Research Article

The Mixed Type Splitting Methods for Solving Fuzzy Linear Systems

H. Saberi Najafi,[1] S. A. Edalatpanah,[1] and S. Shahabi[2]

[1] Department of Mathematics, Faculty of Sciences, University of Guilan, University Campus 2, Rasht, Iran
[2] Department of Mathematics, Faculty of Sciences, Islamic Azad University, Tonekabon Branch, Tonekabon, Iran

Correspondence should be addressed to S. A. Edalatpanah; saedalatpanah@gmail.com

We consider a class of fuzzy linear systems (FLS) and demonstrate some of the existing methods using the embedding approach for calculating the solution. The main aim in this paper is to design a class of mixed type splitting iterative methods for solving FLS. Furthermore, convergence analysis of the method is proved. Numerical example is illustrated to show the applicability of the methods and to show the efficiency of proposed algorithm.

1. Introduction

Solving fuzzy systems has been considered by many researchers, for example, [1–9] and the references therein. In [1, 2] Kandel et al. applied the embedding method for fuzzy linear system (hereafter denoted by FLS) and replaced the FLS by a $2n \times 2n$ crisp linear system. This model has been modified later by some other researchers; see [10–16] and the references therein.

Here, based on mixed type splitting, we introduce a new iterative method to FLS. The mixed type splitting iterative method [17, 18] is given for the linear system of equations $Ax = b$, where A is positive real. Cheng et al. in [19] presented a class of the mixed type splitting iterative methods based on [17, 18] and some convergence conditions were given. They also proposed some sufficient and necessary conditions of convergence when coefficient matrix of the linear system is certain matrices such as M-matrix.

In this paper, the mixed type splitting iterative method for FLS will be established, which is a generalization of mixed type splitting iterative method for linear system. Some sufficient conditions for convergence of the mixed type splitting iterative method will be considered. Moreover, we will discuss a comparison theorem, which describes the influences of the parameters on the convergence rates of the new methods.

2. Preliminaries

In this section we provide some basic notations and definitions of fuzzy number and fuzzy linear system.

Definition 1. An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements (see [2, 3]).

   (i) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function over $[0, 1]$.

   (ii) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function over $[0, 1]$.

   (iii) $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A crisp number α can be simply expressed as $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. The addition and scalar multiplication of fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$ and $y = (\underline{y}(r), \overline{y}(r))$ can be described as follows:

   (i) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$;

   (ii) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$;

   (iii) for all $K \in R$; $Kx = \begin{cases} (K\underline{x}, K\overline{x}), & K \geq 0, \\ (K\overline{x}, K\underline{x}), & K < 0. \end{cases}$

Definition 2. Consider the $n \times n$ linear system of the following equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2,$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n,$$

$$(1)$$

where the coefficient matrix $A = (a_{ij})$, $1 \le i, j \le n$, is a crisp matrix and $b_i \in E^1$, $1 \le i \le n$, is called a FLS.

Definition 3. A fuzzy number vector $X = (x_1, x_2, \ldots, x_n)^T$, given by the parametric form $x_i = (\underline{x_i}(r), \overline{x_i}(r))$, $1 \le i \le n$, $0 \le r \le 1$, is called a solution of the FLS (1) if

$$\underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{b_i},$$

$$\overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{b_i}.$$

$$(2)$$

Friedman et al. [2, 3], in order to solve the system given by (2), have solved a $2n \times 2n$ crisp linear system as follows:

$$SX = B, \qquad (3)$$

where $S = (s_{ij})$ are determined as follows:

$$a_{ij} \ge 0 \longrightarrow s_{ij} = a_{ij}, \quad s_{i+n,j+n} = a_{ij},$$

$$a_{ij} < 0 \longrightarrow s_{i,j+n} = -a_{ij}, \quad s_{i+n,j} = -a_{ij},$$

$$(4)$$

and any (s_{ij}) which is not determined by (4) is zero.

