Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 103864, 7 pages
http://dx.doi.org/10.1155/2013/103864

Research Article

Semistrict G-Preinvexity and Optimality in Nonlinear Programming

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Received 26 October 2012; Accepted 10 April 2013

Academic Editor: Natig M. Atakishiyev

A class of semistrictly G-preinvex functions and optimality in nonlinear programming are further discussed. Firstly, the relationships between semistrictly G-preinvex functions and G-preinvex functions are further discussed. Then, two interesting properties of semistrictly G-preinvexity are given. Finally, two optimality results for nonlinear programming problems are obtained under the assumption of semistrict G-preinvexity. The obtained results are new and different from the corresponding ones in the literature. Some examples are given to illustrate our results.

1. Introduction

It is well known that convexity and generalized convexity have been playing a central role in mathematical programming, economics, engineering, and optimization theory. The research convexity and generalized convexity are one of the most important aspects in mathematical programming and optimization theory in [1–4]. Various kinds of generalized convexity have been introduced by many authors (see, e.g., [5–21] and the references therein). In 1981, Hanson [5] introduced the concept of invexity which is extension of differentiable convex functions and proved the sufficiency of Kuhn-Tucker condition. Later, Weir and Mond [6] considered functions (not necessarily differentiable) for which there exists a vector function $\eta : R^n \times R^n \rightarrow R^n$ such that, for all $x, y \in R^n$, $\lambda \in [0, 1]$, one hasthe following:

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda) f(y), \qquad (1)$$

which has been named as preinvex functions with respect to vector-valued function η. In 2001, Yang and Li [8] obtained some properties of preinvex function. At the same time, Yang and Li [9] introduced the concept of semistrictly preinvex functions and investigated the relationships between semistrictly preinvex functions and preinvex functions. It is worth mentioning that many properties and applications in mathematical programming for invex functions and preinvex functions are discussed by many authors (see, e.g., [6–11, 21] and the references therein).

On the other hand, Avriel et al. [12] introduced a class of G-convex functions which is another generalization of convex functions and obtained some relations with other generalization of convex functions. In [13], Antczak introduced the concept of a class of G-invex fuctions, which is a generalization of G-convex functions and invex functions. Recently, Antczak [14] introduced a class of G-preinvex functions, which is a generalization of G-invex [13], preinvex functions [8] and derived some optimality results for constrained optimization problems under G-preinvexity. Very recently, Luo and Wu introduced a new class of functions called semistrictly G-preinvex functions in [15], which include semistrictly preinvex functions [9] as a special case. They investigated the relationships between semistrictly G-preinvex functions and G-preinvex functions and gave a criterion for semistrict G-preinvexity. Moreover, they also proposed three open questions (just as they said: "an interesting topic for our future research is to": (1) investigate G-preinvex functions and semicontinuity; (2) explore some properties of semistrictly G-preinvex functions; (3) research into some applications in optimization problems under semistrictly G-preinvexity [15]).

However, as far as we know, there are few papers dealing with the properties and applications of the semistrictly G-preinvex functions [16]. The questions above in [15] have not been solved, and one condition in [15] is not mild in researching of relationships between G-preinvexity and semistrict G-preinvexity. So, in this paper, we further discuss semistrictly G-preinvex functions. The rest of the paper is organized as follows. Firstly, we investigate G-preinvex functions and semicontinuity and obtain a criterion of G-preinvexity under semicontinuity. Then, we give new relationships between G-preinvexity and semistrictly G-preinvexity, which are different from the recent ones in the literature [15]. Finally, we get two optimality results under semistrict G-preinvexity for nonlinear programming. The obtained results in this paper improve and extend the existing ones in the literature (e.g., [8, 9, 11, 15, 16]).

2. Preliminaries and Definitions

Throughout this paper, let K be a nonempty subset of R^n. Let $f : K \to R$ be a real-valued function and $\eta : K \times K \to R^n$ a vector-valued function. Let $I_f(K)$ be the range of f, that is, the image of K under f.

Now we recall some definitions.

