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Research Article

Stability of a Functional Equation Deriving from Cubic and Quartic Functions

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We obtain the general solution and the generalized Ulam-Hyers stability of the cubic and quartic functional equation $4(f(3x+y)+f(3x-y)) = -12(f(x+y)+f(x-y))+12(f(2x+y)+f(2x-y)) - 8f(y) - 192f(x) + f(2y) + 30f(2x)$.

1. Introduction

The stability problem of functionalequations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x)*h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta \tag{1.1}$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \delta \tag{1.2}$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. Finally, in 1978, Th. M. Rassias [3] proved the following theorem.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.3}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\left\| f(x) - T(x) \right\| \le \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.4}$$

for all $x \in E$. If $p < 0$, then inequality (1.3) holds for all $x, y \neq 0$, and (1.4) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.

In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [2, 4–13]). On the other hand, J. M. Rassias [14–16] generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J. M. Rassias theorem.

Theorem 1.2. *If it is assumed that there exist constants $\Theta \ge 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : E \to E'$ is a map from a norm space E into a Banach space E' such that the inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \epsilon\|x\|^{p_1}\|y\|^{p_2} \tag{1.3p}$$

for all $x, y \in E$, then there exists a unique additive mapping $T : E \to E'$ such that

$$\left\| f(x) - T(x) \right\| \le \frac{\Theta}{2-2^p}\|x\|^p, \tag{1.5}$$

for all $x \in E$. If in addition for every $x \in E$, $f(tx)$ is continuous in real t for each fixed x, then T is linear (see [14, 15, 17–22]).

The oldest cubic functional equation, and was introduced by J. M. Rassias [23, 24] is as follows:

$$f(x+2y) + 3f(x) = 3f(x+y) + f(x-y) + 6f(y). \tag{1.6}$$

Jun and Kim [25] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.7}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.7). The function $f(x) = x^3$ satisfies the functional equation (1.7), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.7) if and only if there exists a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. The oldest quartic functional equation, and was introducedby J. M. Rassias [16, 26], and then was employed by Park and Bae [27] , such that

$$f(x+2y) + f(x-2y) = 4\big(f(x+y) + f(x-y)\big) + 24f(y) - 6f(x). \tag{1.8}$$

In fact, they proved that a function f between real vector spaces X and Y is a solution of (1.8) if and only if there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \to Y$ such that $f(x) = Q(x,x,x,x)$ for all x (see also [27–33]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.8), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

We deal with the following functional equation deriving from quartic and cubic functions:

$$\begin{aligned} 4\big(f(3x+y) + f(3x-y)\big) = {} & -12\big(f(x+y) + f(x-y)\big) + 12\big(f(2x+y) + f(2x-y)\big) \\ & - 8f(y) - 192f(x) + f(2y) + 30f(2x). \end{aligned} \tag{1.9}$$

It is easy to see that the function $f(x) = ax^4 + bx^3$ is a solution of the functional equation (1.9). In the present paper, we investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.9).

2. General solution

In this section, we establish the general solution of functional equation (1.9).

Theorem 2.1. *Let X, Y be vector spaces, and let $f : X \to Y$ be a function. Then f satisfies* (1.9) *if and only if there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \to Y$ and a unique function $C : X \times X \times X \to Y$ such that C is symmetric for each fixed one variable and is additive for fixed two variables, and that $f(x) = Q(x,x,x,x) + C(x,x,x)$ for all $x \in X$.*

Proof. Suppose there exists a symmetric multiadditive function $Q : X \times X \times X \times X \to Y$ and a function $C : X \times X \times X \to Y$ such that C is symmetric for each fixed one variable and is additive for fixed two variables, and that $f(x) = Q(x,x,x,x) + C(x,x,x)$ for all $x \in X$. Then it is easy to see that f satisfies (1.9). For the convlet f satisfy (1.9). We decompose f into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}\big(f(x) + f(-x)\big), \qquad f_o(x) = \frac{1}{2}\big(f(x) - f(-x)\big) \tag{2.1}$$