Then referring to [2, 3] we have

$$S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_1 \end{bmatrix}, \qquad X = \begin{bmatrix} \underline{x} \\ -\overline{x} \end{bmatrix}, \qquad B = \begin{bmatrix} \underline{b} \\ -\overline{b} \end{bmatrix}$$

$$\implies \begin{cases} s_1 \underline{x} - s_2 \overline{x} = \underline{b}, \\ s_2 \underline{x} - s_1 \overline{x} = -\overline{b}. \end{cases}$$

$$(5)$$

Or

$$S = \begin{bmatrix} s_1 & -s_2 \\ -s_2 & s_1 \end{bmatrix}, \qquad X = \begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix}, \qquad B = \begin{bmatrix} \underline{b} \\ \overline{b} \end{bmatrix}$$

$$\implies \begin{cases} s_1 \underline{x} - s_2 \overline{x} = \underline{b}, \\ -s_2 \underline{x} + s_1 \overline{x} = \overline{b}, \end{cases}$$

$$(6)$$

where $s_1, s_2 \ge 0$ and $A = s_1 - s_2$.

Definition 4 (see [20]). (a) A matrix $A = a_{ij}$ is called a Z-matrix if, for any $i \ne j$, $a_{ij} \le 0$.

(b) A Z-matrix is an L-matrix, if $a_{ii} > 0$.

(c) A Z-matrix is an M-matrix, if A is nonsingular and $A^{-1} \ge 0$.

(d) For any matrix A the comparison matrix $\langle A \rangle = (m_{ij}) \in R^{n \times n}$ is defined by

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad i \ne j, 1 \le i, j \le n. \quad (7)$$

(e) A complex matrix A is an H-matrix if $\langle A \rangle = (m_{ij}) \in R^{n \times n}$ is an M-matrix.

(f) Matrix A is called a Generalized Diagonally Dominant Matrix (GDDM) if there exists a positive diagonal matrix W such that AW is a Strictly Diagonally Dominant Matrix (SDDM).

Lemma 5 (see [19]). *Let A be the coefficient matrix of the linear system $Ax = b$. If A is an M-matrix and conditions of (19) are satisfied, then the mixed type splitting iterative method is convergent.*

Remark 6. For any splitting, $A = M - N$, where M is nonsingular, the iterative method for solving linear systems of $Ax = b$ is

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \quad i = 0, 1, \ldots. \quad (8)$$

This iterative process converges to the unique solution $x = A^{-1}b$ for initial vector value $x^0 \in R^n$ if and only if the spectral radius $\rho(M^{-1}N) < 1$, where $T = M^{-1}N$ is called the iteration matrix [20]. For example, suppose $\text{diag}(A) = I$ and $A = I - L - U$, where L and U are strictly lower and strictly upper triangular part of A, respectively. Then for classical AOR (see; [10]) we have

$$M_{r,w} = \frac{1}{w} (I - rL),$$

$$N_{r,w} = \frac{1}{w} ((1 - w) I + (w - r) L + wU)$$

$$\implies l_{r,w} = M_{r,w}^{-1} N_{r,w}$$

$$= (I - rL)^{-1} ((1 - w) I + (w - r) L + wU).$$

$$(9)$$

And when $w = r$, we have SOR method [20]; that is,

$$M_w = \frac{1}{w} (I - wL), \qquad N_w = \frac{1}{w} ((1 - w) I + wU)$$

$$\implies l_w = M_w^{-1} N_w.$$

$$(10)$$

Lemma 7 (see [19]). *Let A be the coefficient matrix of the linear system $Ax = b$. If A is an L-matrix, conditions of (19) are satisfied and*

$$0 \le D_d \le \frac{1}{w} (1 - w) D,$$

$$0 \le L_l \le \left(1 - \frac{r}{w}\right),$$

$$0 \le r \le w \le 1.$$

$$(11)$$

Then

(i) *If $\rho(l_{r,w}) < 1 \implies \rho(T) \le \rho(L_{r \cdot w})$.*

(ii) If $\rho(l_{r,w}) = 1 \Rightarrow \rho(T) = \rho(L_{r \cdot w})$.

(iii) If $\rho(l_{r,w}) > 1 \Rightarrow \rho(T) \geq \rho(L_{r \cdot w})$.