Definition 1 (see [5, 6]). A set K is said to be invex if there exists a vector-valued function $\eta : K \times K \to R^n$ ($\eta \neq 0$) such that

$$x, y \in K, \quad \lambda \in [0,1] \Longrightarrow y + \lambda\eta(x,y) \in K. \tag{2}$$

Definition 2 (see [6, 8]). Let $K \subseteq R^n$ be an invex set with respect to $\eta : R^n \times R^n \to R^n$ and let $f : K \to R$ be a mapping. One says that f is preinvex if

$$\begin{aligned} & f(y + \lambda\eta(x,y)) \\ & \quad \le \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in K,\ \lambda \in [0,1]. \end{aligned} \tag{3}$$

Remark 3. Any convex function is a preinvex function with $\eta(x,y) = x - y$.

Definition 4 (see [10]). Let $K \subseteq R^n$ be an invex set with respect to $\eta : R^n \times R^n \to R^n$. Let $f : K \to R$. One says that f is prequasi-invex if

$$\begin{aligned} & f(y + \lambda\eta(x,y)) \\ & \quad \le \max\{f(x), f(y)\}, \quad \forall x, y \in K,\ \lambda \in [0,1]. \end{aligned} \tag{4}$$

Definition 5 (see [14]). Let K be a nonempty invex (with respect to η) subset R^n. A function $f : K \to R$ is said to be G-preinvex at u on K if there exists a continuous real-valued increasing function $G : I_f(K) \to R$ such that for all $x \in K$ and $\lambda \in [0,1]$ ($\lambda \in (0,1)$),

$$f(u + \lambda\eta(x,u)) \le G^{-1}(\lambda G(f(x)) + (1-\lambda) G(f(u))). \tag{5}$$

If (5) is satisfied for any $u \in K$ then f is G-preinvex on K, with respect to η.

Definition 6 (see [15]). Let K be a nonempty invex (with respect to η) subset R^n. A function $f : K \to R$ is said to be semistrictly G-preinvex at u on K if there exists a continuous real-valued increasing function $G : I_f(K) \to R$ such that for all $x \in K$ ($f(x) \neq f(u)$) and $\lambda \in (0,1)$,

$$f(u + \lambda\eta(x,u)) < G^{-1}(\lambda G(f(x)) + (1-\lambda) G(f(u))). \tag{6}$$

If (6) is satisfied for any $u \in K$, then f is semistrictly G-preinvex on K with respect to η.

Remark 7. In order to define an analogous class of semistrictly G-preincave functions with respect to η, the direction of the inequality in Definition 6 should be changed to the opposite one.

Remark 8. Every semistrictly preinvex function [8, 9] is semistrictly G-preinvex with respect to the same function η, where $G : I_f(K) \to R$ is defined by $G(x) = x$.

In order to prove our main result, we need Condition C as follows.

Condition C (see [9]). The vector-valued function $\eta : X \times X \to X$ is said to satisfy Condition C if for any $x, y \in X$, and $\lambda \in [0,1]$,

$$\begin{aligned} \eta(y, y + \lambda\eta(x,y)) &= -\lambda\eta(x,y), \\ \eta(x, y + \lambda\eta(x,y)) &= (1-\lambda)\eta(x,y). \end{aligned} \tag{7}$$

Example 9. Let

$$\eta(x,y) = \begin{cases} x - y, & \text{if } x \ge 0,\ y \ge 0, \\ x - y, & \text{if } x < 0,\ y < 0, \\ -\dfrac{5}{2} - y, & \text{if } x > 0,\ y \le 0, \\ \dfrac{5}{2} - y, & \text{if } x \le 0,\ y > 0. \end{cases} \tag{8}$$

It can be verified that η satisfies the Condition C.

3. Relationships with Semistrictly G-Preinvexity

In [15], Luo and Wu obtained a sufficient condition of G-preinvex functions under the condition of intermediate-point G-preinvexity. Now we investigate G-preinvex functions and semicontinuity without the condition of intermediate-point G-preinvexity.

Theorem 10. *Let K be a nonempty invex set with respect to η, where η satisfies Condition C. Let $f : K \to R$ be lower semicontinuous and semistrictly G-preinvex for the same η on K, and let $f(y + \eta(x,y)) \le f(x)$ (for all $x \in K$). Then f is G-preinvex function on K.*

Proof. Let $x, y \in K$. From the assumption of $f(y+\eta(x,y)) \leq f(x)$, when $\lambda = 0, 1$, we can know that

$$f(y+\lambda\eta(x,y)) \leq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))). \tag{9}$$

Then, there are two cases to be considered.