for all $x \in X$. By (1.9), we have

$$
\begin{aligned}
&4f_e(3x+y)+4f_e(3x-y)\\
&\quad=\frac{1}{2}\left[4f(3x+y)+4f(-3x-y)+4f(3x-y)+4f(-3x+y)\right]\\
&\quad=\frac{1}{2}\left[4f(3x+y)+4f(3x-y)\right]+\frac{1}{2}\left[4f\left((-3x)+(-y)\right)+4f\left((-3x)-(-y)\right)\right]\\
&\quad=\frac{1}{2}\left[12f(2x+y)+12f(2x-y)-12f(x+y)-12f(x-y)\right.\\
&\qquad\qquad\left.-8f(y)-192f(x)+f(2y)+30f(2x)\right]\\
&\qquad+\frac{1}{2}\left[12f(-2x-y)+12f\left((-2x)+y\right)-12f(-x-y)-12f(-x+y)\right.\\
&\qquad\qquad\left.-8f(-y)-192f(-x)+f(-2y)+30f(-2x)\right]\\
&\quad=12\left[\frac{1}{2}\left(f(2x+y)+f\left(-(2x+y)\right)\right)\right]+12\left[\frac{1}{2}\left(f(2x-y)+f\left(-(2x-y)\right)\right)\right]\\
&\qquad-12\left[\frac{1}{2}\left(f(x+y)+f\left(-(x+y)\right)\right)\right]-12\left[\frac{1}{2}\left(f(x-y)+f\left(-(x-y)\right)\right)\right]\\
&\qquad-8\left[\frac{1}{2}\left(f(y)+f(-y)\right)\right]-192\left[\frac{1}{2}\left(f(x)+f(-x)\right)\right]\\
&\qquad+\frac{1}{2}\left[f(2y)+f(-2y)\right]+30\left[\frac{1}{2}\left(f(2x)+f(-2x)\right)\right]\\
&\quad=12\left(f_e(2x+y)+f_e(2x-y)\right)-12\left(f_e(x+y)+f_e(x-y)\right)\\
&\qquad-8f_e(y)-192f_e(x)+f_e(2y)+30f_e(2x)
\end{aligned}
\tag{2.2}
$$

for all $x, y \in X$. This means that f_e satisfies (1.9), or

$$
\begin{aligned}
4\left(f_e(3x+y)+f_e(3x-y)\right)&=-12\left(f_e(x+y)+f_e(x-y)\right)+12\left(f_e(2x+y)+f_e(2x-y)\right)\\
&\quad-8f_e(y)-192f_e(x)+f_e(2y)+30f_e(2x).
\end{aligned}
\tag{1.9e}
$$

Now, putting $x = y = 0$ in (1.9*e*), we get $f_e(0) = 0$. Setting $x = 0$ in (1.9*e*), by evenness of f_e we obtain

$$
f_e(2y) = 16f_e(y) \tag{2.3}
$$

for all $y \in X$. Hence, (1.9*e*) can be written as

$$
\begin{aligned}
&f_e(3x+y)+f_e(3x-y)+3\left(f_e(x+y)+f_e(x-y)\right)\\
&\quad=3\left(f_e(2x+y)+f_e(2x-y)\right)+72f_e(x)+2f_e(y)
\end{aligned}
\tag{2.4}
$$

for all $x, y \in X$. With the substitution $y := 2y$ in (2.4), we have

$$\begin{aligned} &f_e(3x+2y)+f_e(3x-2y)+3f_e(x+2y)+3f_e(x-2y) \\ &\quad = 48f_e(x+y)+48f_e(x-y)+72f_e(x)+32f_e(y). \end{aligned} \tag{2.5}$$

Replacing y by $x+2y$ in (2.4), we obtain

$$\begin{aligned} &16f_e(2x+y)+16f_e(x-y)+48f_e(x+y)+48f_e(y) \\ &\quad = 3f_e(3x+2y)+3f_e(x-2y)+2f_e(x+2y)+72f_e(x). \end{aligned} \tag{2.6}$$

Substituting $-y$ for y in (2.6) gives

$$\begin{aligned} &16f_e(2x-y)+16f_e(x+y)+48f_e(x-y)+48f_e(y) \\ &\quad = 3f_e(3x-2y)+3f_e(x+2y)+2f_e(x-2y)+72f_e(x). \end{aligned} \tag{2.7}$$

By utilizing (2.5), (2.6), and (2.7), we obtain

$$4f_e(2x+y)+4f_e(2x-y)+f_e(x+2y)+f_e(x-2y)=20f_e(x+y)+20f_e(x-y)+90f_e(x). \tag{2.8}$$

Interchanging x and y in (2.5), we get

$$\begin{aligned} &f_e(2x+3y)+f_e(2x-3y)+3f_e(2x+y)+3f_e(2x-y) \\ &\quad = 48f_e(x+y)+48f_e(x-y)+32f_e(x)+72f_e(y). \end{aligned} \tag{2.9}$$