Lemma 8 (see [20]). *Matrix A is a GDDM matrix if and only if A is an H-matrix.*

3. The Mixed Type Splitting Iterative Methods for Fuzzy Linear Systems

Let S be nonsingular and $S = D - L - U$, where

$$
\begin{bmatrix} s_1 & -s_2 \\ -s_2 & s_1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix} = \begin{bmatrix} \underline{b} \\ \overline{b} \end{bmatrix},
$$

$$
D = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}, \qquad L = \begin{bmatrix} L_1 & 0 \\ s_2 & L_1 \end{bmatrix},
$$

$$
U = \begin{bmatrix} U_1 & s_2 \\ 0 & U_1 \end{bmatrix}, \qquad D_1 = \text{diag}(s_1) > 0, \tag{12}
$$

$$
D_1 - L_1 - U_1 = S_1,
$$

and $-L_1$, $-U_1$ are strictly lower and upper triangular matrices of s_1, respectively. The iterative method for $SX = B$ is

$$
X^{(i+1)} = M^{-1} N X^{(i)} + M^{-1} B, \quad i = 0, 1, \ldots, \tag{13}
$$

where $S = M - N$, $\det(M) \neq 0$, and $X^{(0)}$ is any initial vector. There are several well-known point iterative methods and block numerical iterative methods for FLS such as Jacobi, Gauss-Seidel, SOR, and AOR; see [9–14]. As a matter of fact, these methods are generalization of iterative methods for crisp linear systems $Ax = b$. For instance, in AOR method for FLS [12] we have

$$
x^{(i+1)} = T_{r,w} x^{(i)} + (D - rL)^{-1} wb, \quad i = 0, 1, \ldots, \tag{14}
$$

where the iterative matrix is

$$
T_{r,w} = (D - rL)^{-1} [(1 - w) D + (w - r) L + wU]. \tag{15}
$$

Therefore, we obtain

$$
M = (D - rL) = \begin{pmatrix} D_1 - rL_1 & 0 \\ -rS_2 & D_1 - rL_1 \end{pmatrix} \Longrightarrow M^{-1} = \begin{pmatrix} \underbrace{(D_1 - rL_1)^{-1}}_{k} & \underbrace{0}_{\beta} \\ \underbrace{r(D_1 - rL_1)^{-1} S_2 (D_1 - rL_1)^{-1}}_{\eta} & \underbrace{(D_1 - rL_1)^{-1}}_{\gamma} \end{pmatrix}. \tag{16}
$$

And we have

$$
T_{r,w} = M^{-1} N = \begin{pmatrix} k & 0 \\ rkS_2k & k \end{pmatrix} \left\{ \begin{pmatrix} (1-w)D_1 & 0 \\ 0 & (1-w)D_1 \end{pmatrix} + \begin{pmatrix} (w-r)L_1 & 0 \\ (w-r)S_2 & (w-r)L_1 \end{pmatrix} + \begin{pmatrix} wU_1 & wS_2 \\ 0 & wU_1 \end{pmatrix} \right\}
$$

$$
\Longrightarrow T_{r,w} = \begin{bmatrix} \underbrace{k[(1-w)D_1 + (w-r)L_1 + wU_1]}_{\Phi} & \underbrace{wkS_2}_{\Gamma} \\ \underbrace{wkS_2k[(1-r)D_1 + rU_1]}_{\Omega} & \underbrace{k[(1-w)D_1 + (w-r)L_1 + rwS_2kS_2 + wU_1]}_{\Psi} \end{bmatrix}. \tag{17}
$$

Other methods are the same and we know that, by choosing special parameters, the similar results can be obtained, for example,

(1) Jacobi method for $w = 1$, $r = 0$;

(2) JOR (Jacobi Overrelaxation) method for $r = 0$;

(3) Gauss-Seidel method for $r = w = 1$;

(4) SOR method for $r = w$.

Now, from $X^{(i)} = [\frac{\underline{x}^{(i)}}{\overline{x}^{(i)}}]$ we have the following algorithms (mentioned in [12] for the first time).

Algorithm 9. AOR iterative method for FLS.