(i) If $f(x) \neq f(y)$, then by the semistrict G-preinvexity of f, we have the following:

$$\begin{aligned} &f(y+\lambda\eta(x,y)) \\ &\quad < G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))), \quad \forall \lambda \in (0,1). \end{aligned} \tag{10}$$

(ii) If $f(x) = f(y)$, to show that f is a G-preinvex function, we need to show that

$$\begin{aligned} &f(y+\lambda\eta(x,y)) \\ &\quad \leq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) = f(x). \end{aligned} \tag{11}$$

By contradiction, suppose that there exists an $\alpha \in (0,1)$ such that

$$f(y+\alpha\eta(x,y)) > f(x). \tag{12}$$

Let $z_\alpha = y + \alpha\eta(x,y)$. Since f is lower semicontinuous, there exists $\beta : \alpha < \beta < 1$, such that

$$f(z_\beta) = f(y+\beta\eta(x,y)) > f(x) = f(y). \tag{13}$$

From Condition C,

$$z_\beta = z_\alpha + \frac{\beta-\alpha}{1-\alpha}\eta(x, z_\alpha). \tag{14}$$

By the semistrict G-preinvexity of f and (12), we have the following:

$$\begin{aligned} &f(z_\beta) \\ &\quad = f\left(z_\alpha + \frac{\beta-\alpha}{1-\alpha}\eta(x,z_\alpha)\right) \\ &\quad < G^{-1}\left[\frac{\beta-\alpha}{1-\alpha}G(f(x)) + \left(1-\frac{\beta-\alpha}{1-\alpha}\right)G(f(z_\alpha))\right] \\ &\quad < G^{-1}\left[\frac{\beta-\alpha}{1-\alpha}G(f(z_\alpha)) + \left(1-\frac{\beta-\alpha}{1-\alpha}\right)G(f(z_\alpha))\right] \\ &\quad = f(z_\alpha). \end{aligned} \tag{15}$$

On the other hand, from Condition C, one can obtain the following:

$$z_\alpha = z_\beta + \left(1-\frac{\alpha}{\beta}\right)\eta(y, z_\beta). \tag{16}$$

According to (13) and the semistrictly G-preinvexity of f, we have the following:

$$\begin{aligned} &f(z_\alpha) \\ &\quad = f\left(z_\beta + \left(1-\frac{\alpha}{\beta}\right)\eta(y,z_\beta)\right) \\ &\quad < G^{-1}\left[\left(1-\frac{\alpha}{\beta}\right)G(f(y)) + \frac{\alpha}{\beta}G(f(z_\beta))\right] \\ &\quad < G^{-1}\left[\left(1-\frac{\alpha}{\beta}\right)G(f(z_\beta)) + \frac{\alpha}{\beta}G(f(z_\beta))\right] \\ &\quad = f(z_\beta) \end{aligned} \tag{17}$$

which contradicts (15). This completes the proof. □

Remark 11. In Theorem 10, we investigate G-preinvex functions and lower semicontinuity, and establish a new sufficient condition for G-preinvexity without the condition of intermediate-point G-preinvexity. Therefore, we also answer the open question (1) which proposed in [15] ("(1) investigate G-preinvex functions and semicontinuity" [15]).

Now, we give a new sufficient condition for semistrictly G-preinvexity.

Theorem 12. *Let K be a nonempty invex set with respect to η, where η satisfies Condition C. Let $f : K \to R$ be a G-preinvex function for the same η on K. Suppose that for any $x, y \in K$ with $f(x) \neq f(y)$, there exists an $\alpha \in (0,1)$ such that*

$$f(y+\alpha\eta(x,y)) < G^{-1}(\alpha G(f(x)) + (1-\alpha)G(f(y))). \tag{18}$$

Then, f is semistrictly G-preinvex function on K.