If we add (2.5) to (2.9), we have

$$\begin{aligned} &f_e(2x+3y)+f_e(3x+2y)+f_e(2x-3y)+f_e(3x-2y)+3f_e(2x+y) \\ &\quad +3f_e(x+2y)+3f_e(2x-y)+3f_e(x-2y) \\ &= 96f_e(x+y)+96f_e(x-y)+104f_e(x)+104f_e(y). \end{aligned} \tag{2.10}$$

And by utilizing (2.6), (2.7), and (2.10), we arrive at

$$\begin{aligned} &3f_e(2x+3y)+3f_e(2x-3y) \\ &\quad = -25f_e(2x+y)-25f_e(2x-y)-4f_e(x-2y)-4f_e(x+2y) \\ &\qquad +224f_e(x+y)+224f_e(x-y)+456f_e(x)+216f_e(y). \end{aligned} \tag{2.11}$$

Let us interchange x and y in (2.11). Then we see that

$$\begin{aligned}3f_e(3x+2y)&+3f_e(3x-2y)\\&=-25f_e(x+2y)-25f_e(x-2y)-4f_e(2x-y)-4f_e(2x+y)\\&\quad+224f_e(x+y)+224f_e(x-y)+456f_e(y)+216f_e(x).\end{aligned}\tag{2.12}$$

Comparing (2.12) with (2.5), we get

$$\begin{aligned}4f_e(2x-y)+4f_e(2x+y)&=-16f_e(x+2y)-16f_e(x-2y)+80f_e(x+y)\\&\quad+80f_e(x-y)+360f_e(y).\end{aligned}\tag{2.13}$$

If we compare (2.13) and (2.8), we conclude that

$$f_e(x+2y)+f_e(x-2y)+6f_e(x)=4f_e(x+y)+4f_e(x-y)+24f_e(y).\tag{2.14}$$

This means that f_e is quartic function. Thus, there exists a unique symmetric multiadditive function $Q : X\times X\times X\times X \to Y$ such that $f_e(x)=Q(x,x,x,x)$ for all $x\in X$. On the other hand, we can show that f_o satisfies (1.9), or

$$\begin{aligned}4\left(f_o(3x+y)+f_o(3x-y)\right)&=-12\left(f_o(x+y)+f_o(x-y)\right)+12\left(f_o(2x+y)+f_o(2x-y)\right)\\&\quad-8f_o(y)-192f_o(x)+f_o(2y)+30f_o(2x).\end{aligned}\tag{1.9o}$$

Now setting $x=y=0$ in (1.9o) gives $f_o(0)=0$. Putting $x=0$ in (1.9o), then by oddness of f_o, we have

$$f_o(2y)=8f_o(y).\tag{2.15}$$

Hence, (1.9o) can be written as

$$f_o(3x+y)+f_o(3x-y)+3f_o(x+y)+3f_o(x-y)=3f_o(2x+y)+3f_o(2x-y)+12f_o(x)\tag{2.16}$$

for all $x,y\in X$. Replacing x by $x+y$, and y by $x-y$ in (2.16) we have

$$8f_o(2x+y)+8f_o(x+2y)+24f_o(x)+24f_o(y)=3f_o(3x+y)+3f_o(x+3y)+12f_o(x+y)\tag{2.17}$$

and interchanging x and y in (2.16) yields

$$f_o(x+3y)-f_o(x-3y)+3f_o(x+y)-3f_o(x-y)=3f_o(x+2y)-3f_o(x-2y)+12f_o(y).\tag{2.18}$$

Which on substitution of $-y$ for y in (2.16) gives

$$f_o(3x-y)+f_o(3x+y)+3f_o(x-y)+3f_o(x+y)=3f_o(2x-y)+3f_o(2x+y)+12f_o(x). \tag{2.19}$$

Replace y by $x+2y$ in (2.16). Then we have

$$8f_o(2x+y)+8f_o(x-y)+24f_o(x+y)-24f_o(y)=3f_o(3x+2y)+3f_o(x-2y)+12f_o(x). \tag{2.20}$$

From the substitution $y:=-y$ in (2.20) it follows that

$$8f_o(2x-y)+8f_o(x+y)+24f_o(x-y)+24f_o(y)=3f_o(3x-2y)+3f_o(x+2y)+12f_o(x). \tag{2.21}$$