Step 1. Choose an initial vector $X^{(0)} = [\frac{\underline{x}^{(0)}}{\overline{x}^{(0)}}]$ and parameters r and w.

Step 2. For $k = 0, 1, 2, \ldots$ do

$$
\underline{x}^{(i+1)} = \Phi \underline{x}^{(i)} + \Gamma \overline{x}^{(i)} + w \left(k\underline{b} + \beta \overline{b} \right)
$$

$$
\overline{x}^{(i+1)} = \Psi \overline{x}^{(i)} + \Omega \underline{x}^{(i)} + w \left(\eta \underline{b} + \gamma \overline{b} \right). \tag{18}
$$

Step 3. If $\|\overline{x}^{(i+1)} - \overline{x}^{(i)}\| / \|\overline{x}^{(i+1)}\| < \varepsilon$ or $\|\underline{x}^{(i+1)} - \underline{x}^{(i)}\| / \|\underline{x}^{(i+1)}\| < \varepsilon$, then stop; otherwise set $k = k + 1$ and go to Step 2.

Step 4. End for.

Step 5. Return to $X^{(k+1)}$.

Next, we will establish the mixed type splitting iterative method for FLS. The mixed type splitting iterative methods [19] for solving $Ax = b$ are given by the following:

$$\left(D + D_d + L_s - L \right) x^{(i+1)} = \left(D_d + L_s + U \right) x^{(i)} + b, \tag{19}$$

$$i = 0, 1, 2, \ldots;$$

whose iteration matrix is

$$T = \left(D + D_d + L_s - L \right)^{-1} \left(D_d + L_s + U \right), \tag{20}$$

$$A = D - L - U, \quad D_d \geq 0, \quad 0 \leq L_s \leq L.$$

Now, from $X^{(i)} = \left[\frac{\underline{x}^{(i)}}{\overline{x}^{(i)}} \right]$ we have the following algorithms (mentioned in [12] for the first time).

Algorithm 10. Mixed type splitting iterative methods_1 for FLS.

Step 1. Choose an initial vector $X^{(0)} = \left[\frac{\underline{x}^{(0)}}{\overline{x}^{(0)}} \right]$ and parameters r and w.

Step 2. For $k = 0, 1, 2, \ldots$ do

$$\underline{x}^{(i+1)} = \widehat{\Phi}\underline{x}^{(i)} + \widehat{\Gamma}\overline{x}^{(i)} + \left(k\underline{b} \right)$$

$$\overline{x}^{(i+1)} = \widehat{\Psi}\overline{x}^{(i)} + \widehat{\Omega}\underline{x}^{(i)} + \left(\widehat{\eta}\underline{b} + k\overline{b} \right). \tag{25}$$

Step 3. If $\|\overline{x}^{(i+1)} - \overline{x}^{(i)}\| / \|\overline{x}^{(i+1)}\| < \varepsilon$ and $\|\underline{x}^{(i+1)} - \underline{x}^{(i)}\| / \|\underline{x}^{(i+1)}\| < \varepsilon$, then stop; otherwise set $k = k + 1$ and go to Step 2.

Step 4. End for.

Step 5. Return to $X^{(k+1)}$.

Note that when

$$D_d = \frac{1}{w}\left(1 - w \right) D_1, \quad L_s = 0, \quad L_{2s} = 0, \tag{26}$$

we have SOR method for FLS (see [11]) and when

$$D_d = \frac{1}{w}\left(1 - w \right) D_1, \quad L_s = \frac{1}{w}\left(w - r \right) L_1, \quad L_{2s} = 0, \tag{27}$$