Proof. For any $x, y \in K$ with $f(x) \neq f(y)$ and $\lambda \in (0,1)$, by assumption, we have the following:

$$f(y+\lambda\eta(x,y)) \leq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))). \tag{19}$$

(i) Let $\lambda \leq \alpha$. From Condition C, we can obtain the following:

$$\begin{aligned} &y + \frac{\lambda}{\alpha}\eta(y+\alpha\eta(x,y), y) \\ &\quad = y + \frac{\lambda}{\alpha}\eta(y+\alpha\eta(x,y), y+\alpha\eta(x,y) - \alpha\eta(x,y)) \\ &\quad = y + \frac{\lambda}{\alpha}\eta(y+\alpha\eta(x,y), y+\alpha\eta(x,y) \\ &\qquad\qquad + \eta(y, y+\alpha\eta(x,y))) \\ &\quad = y - \frac{\lambda}{\alpha}\eta(y, y+\alpha\eta(x,y)) \\ &\quad = y + \lambda\eta(x,y). \end{aligned} \tag{20}$$

Using (18) and the G-preinvexity of f, we have the following:

$$\begin{aligned}
&f(y+\lambda\eta(x,y)) \\
&\quad = f\left[y+\frac{\lambda}{\alpha}\eta(y+\alpha\eta(x,y),y)\right] \\
&\quad \leq G^{-1}\left[\frac{\lambda}{\alpha}G\left(f(y+\alpha\eta(x,y))\right.\right. \\
&\qquad\qquad \left.+\left(1-\frac{\lambda}{\alpha}\right)G(f(y))\right] \\
&\quad < G^{-1}\left[\frac{\lambda}{\alpha}G\left(G^{-1}(\alpha G(f(x))+(1-\alpha)G(f(y)))\right)\right. \\
&\qquad\qquad \left.+\left(1-\frac{\lambda}{\alpha}\right)G(f(y))\right] \\
&\quad = G^{-1}\left[\frac{\lambda}{\alpha}(\alpha G(f(x))+(1-\alpha)G(f(y)))\right. \\
&\qquad\qquad \left.+\left(1-\frac{\lambda}{\alpha}\right)G(f(y))\right] \\
&\quad = G^{-1}(\lambda G(f(x))+(1-\lambda)G(f(y))).
\end{aligned} \tag{21}$$

(ii) Let $\alpha < \lambda$; that is,

$$0 < \frac{1-\lambda}{1-\alpha} < 1. \tag{22}$$

From Condition C, we have the following:

$$\begin{aligned}
&y+\alpha\eta(x,y)+\left(1-\frac{1-\lambda}{1-\alpha}\right)\eta(x,y+\alpha\eta(x,y)) \\
&\quad = y+\left[\alpha+\left(1-\frac{1-\lambda}{1-\alpha}\right)(1-\alpha)\right]\eta(x,y) \\
&\quad = y+\lambda\eta(x,y).
\end{aligned} \tag{23}$$

According to (18) and the G-preinvexity of f, we get the following:

$$\begin{aligned}
&f(y+\lambda\eta(x,y)) \\
&\quad = f\left[y+\alpha\eta(x,y)+\left(1-\frac{1-\lambda}{1-\alpha}\right)\eta(x,y+\alpha\eta(x,y))\right] \\
&\quad \leq G^{-1}\left[\left(1-\frac{1-\lambda}{1-\alpha}\right)G(f(x))\right. \\
&\qquad\qquad \left.+\frac{1-\lambda}{1-\alpha}G(f(y+\alpha\eta(x,y)))\right] \\
&\quad < G^{-1}\left[\left(1-\frac{1-\lambda}{1-\alpha}\right)G(f(x))\right. \\
&\qquad\qquad \left.+\frac{1-\lambda}{1-\alpha}G\left(G^{-1}(\alpha G(f(x))+(1-\alpha)G(f(y)))\right)\right] \\
&\quad = G^{-1}\left[\left(1-\frac{1-\lambda}{1-\alpha}\right)G(f(x))\right. \\
&\qquad\qquad \left.+\frac{1-\lambda}{1-\alpha}(\alpha G(f(x))+(1-\alpha)G(f(y)))\right] \\
&\quad = G^{-1}(\lambda G(f(x))+(1-\lambda)G(f(y))),
\end{aligned} \tag{24}$$

(21) and (24) imply that f is a semistrictly G-preinvex function on K. □

Remark 13. Theorem 12 extends and improves Theorem 1 in [15]. In Theorem 1 of [15], a uniform $\alpha \in (0,1)$ is needed; while in assumption (18) of Theorem 12, this condition has been weakened, where a uniform $\alpha \in (0,1)$ is not necessary. Moreover, our proof is also different from the corresponding result of [15].