If we add (2.20) to (2.21), we have

$$\begin{aligned}3f_o(3x-2y)+3f_o(3x+2y)&=8f_o(2x+y)+8f_o(2x-y)-3f_o(x+2y)-3f_o(x-2y)\\&\quad+32f_o(x-y)+32f_o(x+y)-24f_o(x).\end{aligned} \tag{2.22}$$

Let us interchange x and y in (2.22). Then we see that

$$\begin{aligned}3f_o(2x+3y)-3f_o(2x-3y)&=8f_o(x+2y)-8f_o(x-2y)-3f_o(2x+y)+3f_o(2x-y)\\&\quad+32f_o(x+y)-32f_o(x-y)-24f_o(y).\end{aligned} \tag{2.23}$$

With the substitution $y:=x+y$ in (2.16), we have

$$f_o(4x+y)+f_o(2x-y)+3f_o(2x+y)-3f_o(y)=3f_o(3x+y)+3f_o(x-y)+12f_o(x), \tag{2.24}$$

and replacing $-y$ by y gives

$$f_o(4x-y)+f_o(2x+y)+3f_o(2x-y)+3f_o(y)=3f_o(3x-y)+3f_o(x+y)+12f_o(x). \tag{2.25}$$

If we add (2.24) to (2.25), we have

$$\begin{aligned}f_o(4x+y)+f_o(4x-y)&=3f_o(3x+y)+3f_o(3x-y)-4f_o(2x-y)-4f_o(2x+y)\\&\quad+3f_o(x-y)+3f_o(x+y)+24f_o(x).\end{aligned} \tag{2.26}$$

By comparing (2.19) with (2.26), we arrive at

$$f_o(4x+y)+f_o(4x-y)=5f_o(2x+y)+5f_o(2x-y)-6f_o(x+y)-6f_o(x-y)+60f_o(x) \tag{2.27}$$

and replacing y by $2y$ in (2.16) gives

$$f_o(3x+2y)+f_o(3x-2y)=24f_o(x+y)+24f_o(x-y)-3f_o(x+2y)-3f_o(x-2y)+12f_o(x). \tag{2.28}$$

By comparing (2.28) with (2.22), we arrive at

$$3f_o(x+2y)+3f_o(x-2y)=20f_o(x+y)+20f_o(x-y)-4f_o(2x+y)-4f_o(2x-y)+30f_o(x). \tag{2.29}$$

Let us interchange x and y in (2.28). Then we see that

$$f_o(2x+3y)-f_o(2x-3y)=24f_o(x+y)-24f_o(x-y)-3f_o(2x+y)+3f_o(2x-y)+12f_o(y). \tag{2.30}$$

Thus combining (2.30) with (2.23) yields

$$4f_o(x+2y)-4f_o(x-2y)=3f_o(2x-y)-3f_o(2x+y)+20f_o(x+y)-20f_o(x-y)+30f_o(y). \tag{2.31}$$

By comparing (2.31) with (2.18), we arrive at

$$4f_o(x+3y)-4f_o(x-3y)=9f_o(2x-y)-9f_o(2x+y)+48f_o(x+y)-48f_o(x-y)+138f_o(y). \tag{2.32}$$

Which, by putting $y:=2y$ in (2.17), leads to

$$64f_o(x+y)+8f_o(x+4y)+24f_o(x)+192f_o(y)=3f_o(3x+2y)+3f_o(x+6y)+12f_o(x+2y). \tag{2.33}$$

Replacing y by $-y$ in (2.33) gives

$$64f_o(x-y)+8f_o(x-4y)+24f_o(x)-192f_o(y)=3f_o(3x-2y)+3f_o(x-6y)+12f_o(x-2y). \tag{2.34}$$

If we subtract (2.33) from (2.34), we obtain

$$\begin{aligned} 8f_o(x+4y)-8f_o(x-4y) &= 3f_o(3x+2y)-3f_o(3x-2y)+3f_o(x+6y) \\ &\quad -3f_o(x-6y)+12f_o(x+2y)-12f_o(x-2y) \\ &\quad +64f_o(x-y)-64f_o(x+y)-384f_o(y). \end{aligned} \tag{2.35}$$

Setting x instead of y and y instead of x in (2.27), we get

$$f_o(x+4y)-f_o(x-4y)=5f_o(x+2y)-5f_o(x-2y)+6f_o(x-y)-6f_o(x+y)+60f_o(y). \tag{2.36}$$