Now, we consider the mixed type splitting iterative methods for solving FLS. Based on above demonstration, we have

$$\underbrace{\left[\begin{array}{cc} \underbrace{D_1 + D_d + L_s - L_1}_{E} & 0 \\ L_{2s} - s_2 & D_1 + D_d + L_s - L_1 \end{array} \right]}_{M} \left[\begin{array}{c} \underline{x}^{(i+1)} \\ \overline{x}^{(i+1)} \end{array} \right]$$

$$= \underbrace{\left[\begin{array}{cc} D_d + L_s + U_1 & s_2 \\ L_{2s} & D_d + L_s + U_1 \end{array} \right]}_{N} \left[\begin{array}{c} \underline{x}^{(i)} \\ \overline{x}^{(i)} \end{array} \right] + \left[\begin{array}{c} \underline{b} \\ \overline{b} \end{array} \right], \tag{21}$$

where $0 \leq L_s \leq L$ and $0 \leq L_{2s} \leq s_2$. Now, let

$$k = \left(E + L_s - L_1 \right)^{-1}. \tag{22}$$

Therefore, we obtain

$$M^{-1} = \left(\begin{array}{cc} k & 0 \\ \underbrace{-k(L_{2s} - s_2)k}_{\widehat{\eta}} & k \end{array} \right). \tag{23}$$

And we have

$$T = M^{-1}N = \left(\begin{array}{cc} \underbrace{k\left[D_d + L_s + U_1 \right]}_{\widehat{\Phi}} & \underbrace{ks_2}_{\widehat{\Gamma}} \\ \underbrace{-k\left(L_{2s} - s_2 \right)k\left(D_d + L_s + U_1 \right) + kL_{2s}}_{\widehat{\Omega}} & \underbrace{-k\left(L_{2s} - s_2 \right)ks_2 + k\left(D_d + L_s + U_1 \right)}_{\widehat{\Psi}} \end{array} \right). \tag{24}$$

where (w, r) are real parameters with $w \neq 0$, we have AOR method for FLS (see [12]).

Theorem 11. *Let A be an H-matrix and let conditions of Lemma 7 be satisfied.*
   Then we have

$$(i): \rho(T) \leq \rho(\langle T \rangle) \leq \rho(\langle l_{r,w} \rangle) < 1, \tag{28}$$

$$(ii): \rho(T) \leq \rho(\langle T \rangle) \leq \rho(\langle l_w \rangle) < 1.$$

Proof. We only prove (i); (ii) can be similarly verified.
   Let A be an H-matrix. Then $\langle A \rangle$ is an M-matrix and by Lemmas 5 and 7,

$$\rho(\langle T \rangle) \leq \rho(\langle l_{r,w} \rangle) < 1. \tag{29}$$

By definition of mixed splitting method, we have

$$|T| = \left| \left(D + D_d + L_l - L \right)^{-1} \left(D_d + L_l + U \right) \right|$$

$$= \left| E\left(I - E^{-1}\left(L - L_l \right) \right)^{-1} \right.$$

$$\left. \times \left[E\left\{ E^{-1}D_d + E^{-1}L_l + E^{-1}U \right\} \right] \right|$$

$$= \left| \left(I - E^{-1} \left(L - L_l \right) \right)^{-1} \left[E^{-1} D_d + E^{-1} L_l + E^{-1} U \right] \right|$$

$$= \left| \left(I + \left(E^{-1} \left(L - L_l \right) \right) \right. \right.$$

$$\left. + \left(E^{-1} \left(L - L_l \right) \right)^2 + \cdots \right) \left[E^{-1} D_d + E^{-1} L_l + E^{-1} U \right] \right|$$

$$= \left| \left(\left(E^{-1} \left(L - L_l \right) \right) + \left(E^{-1} \left(L - L_l \right) \right)^2 + \cdots \right) \right.$$

$$\times \left[E^{-1} D_d + E^{-1} L_l + E^{-1} U \right]$$

$$\left. + \left[E^{-1} D_d + E^{-1} L_l + E^{-1} U \right] \right|$$

$$\leq \left(\left\{ \left| E^{-1} \left(L - L_l \right) \right| + \left(\left| E^{-1} \left(L - L_l \right) \right| \right)^2 + \cdots \right\} \right.$$

$$\times \left[\left| E^{-1} D_d \right| + \left| E^{-1} L_l \right| + \left| E^{-1} U \right| \right] \right)$$

$$\left. + \left(\left[\left| E^{-1} D_d \right| + \left| E^{-1} L_l \right| + \left| E^{-1} U \right| \right] \right) \right.$$

$$= \left(I - \left| E^{-1} \left(L - L_l \right) \right| \right)^{-1} \left(\left| E^{-1} D_d \right| + \left| E^{-1} L_l \right| + \left| E^{-1} U \right| \right)$$

$$= \langle T \rangle \implies \rho \left(|T| \right) \leq \rho \left(\langle T \rangle \right). \tag{30}$$