The following example illustrates that assumption (18) in Theorem 12 is essential.

Example 14. Let

$$\begin{gathered}
f(x) = \ln\left(\frac{3}{2}|x|+\frac{7}{4}\right), \\
\eta(x,y) = \begin{cases} x-y, & \text{if } x \geq 0,\ y \geq 0; \\ x-y, & \text{if } x \leq 0,\ y \leq 0; \\ -x-y, & \text{if } x > 0,\ y < 0; \\ -x-y, & \text{if } x < 0,\ y > 0. \end{cases}
\end{gathered} \tag{25}$$

It is obvious that f is a G-preinvex function with respect to η, where $G(t) = e^t$, $K = R$.

However, from Definition 6, we can verify that f is a semistrictly G-preinvex function. The reason is that the assumption (18) is violated. Indeed, there exist $x = 1/2$, $y = 4$ (with $f(x) \neq f(y)$), for any $\lambda \in (0,1)$, we have the following:

$$\begin{aligned}
&f(y+\lambda\eta(x,y)) \\
&\quad = f\left(4-\frac{7}{2}\lambda\right) = \ln\left(\frac{3}{2}\left|4-\frac{7}{2}\lambda\right|+\frac{7}{4}\right) \\
&\quad = \ln\left(\frac{31}{4}-\frac{21}{4}\lambda\right) \\
&\quad = G^{-1}\left(\frac{3}{2}(\lambda|x|+(1-\lambda)|y|)+\frac{7}{4}\right) \\
&\quad = G^{-1}(\lambda G(f(x))+(1-\lambda)G(f(y))).
\end{aligned} \tag{26}$$

Therefore, (18) is essential.

Lemma 15 (see [16]). *Let K be a nonempty invex set with respect to η. Suppose that f is a semistrictly G_1-preinvex function with respect to η and G_2 is a continuous and strictly increasing function on $I_f(X)$. If $g(t) = G_2G_1^{-1}$ is convex on the image under G_1 of the range of f, then f is also semistrictly G_2-preinvex function with respect to the same η on K.*

Theorem 16. *Let K be an invex set with respect to η and $f : K \to R$ be a semistrictly G-preinvex function on K. If G is* *concave on $I_f(X)$, then f is a preinvex function with respect to the same η on K.*

Proof. Let y and z be two points in $I_f(X)$. Because G is concave on $I_f(X)$, the following inequality

$$G(\lambda y + (1-\lambda) z) \geq \lambda G(y) + (1-\lambda) G(z) \tag{27}$$

holds for any $\lambda \in [0,1]$. Let $G(y) = x$ and $G(z) = u$. Then, for each pair of points x and u in image G of $I_f(X)$, that is, $G^{-1}(x) = y$ and $G^{-1}(u) = z$, we have the following:

$$\begin{aligned} &G\left(\lambda G^{-1}(x) + (1-\lambda) G^{-1}(u)\right) \\ &\quad \geq \lambda G\left(G^{-1}(x)\right) + (1-\lambda) G\left(G^{-1}(u)\right) \\ &\quad = \lambda x + (1-\lambda) u. \end{aligned} \tag{28}$$

It follows from (28) and the increasing property of G^{-1} that

$$\begin{aligned} &G^{-1} G\left(\lambda G^{-1}(x) + (1-\lambda) G^{-1}(u)\right) \\ &\quad \geq G^{-1}\left(\lambda G\left(G^{-1}(x)\right) + (1-\lambda) G\left(G^{-1}(u)\right)\right) \\ &\quad = G^{-1}(\lambda x + (1-\lambda) u). \end{aligned} \tag{29}$$

Thus,

$$\lambda G^{-1}(x) + (1-\lambda) G^{-1}(u) \geq G^{-1}(\lambda x + (1-\lambda) u). \tag{30}$$

This means that G^{-1} is convex. Letting $G_1 = G, G_2(t) = t$, then $g(t) = G_2 G_1^{-1}(t)$ is convex. Hence, by virtue of Lemma 15, f is a semistrictly G_2-preinvex function with respect to η. Because $G_2(t) = t$ is identity function, f is a preinvex function with respect to the same η on K. □

From Theorem 16 and Definitions 2-4, we can obtain the following Corollary easily.

