dt \le d_n\frac{\|f\|^p_{H^p(\Pi_+)}}{y^{np+1}},\end{aligned} \tag{3.19}$$

where

$$d_n = \left(\frac{c_n n!}{2\pi}\right)^p, \tag{3.20}$$

from which it follows that

$$\left|f^{(n)}(z)\right| \le C\frac{\|f\|_{H^p(\Pi_+)}}{y^{n+1/p}}. \tag{3.21}$$

Assume that $f \in H^p(\Pi_+)$. By applying (3.21), and Lemma 1 in [1, page 188], we have

$$\begin{aligned}\|C_\varphi f\|_{\mathcal{Z}(\Pi_+)} &= |f(\varphi(i))| + |(f\circ\varphi)'(i)| + \sup_{z\in\Pi_+}\operatorname{Im} z\left|(C_\varphi f)''(z)\right| \\ &= |f(\varphi(i))| + |f'(\varphi(i))||\varphi'(i)| + \sup_{z\in\Pi_+}\operatorname{Im} z\left|f''(\varphi(z))(\varphi'(z))^2 + f'(\varphi(z))\varphi''(z)\right| \\ &\le C\|f\|_{H^p(\Pi_+)}\left(1 + \sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{2+1/p}}|\varphi'(z)|^2 + \sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{1+1/p}}|\varphi''(z)|\right).\end{aligned} \tag{3.22}$$

From this and by conditions (3.1) and (3.2), it follows that the operator $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}(\Pi_+)$ is bounded. Moreover, if we consider the space $\mathcal{Z}/\mathbb{P}_1(\Pi_+)$, we have that

$$\|C_\varphi\|_{H^p(\Pi_+)\to\mathcal{Z}/\mathbb{P}_1(\Pi_+)} \le C\left(\sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{2+1/p}}|\varphi'(z)|^2 + \sup_{z\in\Pi_+}\frac{\operatorname{Im} z}{(\operatorname{Im}\varphi(z))^{1+1/p}}|\varphi''(z)|\right). \tag{3.23}$$

From (3.14) and (3.23), we obtain the asymptotic relation (3.3). □

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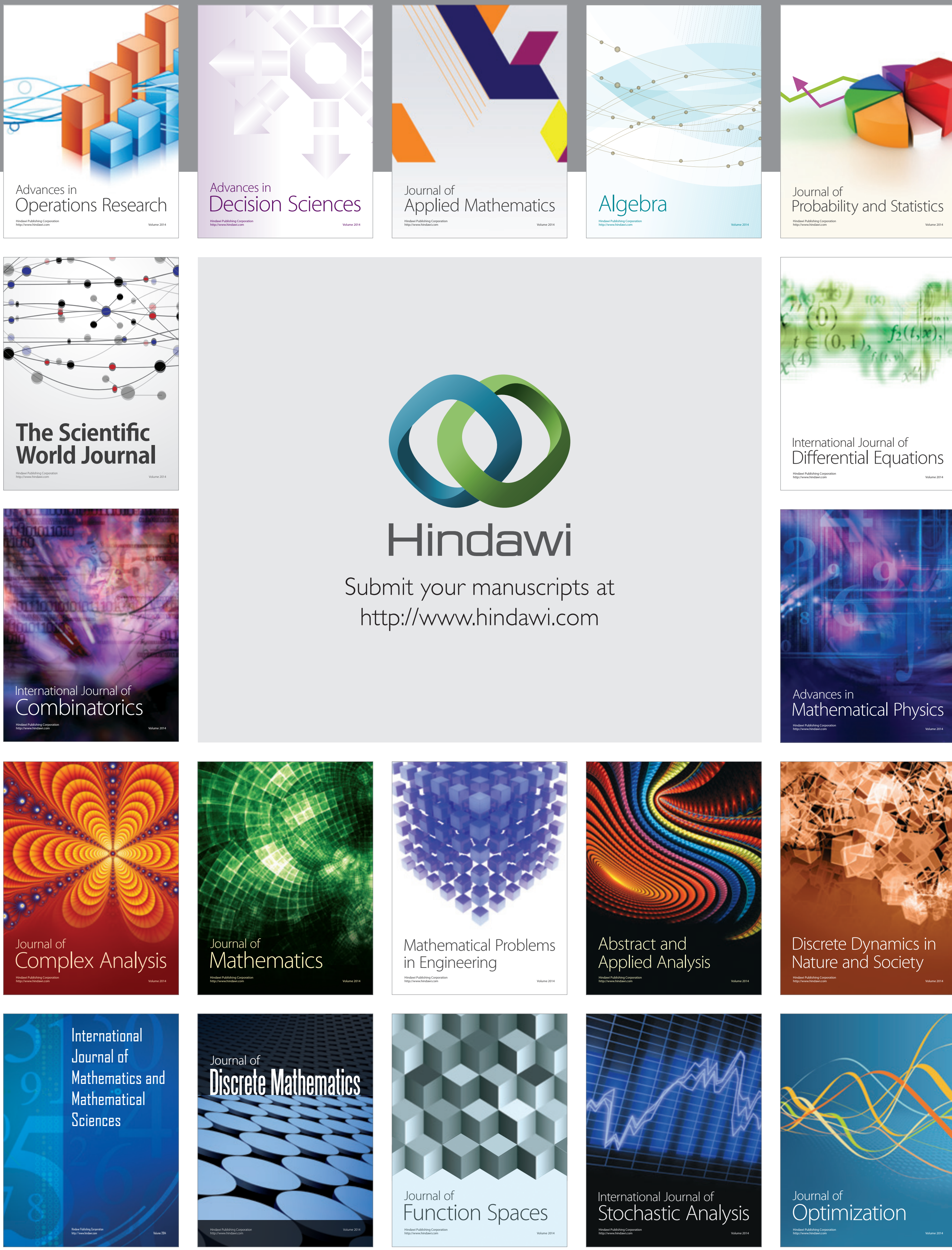
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