Combining (2.35) and (2.36) yields

$$\begin{aligned}3f_o(3x+2y)-3f_o(3x-2y)&=28f_o(x+2y)-28f_o(x-2y)+3f_o(x-6y)-3f_o(x+6y)\\&\quad+16f_o(x+y)-16f_o(x-y)+864f_o(y)\end{aligned} \tag{2.37}$$

and subtracting (2.21) from (2.20), we obtain

$$\begin{aligned}3f_o(3x+2y)-3f_o(3x-2y)&=3f_o(x+2y)-3f_o(x-2y)+8f_o(2x+y)-8f_o(2x-y)\\&\quad+16f_o(x+y)-16f_o(x-y)-48f_o(y).\end{aligned} \tag{2.38}$$

By comparing (2.37) with (2.38), we arrive at

$$\begin{aligned}3f_o(x+6y)-3f_o(x-6y)&=25f_o(x+2y)-25f_o(x-2y)+8f_o(2x-y)\\&\quad-8f_o(2x+y)+912f_o(y).\end{aligned} \tag{2.39}$$

Interchanging y with $2y$ in (2.32) gives the equation

$$\begin{aligned}4f_o(x+6y)-4f_o(x-6y)&=48f_o(x+2y)-48f_o(x-2y)+72f_o(x-y)\\&\quad-72f_o(x+y)+1104f_o(y).\end{aligned} \tag{2.40}$$

We obtain from (2.39) and (2.40)

$$\begin{aligned}44f_o(x+2y)-44f_o(x-2y)&=32f_o(2x-y)-32f_o(2x+y)+216f_o(x+y)\\&\quad-216f_o(x-y)+336f_o(y).\end{aligned} \tag{2.41}$$

By using (2.31) and (2.41), we lead to

$$f_o(2x+y)-f_o(2x-y)=4f_o(x+y)-4f_o(x-y)-6f_o(y). \tag{2.42}$$

And interchanging x with y in (2.42) gives

$$f_o(x+2y)+f_o(x-2y)=4f_o(x+y)+4f_o(x-y)-6f_o(x). \tag{2.43}$$

If we compare (2.43) and (2.29), we conclude that

$$8f_o(x+y)+8f_o(x-y)+48f_o(x)=4f_o(2x+y)+4f_o(2x-y). \tag{2.44}$$

This means that f_o is cubic function and that there exits a unique function $C : X\times X\times X \to Y$ such that $f_o(x) = C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. Thus for all $x \in X$, we have

$$f(x)=f_e(x)+f_o(x)=C(x,x,x)+Q(x,x,x,x). \tag{2.45}$$

This completes the proof of theorem. □

The following corollary is an alternative result of Theorem 2.1.

Corollary 2.2. *Let X, Y be vector spaces, and let $f : X \to Y$ be a function satisfying* (1.9). *Then the following assertions hold.*

(a) *If f is even function, then f is quartic.*

(b) *If f is odd function, then f is cubic.*

3. Stability

We now investigate the generalized Hyers-Ulam-Rassias stability problem for functional equation (1.9). From now on, let X be a real vector space and let Y be a Banach space. Now before taking up the main subject, given $f : X \to Y$, we define the difference operator $D_f : X\times X \to Y$ by

$$\begin{aligned} D_f(x,y) &= 4\left[f(3x+y)+f(3x-y)\right]-12\left[f(2x+y)+f(2x-y)\right]+12\left[f(x+y)+f(x-y)\right] \\ &\quad - f(2y)+8f(y)-30f(2x)+192f(x) \end{aligned} \tag{3.1}$$

for all $x, y \in X$. We consider the following functional inequality:

$$\left\|D_f(x,y)\right\| \le \phi(x,y) \tag{3.2}$$

for an upper bound $\phi : X\times X \to [0,\infty)$.