Corollary 17. *Let K be an invex set with respect to η and $f : K \to R$ is a semistrictly G-preinvex function on K. If G is concave on $I_f(X)$ then f be a prequasi-invex function with respect to the same η on K.*

4. Semistrictly G-Preinvexity and Optimality

In order to solve the open question (3) proposed in [15] (see, the part of Introduction), in this section, we consider nonlinear programming problems with constraint and obtain two optimality results under semistrict G-preinvexity.

We consider the following nonlinear programming Problem (P) with inequality constraint:

$$\min f(x) \tag{P}$$

$$g_i(x) \leq 0, \quad i \in J = 1, \ldots, m, \tag{31}$$

where $f : K \to R$, $g_i : K \to R$, $i \in J$, and K is a nonempty subset of R^n. We denote the set of all feasible solutions in (P) by the following:

$$D = \{x \in K : g_i(x) \leq 0, i \in J\}. \tag{32}$$

Theorem 18. *Suppose the set of all feasible solutions D of problem (P) is an invex set with respect to η, and D at least contains two points with nonempty interior. Let f be a nonconstant semistrictly G-preincave function with respect to η on D. Then no interior of D is an optimal solution of (P), or equivalently, any optimal solution $\overline{x}$ in problem (P), if exists, must be a boundary point of D.*

Proof. If problem (P) has no solution the theorem is trivially true. Let $\overline{x}$ be an optimal solution in problem (P). By assumption, f is a nonconstant on D. Then, there exists a feasible point $x^* \in D$ such that

$$f(x^*) > f(\overline{x}). \tag{33}$$

Let z $(z \neq x^*)$ be an interior point of D. By assumption, D is an invex set with respect to η. It follows from the definition of invex set D that there exists $y \in D$ such that for some $\lambda \in (0,1)$,

$$z = x^* + \lambda \eta(y, x^*). \tag{34}$$

By assumption, f is semistrictly G-preincave with respect to η on D. Then, we have the following:

$$\begin{aligned} f(z) &= f(x^* + \lambda \eta(y, x^*)) \\ &> G^{-1}(\lambda G(f(y)) + (1-\lambda) G(f(x^*))) \\ &> G^{-1}(\lambda G(f(\overline{x})) + (1-\lambda) G(f(\overline{x}))) = f(\overline{x}), \end{aligned} \tag{35}$$

where z is an interior point of D.

From the inequality above, we conclude that no interior of D is an optimal solution of (P), that is, any optimal solution $\overline{x}$ in problem (P), if exists, must be a boundary point of D. This completes the proof. □

Theorem 19. *Let $\overline{x} \in D$ be local optimal in problem (P). Moreover, we assume that f is semistrictly G-preinvex with respect to η at $\overline{x}$ on D and the constraint functions g_i, $i \in J$, are G-preinvex with respect to η at $\overline{x}$ on D. Then, $\overline{x}$ is a global optimal solution in problem (P).*

Proof. Assume that $\overline{x} \in K$ is a local optimal solution in (P). Hence, there is an neighborhood $N_\epsilon(\overline{x})$ around $\overline{x}$ such that

$$f(\overline{x}) \leq f(x), \quad \forall x \in K \cap N_\epsilon(\overline{x}). \tag{36}$$

Suppose to the contrary, $\overline{x}$ is not a global minimum in (P), there exists an $x^* \in D$ such that

$$f(x^*) < f(\overline{x}). \tag{37}$$

By assumption, the constraint functions g_i, $i \in J$, are G-preinvex with respect to the same η at $\overline{x}$ on D. By using Definition 5 together with $x^*, \overline{x} \in D$, then for all $i \in J$ and any $\lambda \in [0,1]$, we have the following:

$$\begin{aligned} g_i(\overline{x} + \lambda \eta(x^*, \overline{x})) &\leq G_i^{-1}(\lambda G_i(g_i(x^*)) + (1-\lambda) G_i(g_i(\overline{x}))) \\ &\leq G_i^{-1}(\lambda G_i(0) + (1-\lambda) G_i(0)) = 0. \end{aligned} \tag{38}$$