Therefore we have

$$\rho \left(T \right) \leq \rho \left(|T| \right) \leq \rho \left(\langle T \rangle \right). \tag{31}$$

And the proof is completed. □

Lemma 12. *The Matrix S in (5) or (6) is an H-matrix if and only if A in (1) is H-matrix.*

Proof. Let A be an H-matrix; then by Lemma 8, there exists a positive diagonal matrix W such that AW is strictly diagonally dominant matrix. Without loss of generality, let AW be row strictly diagonally dominant; that is,

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \left| a_{ij} w_{jj} \right| < \left| a_{ii} w_{ii} \right|. \tag{32}$$

Now, let

$$j^+ = \left\{ a_{ij} \mid a_{ij} \geq 0 \right\}, \quad j^- = \left\{ a_{ij} \mid a_{ij} < 0 \right\}, \quad 1 \leq i, j \leq n; \tag{33}$$

then we have

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \left| a_{ij} w_{jj} \right| = \sum_{\substack{j \in j^+ \\ j \neq i}} \left| a_{ij} w_{jj} \right| + \sum_{\substack{j \in j^- \\ j \neq i}} \left| a_{ij} w_{jj} \right| < \left| a_{ii} w_{ii} \right|. \tag{34}$$

By considering the structure of S and since, for all $i = 1, \ldots, n$, $a_{ii} \geq 0$, we have

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \left| s_{ij} \right| = \sum_{\substack{j \in j^+ \\ j \neq i}} \left| a_{ij} \right|, \qquad \sum_{j=n+1}^{2n} \left| s_{ij} \right| = \sum_{j \in j^-} \left| a_{ij} \right|,$$

$$1 \leq i \leq n, \tag{35}$$

$$\sum_{j=1}^{n} \left| s_{ij} \right| = \sum_{j \in j^-} \left| a_{ij} \right|, \qquad \sum_{\substack{j=n+1 \\ j \neq i}}^{2n} \left| s_{ij} \right| = \sum_{\substack{j \in j^+ \\ j \neq i}} \left| a_{ij} \right|,$$

$$n + 1 \leq i \leq 2n.$$

Therefore

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \left| s_{ij} w_{jj} \right| = \sum_{\substack{j \in j^+ \\ j \neq i}} \left| a_{ij} w_{jj} \right| + \sum_{\substack{j \in j^- \\ j \neq i}} \left| a_{ij} w_{jj} \right| < \left| a_{ii} w_{ii} \right|$$

$$1 \leq i \leq n,$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \left| s_{ij} w_{jj} \right| = \sum_{\substack{j \in j^- \\ j \neq i}} \left| a_{ij} w_{jj} \right| + \sum_{\substack{j \in j^+ \\ j \neq i}} \left| a_{ij} w_{jj} \right| < \left| a_{ii} w_{ii} \right| \tag{36}$$

$$n + 1 \leq i \leq 2n,$$

$$\implies \sum_{\substack{j=1 \\ j \neq i}}^{2n} \left| s_{ij} w_{jj} \right| < \left| a_{ii} w_{ii} \right| = \left| s_{ii} w_{ii} \right|.$$

Then, by choice of

$$T = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}_{2n \times 2n} \tag{37}$$

ST is row $SDDM$. Therefore, by Lemma 8, S is also an H-matrix. Conversely, if S is an H-matrix, then by reasoning similar to that above, it can be seen that A is an H-matrix too. □

Therefore, we can obtain the following theorem.

Theorem 13. *Let A be an H-matrix and the following conditions are satisfied:*

$$0 \leq D_d \leq \frac{1}{w} \left(1 - w \right) D_1, \qquad 0 \leq L_l \leq \frac{1}{w} \left(w - r \right) L_1,$$

$$0 \leq L_{2l} \leq s_2. \tag{38}$$

Then the speed of convergence of Algorithm 9 is faster than the speed of convergence of SOR and AOR methods for FLS.