Theorem 3.1. *Let $s \in \{1,-1\}$ be fixed. Suppose that an even mapping $f : X \to Y$ satisfies $f(0) = 0$, and*

$$\left\|D_f(x,y)\right\| \le \phi(x,y) \tag{3.3}$$

for all $x, y \in X$. If the upper bound $\phi : X\times X \to [0,\infty)$ is a mapping such that the series $\sum_{i=0}^{\infty} 2^{4si}\phi(0, x/2^{si})$ converges, and that $\lim_{n\to\infty} 2^{4sn}\phi(x/2^{sn}, y/2^{sn}) = 0$ for all $x, y \in X$, then *the limit* $Q(x) = \lim_{n\to\infty} 2^{4sn} f(x/2^{sn})$ *exists for all* $x \in X$, *and* $Q : X \to Y$ *is a unique quartic function satisfying* (1.9), *and*

$$\|f(x) - Q(x)\| \leq \frac{1}{16} \sum_{i=(s-1)/2}^{\infty} 2^{4s(i+1)} \phi\left(0, \frac{x}{2^{s(i+1)}}\right) \tag{3.4}$$

for all $x \in X$.

Proof. Let $s = 1$. Putting $x = 0$ in (3.3), we get

$$\|f(2y) - 16f(y)\| \leq \phi(0, y). \tag{3.5}$$

Replacing y by $x/2$ in (3.5), yields

$$\left\|f(x) - 16f\left(\frac{x}{2}\right)\right\| \leq \phi\left(0, \frac{x}{2}\right). \tag{3.6}$$

Interchanging x with $x/2$ in (3.6), and multiplying by 16 it follows that

$$\left\|16f\left(\frac{x}{2}\right) - 16^2 f\left(\frac{x}{4}\right)\right\| \leq 16\phi\left(0, \frac{x}{4}\right). \tag{3.7}$$

Combining (3.6) and (3.7), we lead to

$$\left\|16^2 f\left(\frac{x}{4}\right) - f(x)\right\| \leq \phi\left(0, \frac{x}{2}\right) + 16\phi\left(0, \frac{x}{4}\right). \tag{3.8}$$

From the inequality (3.6) we use iterative methods and induction on n to prove our next relation:

$$\left\|16^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \leq \frac{1}{16} \sum_{i=0}^{n-1} 16^{i+1} \phi\left(0, \frac{x}{2^{i+1}}\right). \tag{3.9}$$

We multiply (3.9) by 16^m and replace x by $x/2^m$ to obtain that

$$\left\|16^{m+n} f\left(\frac{x}{2^{m+n}}\right) - 16^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{i=0}^{n-1} 16^{m+i} \phi\left(0, \frac{x}{2^{i+m+1}}\right). \tag{3.10}$$

This shows that $\{16^n f(x/2^n)\}$ is a Cauchy sequence in Y by taking the limit $m \to \infty$. Since Y is a Banach space, it follows that the sequence $\{16^n f(x/2^n)\}$ converges. We define $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty} 2^{4n} f(x/2^n)$ for all $x \in X$. It is clear that $Q(-x) = Q(x)$ for all $x \in X$, and it follows from (3.3) that

$$\|D_Q(x, y)\| = \lim_{n\to\infty} 16^n \left\|D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \leq \lim_{n\to\infty} 16^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.11}$$

for all $x, y \in X$. Hence, by Corollary 2.2, Q is quartic. It remains to show that Q is unique. Suppose that there exists another quartic function $Q' : X \to Y$ which satisfies (1.9) and (3.4). Since $Q(2^n x) = 16^n Q(x)$, and $Q'(2^n x) = 16^n Q'(x)$ for all $x \in X$, we conclude that

$$\begin{aligned} \|Q(x) - Q'(x)\| &= 16^n \left\| Q\left(\frac{x}{2^n}\right) - Q'\left(\frac{x}{2^n}\right) \right\| \\ &\le 16^n \left\| Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + 16^n \left\| Q'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \\ &\le 2\sum_{i=0}^{\infty} 16^{n+i} \phi\left(0, \frac{x}{2^{n+i+1}}\right) \end{aligned} \tag{3.12}$$

for all $x \in X$. By letting $n \to \infty$ in this inequality, it follows that $Q(x) = Q'(x)$ for all $x \in X$, which gives the conclusion. For $s = -1$, we obtain

$$\left\| \frac{f(2^m x)}{16^m} - f(x) \right\| \le \frac{1}{16} \sum_{i=-1}^{n-2} \frac{\phi(0, 2^{i+1} x)}{16^{i+1}}, \tag{3.13}$$

from which one can prove the result by a similar technique. □

Theorem 3.2. *Let $s \in \{1, -1\}$ be fixed. Suppose that an odd mapping $f : X \to Y$ satisfies*