Thus, for all $i \in J$ and any $\lambda \in [0,1]$,

$$\overline{x} + \lambda\eta(x^*, \overline{x}) \in D. \tag{39}$$

By assumption, f is semistrictly G-preinvex with respect to the same η at $\overline{x}$ on D. Therefore, by Definition 6, we have the following:

$$\begin{aligned} & f(\overline{x} + \lambda\eta(x^*, \overline{x})) \\ & \quad < G^{-1}(\lambda G(f(x^*)) + (1-\lambda) G(f(\overline{x}))), \quad \forall\lambda \in (0,1). \end{aligned} \tag{40}$$

From (37) and (40), we have the following:

$$\begin{aligned} & f(\overline{x} + \lambda\eta(x^*, \overline{x})) \\ & \quad < G^{-1}(\lambda G(f(\overline{x})) + (1-\lambda) G(f(\overline{x}))) \\ & \quad = f(\overline{x}), \quad \forall\lambda \in (0,1). \end{aligned} \tag{41}$$

For a sufficiently small $\lambda > 0$, it follows that

$$\overline{x} + \lambda\eta(x^*, \overline{x}) \in K \cap N_\epsilon(\overline{x}), \tag{42}$$

which contradicts that $\overline{x}$ is local optimal in problem (P). This completes the proof. □

Now, we give an example to illustrate Theorem 19.

Example 20. Let $f : K \to R$ be defined as follows:

$$f(x) = \begin{cases} \ln(5 - 4|x|), & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 1, \end{cases} \tag{43}$$

and let

$$g_i(x) = 3x - 1, \quad i \in J = 1, \ldots, m,$$

$$\eta(x,y) = \begin{cases} \dfrac{3}{4} - y, & \text{if } x > 1,\ 0 \leq y < 1, \\ & \text{or } x < -1,\ -1 < y \leq 0; \\ 1, & \text{if } y > 1,\ 0 \leq x < 1; \\ x - y, & \text{if } y < -1,\ -1 < x \leq 0; \\ -6x - y, & \text{if } 0 \leq x < 1,\ 0 < y < 1; \\ 1 - y, & \text{if } 0 \leq x < 1,\ y = 1; \\ x - y, & \text{if } |x| \geq 1,\ |y| \geq 1; \\ y - x - 2, & \text{if } x > 1,\ -1 < y < 0; \\ y - x, & \text{if } x > 1,\ -1 < y < 0, \\ & \text{or } y > 1,\ -1 < x < 0; \\ y - x, & \text{if } y < -1,\ 0 < x < 1; \\ -5 - \dfrac{1}{2}y, & \text{if } |x| < 1,\ -1 \leq y \leq 0; \\ -4x + \dfrac{3}{2}y, & \text{if } x = 1,\ 0 < y < 1; \\ 3x + \dfrac{3}{2}y, & \text{if } x = -1,\ -1 < y < 0; \\ -y - x^2 - x - \dfrac{15}{2}, & \text{if } -1 < x < 0,\ 0 < y \leq 1; \\ y - 2x, & \text{if } x = 1,\ -1 < y \leq 0; \\ y - 2x - 6, & \text{if } x = -1,\ 0 \leq y < 1. \end{cases} \tag{44}$$

From Definitions 5-6, we can get that f is semistrictly G-preinvex with respect to η and the constraint functions g_i, $i \in J$, are G-preinvex with respect to η, respectively, where $G(t) = e^t$, $K = R$. Obviously, $\overline{x} = -1$ is a local optimal solution of (P). Then, all the conditions in Theorem 19 are satisfied. By virtue of Theorem 19, $\overline{x} = -1$ is a global optimal solution in problem (P).

Acknowledgments

The authors would like to express their thanks to Professor X. M. Yang and the anonymous referees for their valuable comments and suggestions which helped to improve the paper. This work was supported by the Natural Science Foundation of China (nos. 11271389, 11201509, and 71271226), the Natural Science Foundation Project of ChongQing (nos. CSTC, 2012jjA00016, and 2011AC6104), and the Research Grant of Chongqing Key Laboratory of Operations and System Engineering.

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