Proof. Let A be an H-matrix; then by Lemma 12, S is an H-matrix too. Furthermore, by Theorem 11, $\rho(\langle l_{r,w} \rangle) < 1$. Moreover, similar to the proving process of Theorem 11 and (31), we can obtain

$$\rho \left(l_{r,w} \right) \leq \rho \left(\left| l_{r,w} \right| \right) \leq \rho \left(\langle l_{r,w} \rangle \right). \tag{39}$$

TABLE 1: $(D_d = 0.1(-1 + 1/w)D_1; L_s = 0.5(1 - r/w)L_1; L_{2s} = 0.01s_2)$.

Method			AOR or SOR iterative methods			Algorithm 10			Algorithm 14		
n	w	r	Iter.	ELP	ρ	Iter.	ELP	ρ	Iter.	ELP	ρ
	0.5	0.1	64	0.003005	0.6739	20	0.001933	0.3764	20	0.001228	0.3228
5	0.5	0.5	62	0.003032	0.6322	17	0.001876	0.2278	15	0.001183	0.1476
	0.7	0.3	34	0.001592	0.5191	16	0.001545	0.2530	16	0.001199	0.1953
	0.9	0.6	21	0.001523	0.3192	13	0.001109	0.1399	12	0.001056	0.1084
	0.5	0.1	528	0.034716	0.9173	219	0.024478	0.9261	188	0.019920	0.5229
50	0.5	0.5	460	0.032090	0.8419	142	0.020052	0.8847	126	0.015032	0.6006
	0.7	0.3	359	0.029076	0.8241	175	0.019944	0.9072	152	0.018738	0.4894
	0.9	0.6	189	0.019587	0.9227	140	0.015078	0.8831	125	0.013490	0.6094
	0.5	0.1	1123	0.069947	0.9290	403	0.053815	0.9611	343	0.069539	0.5275
100	0.5	0.5	652	0.060993	0.8534	264	0.041211	0.9392	233	0.060378	0.6449
	0.7	0.3	753	0.057825	0.8544	324	0.046366	0.9511	279	0.053829	0.5134
	0.9	0.6	452	0.051150	0.6884	260	0.037807	0.9383	232	0.053185	0.6554
	0.5	0.1	2578	3.042835	0.9672	920	0.898741	0.9840	780	0.795855	0.5320
250	0.5	0.5	1993	1.967367	0.9302	619	0.635806	0.9749	549	0.582322	0.6738
	0.7	0.3	1553	1.566464	0.9419	746	0.761868	0.9798	644	0.656939	0.5342
	0.9	0.6	882	0.885911	0.8819	607	0.617928	0.9745	543	0.571853	0.6854

Therefore,

$$\rho\left(l_{r,w}\right) < 1, \tag{40}$$

and by Lemma 7 the proof is completed. □

In the sequel, we will improve Algorithm 10 by the following updating technique.

(I) First, use $\underline{x}^i$ and $\overline{x}^i$ from our previous information to compute the new $\underline{x}^{(i+1)}$ exactly as in the first line of Step 2.

(II) Then, replace $\underline{x}^{(i)}$ in the second line of Step 2 with $\underline{x}^{(i+1)}$, obtained just as in the above, to compute the new $\overline{x}^{(i+1)}$.

Algorithm 14. Mixed type splitting iterative methods_2 for FLS.

Step 1. Choose an initial vector $X^{(0)} = [\frac{\underline{x}^{(0)}}{\overline{x}^{(0)}}]$ and parameters r and w.