$$\|D_f(x, y)\| \le \phi(x, y) \tag{3.14}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that $\sum_{i=0}^{\infty} 2^{3si} \phi(0, x/2^{si})$ converges, and that $\lim_{n\to\infty} 2^{3si} \phi(x/2^{si}, y/2^{si}) = 0$ for all $x, y \in X$, then the limit $C(x) = \lim_{n\to\infty} 2^{3sn} f(x/2^{sn})$ exists for all $x \in X$, and $C : X \to Y$ is a unique cubic function satisfying (1.9), *and*

$$\|f(x) - C(x)\| \le \frac{1}{8} \sum_{i=(s-1)/2}^{\infty} 2^{3s(i+1)} \phi\left(0, \frac{x}{2^{s(i+1)}}\right) \tag{3.15}$$

for all $x \in X$.

Proof. Let $s = 1$. Set $x = 0$ in (3.14). We obtain

$$\|8f(y) - f(2y)\| \le \phi(0, y). \tag{3.16}$$

Replacing y by $x/2$ in (3.16) to get

$$\left\| 8f\left(\frac{x}{2}\right) - f(x) \right\| \le \phi\left(0, \frac{x}{2}\right). \tag{3.17}$$

An induction argument now implies

$$\left\| 8^n f\left(\frac{x}{2^n}\right) - f(x) \right\| \leq \frac{1}{8} \sum_{i=0}^{n-1} 8^{i+1} \phi\left(0, \frac{x}{2^{i+1}}\right). \tag{3.18}$$

Multiply (3.18) by 8^m and replace x by $x/2^m$, we obtain that

$$\left\| 8^{m+n} f\left(\frac{x}{2^{m+n}}\right) - 8^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{i=0}^{n-1} 8^{m+i} \phi\left(0, \frac{x}{2^{m+i+1}}\right). \tag{3.19}$$

The right hand side of the inequality (3.19) tends to 0 as $m \to \infty$ because of

$$\sum_{i=0}^{\infty} 8^i \phi\left(0, \frac{x}{2^{i+1}}\right) < \infty \tag{3.20}$$

by assumption, and thus the sequence $\{2^{3n} f(x/2^n)\}$ is Cauchy in Y, as desired. Therefore we may define a mapping $C : X \to Y$ as $C(x) = \lim_{n\to\infty} 2^{3n} f(x/2^n)$. The rest of proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let $s \in \{1, -1\}$ be fixed. Suppose a mapping $f : X \to Y$ satisfies $f(0) = 0$, and $\|D_f(x, y)\| \leq \phi(x, y)$ for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that*

$$\begin{gathered} \sum_{i=0}^{\infty} \left[(|s| + s) 2^{4si} \phi\left(0, \frac{x}{2^{si-1}}\right) + (|s| - s) 2^{3si} \phi\left(0, \frac{x}{2^{si-1}}\right) \right] < \infty, \\ \lim_{n\to\infty} \left[(|s| + s) 2^{(4sn-1)} \phi\left(\frac{x}{2^{sn}}, \frac{y}{2^{sn}}\right) + (|s| + s) 2^{3sn} \phi\left(\frac{x}{2^{sn}}, \frac{y}{2^{sn}}\right) \right] = 0 \end{gathered} \tag{3.21}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \sum_{i=(s-1)/2}^{\infty} \left\{ \left(\frac{2^{4s(i+1)}}{32} + \frac{2^{3s(i+1)}}{16} \right) \left[\phi\left(0, \frac{x}{2^{s(i+1)}}\right) + \phi\left(0, \frac{-x}{2^{s(i+1)}}\right) \right] \right\} \tag{3.22}$$

for all $x \in X$.

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0$ and f_e is even function satisfying $\|D_{f_e}(x, y)\| \leq (1/2)[\phi(x, y) + \phi(-x, -y)]$ for all $x, y \in X$. From Theorem 3.1, it follows that there exists a unique quartic function $Q : X \to Y$ satisfies

$$\|f_e(x) - Q(x)\| \leq \frac{1}{32} \sum_{i=(s-1)/2}^{\infty} \left\{ 2^{4s(i+1)} \phi\left(0, \frac{x}{2^{s(i+1)}}\right) + 2^{4s(i+1)} \phi\left(0, \frac{-x}{2^{s(i+1)}}\right) \right\} \tag{3.23}$$

for all $x \in X$. Let now $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then f_o is odd function satisfying

$$\|D_{f_o}(x,y)\| \le \frac{1}{2}[\phi(x,y) + \phi(-x,-y)] \tag{3.24}$$

for all $x, y \in X$. Hence, in view of Theorem 3.2, it follows that there exists a unique cubic function $C : X \to Y$ such that

$$\|f_o(x) - C(x)\| \le \frac{1}{16}\sum_{i=(s-1)/2}^{\infty}\left\{2^{3s(i+1)}\phi\left(0, \frac{x}{2^{s(i+1)}}\right) + 2^{3s(i+1)}\phi\left(0, \frac{-x}{2^{s(i+1)}}\right)\right\} \tag{3.25}$$

for all $x \in X$. On the other hand, we have $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. Then by combining (3.23) and (3.25), it follows that

$$\begin{aligned}\|f(x) - C(x) - Q(x)\| &\le \|f_e(x) - Q(x)\| + \|f_o(x) - C(x)\| \\ &\le \sum_{i=(s-1)/2}^{\infty}\left\{\left(\frac{2^{4s(i+1)}}{32} + \frac{2^{3s(i+1)}}{16}\right)\left[\phi\left(0, \frac{x}{2^{s(i+1)}}\right) + \phi\left(0, \frac{-x}{2^{s(i+1)}}\right)\right]\right\}\end{aligned} \tag{3.26}$$

for all $x \in X$, and the proof of theorem is complete. □

We are going to investigate the Hyers-Ulam-Rassias stability problem for functional equation (1.9).

Corollary 3.4. *Let $p \in (-\infty, 3) \cup (4, +\infty)$, $\theta > 0$. Suppose $f : X \to Y$ satisfies $f(0) = 0$, and inequality*

$$\|D_f(x,y)\| \le \theta(\|x\|^p + \|y\|^p), \tag{3.27}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$, and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \le \begin{cases}\theta\|x\|^p\left(\dfrac{1}{2^p - 2^4} + \dfrac{1}{2^p - 2^3}\right), & p > 4, \\ \theta\|x\|^p\left(\dfrac{1}{2^4 - 2^p} + \dfrac{1}{2^3 - 2^p}\right), & p < 3\end{cases} \tag{3.28}$$

for all $x \in X$.

Proof. Let $s = 1$ in Theorem 3.3. Then by taking $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, the relations (3.21) hold for $p > 4$. Then there exists a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \le \theta\left\|\frac{x}{2}\right\|^p\left(\frac{1}{1 - 2^{4-p}} + \frac{1}{1 - 2^{3-p}}\right) \tag{3.29}$$

for all $x \in X$. Let now $s = -1$ in Theorem 3.3 and put $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then the relations (3.21) hold for $p < 3$. Then there exists a unique quartic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \theta\|x\|^p \left(\frac{1}{2^4 - 2^p} + \frac{1}{2^3 - 2^p}\right) \tag{3.30}$$

for all $x \in X$. □

Similarly, we can prove the following Ulam stability problem for functional equation (1.9) controlled by the mixed type product-sum function

$$(x, y) \longmapsto \theta\left(\|x\|_X^u \|y\|_X^v + \|x\|^p + \|y\|^p\right) \tag{3.31}$$

introduced by J. M. Rassias (e.g., [34]).

Corollary 3.5. *Let u, v, p be real numbers such that $u + v, p \in (-\infty, 3) \cup (4, +\infty)$, and let $\theta > 0$. Suppose $f : X \to Y$ satisfies $f(0) = 0$, and inequality*

$$\|D_f(x, y)\| \leq \theta\left(\|x\|_X^u \|y\|_X^v + \|x\|^p + \|y\|^p\right), \tag{3.32}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$, and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \begin{cases} \theta\|x\|^p \left(\dfrac{1}{2^p - 2^4} + \dfrac{1}{2^p - 2^3}\right), & p > 4, \\ \theta\|x\|^p \left(\dfrac{1}{2^4 - 2^p} + \dfrac{1}{2^3 - 2^p}\right), & p < 3 \end{cases} \tag{3.33}$$

for all $x \in X$.

By Corollary 3.4, we solve the following Hyers-Ulam stability problem for functional equation (1.9).

Corollary 3.6. *Let ϵ be a positive real number. Suppose $f : X \to Y$ satisfies $f(0) = 0$, and $\|D_f(x, y)\| \leq \epsilon$, for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$, and a unique cubic function $C : X \to Y$ satisfying*

$$\|f(x) - Q(x) - C(x)\| \leq \frac{22}{105}\epsilon \tag{3.34}$$

for all $x \in X$.

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