Step 2. For $k = 0, 1, 2, \ldots$ do

$$\underline{x}^{(i+1)} = \widehat{\Phi}\underline{x}^{(i)} + \widehat{\Gamma}\overline{x}^{(i)} + (k\underline{b}),$$

$$upd = \widehat{\Omega}\underline{x}^{(i+1)} \tag{41}$$

$$\overline{x}^{(i+1)} = \widehat{\Psi}\overline{x}^{(i)} + upd + \left(\widehat{\eta}\underline{b} + k\overline{b}\right).$$

Step 3. If $\|\overline{x}^{(i+1)} - \overline{x}^{(i)}\|/\|\overline{x}^{(i+1)}\| < \varepsilon$ or $\|\underline{x}^{(i+1)} - \underline{x}^{(i)}\|/\|\underline{x}^{(i+1)}\| < \varepsilon$, then stop; otherwise set $k = k + 1$ and go to Step 2.

Step 4. End for.

Step 5. Return to $X^{(k+1)}$.

4. Numerical Example

In this section, we give an example of FLS to illustrate the results obtained in previous sections.

Example 1. Consider the $n \times n$ fuzzy system

$$2x_1 - x_3 = (2 + r, 4 - r),$$

$$x_2 + 2x_3 - x_4 = (2 + r, 4 - r),$$

$$x_3 + 2x_4 - x_5 = (2 + r, 4 - r),$$

$$\vdots \tag{42}$$

$$x_{n-3} + 2x_{n-2} = (2 + r, 4 - r),$$

$$x_{n-2} + 2x_{n-1} = (2 + r, 4 - r),$$

$$x_{n-1} + 2x_n - x_1 = (2 + r, 4 - r).$$

The extended $2n \times 2n$ matrix is

$$S = \begin{bmatrix} s_1 & -s_2 \\ -s_2 & s_1 \end{bmatrix}, \tag{43}$$

where

$$s_1 = \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 2 & 0 & 0 \\ 0 & \cdots & \ddots & 0 & 1 & 2 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 2 \end{bmatrix},$$

$$s_2 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(44)$$

Evidently, A is an H-matrix and therefore S is also an H-matrix. Table 1 shows the numerical results of the above example with the tolerance $\varepsilon = 10^{-10}$. Moreover, the initial approximation is zero vector. In the Table 1, we reported the number of iterations (Iter), Elapsed time (ELP), and associated spectral radii (ρ) for the iterative methods with different n, w, and r. This results show that our algorithms can be applied to a large class of FLS. Furthermore, from Table 1, we can see that for solving FLS, Algorithm 14 is superior to the other methods from point of view rate of convergence.

This example is computed with MATLAB7 on a personal computer Pentium 4-256 MHZ.

5. Conclusion

In this paper, we have proposed a class of splitting methods, called mixed type splitting iterative method for fuzzy linear systems that contains auxiliary matrices. Furthermore, our method with some other iterative methods is compared in the frame of fuzzy linear systems and it is shown that by proper choice of these auxiliary matrices, the new iterative method is faster than these mentioned methods form point of view of the convergence speed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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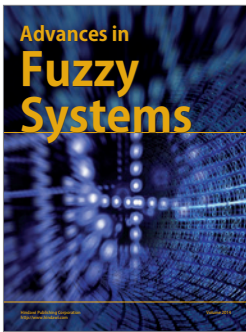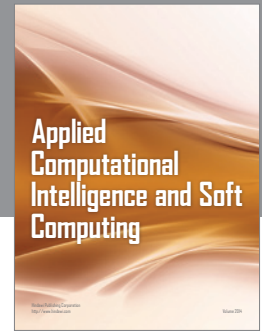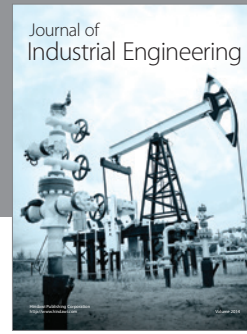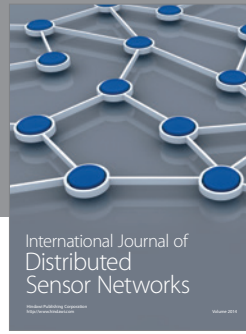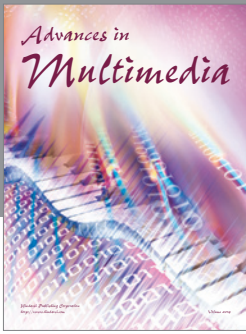
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