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Research Article

Stability and Bogdanov-Takens Bifurcation of an SIS Epidemic Model with Saturated Treatment Function

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This paper introduces the global dynamics of an SIS model with bilinear incidence rate and saturated treatment function. The treatment function is a continuous and differential function which shows the effect of delayed treatment when the rate of treatment is lower and the number of infected individuals is getting larger. Sufficient conditions for the existence and global asymptotic stability of the disease-free and endemic equilibria are given in this paper. The first Lyapunov coefficient is computed to determine various types of Hopf bifurcation, such as subcritical or supercritical. By some complex algebra, the Bogdanov-Takens normal form and the three types of bifurcation curves are derived. Finally, mathematical analysis and numerical simulations are given to support our theoretical results.

1. Introduction

In the mathematical modeling of epidemic transmission, there are several factors that substantially affect the dynamical behavior of the models. Incidence rate functions are seen as a major factor in producing the rich dynamics of epidemic models in many literatures (see [1–25]). In most classical models of epidemics, the incidence rate is taken to be mass action incidence with bilinear interaction, that is, βIS, where β is the probability of transmission per contact and S and I represent the number of susceptible and infected individuals, respectively. Epidemic models with such bilinear incidence rates usually have at most one endemic equilibrium, and then the diseases will be eradicated if the basic reproduction number is less than one and will persist otherwise. Besides, there are also many other types of incidence rate functions, such as nonlinear incidence rate, standard incidence rate, and saturated incidence rate. Recently, there are many studies that have demonstrated the nonlinear incidence rate which is one of the key factors that induce periodic oscillations in epidemic models (see [1, 2, 9, 10, 15]). Moreover, Liu et al. [12] introduced a nonlinear incidence rate of the form

$$f(I)S = \frac{\beta I^p S}{1 + \alpha I^q}, \tag{1}$$

where βI^p means the infection force of the disease and $1/(1+\alpha I^q)$ measures the inhibition effect from the behavioral change of the susceptible individuals when the number of infective individuals increases. So we can see that the bilinear incidence rate βSI is a special case of (1) with $p = 1$ and $\alpha = 0$ or $q = 0$. Furthermore, Wang and Ruan in [16] studied the global dynamics of an SIRS model with the incidence function $f(I) = \beta I^2/(1 + \alpha I^2)$; that is, $p = q = 2$; they also showed that the SIRS epidemic model undergoes a Bogdanov-Takens bifurcation, that is, saddle-node, Hopf, and homoclinic bifurcations. To have a better understanding of the dynamics of the system, Tang et al. [17] calculated higher order Lyapunov values of the weak focus and reduced the system to a universal unfolding form for a cusp of codimension 3, and they gave the bifurcation surfaces and displayed all limit cycles and monoclinic loops of order up to 2. Wei and Cui explored an SIS epidemic model with standard incidence rate in [19]. They took the incidence rate form $f(I)S = \beta IS/(I + S)$ and showed the dynamics and backward bifurcation of the SIS epidemic model.

Recently, in order to prevent and control the spread of the infectious diseases such as measles, tuberculosis, and flu, many mathematicians (see [7, 11, 16, 20, 23, 25–29]) have begun to investigate the role of treatment functions in epidemiological models. In some classical epidemic models, the treatment function is an important method to decrease the spread of the epidemiological diseases. Generally speaking, the treatment function of the infective individuals is assumed to be proportional to the number of the infective individuals. But every community should have a maximal capacity for the treatment of a disease and the resources for treatment should be very large. Therefore, it is very important to adopt a suitable treatment function. In [16], Wang and Ruan introduced a constant treatment function of diseases in an SIR model; that is,

$$T(I) = \begin{cases} r, & I > 0, \\ 0, & I = 0. \end{cases} \tag{2}$$

This means that they use the maximal treatment capacity to cure infective individuals so that the disease can be eradicated. They also found that the model undergoes saddle-node bifurcation, Hopf bifurcation, and Bogdanov-Takens bifurcation. Further, a piecewise linear treatment function was considered in [20]; that is,

$$T(I) = \begin{cases} kI, & 0 \le I \le I_0, \\ m, & I > I_0, \end{cases} \tag{3}$$

where $m = kI_0$ and k and I_0 are positive constants. This means that the treatment rate is proportional to the number of the infective individuals when the capacity of treatment has not been reached; otherwise it takes the maximal capacity of treatment kI_0. By considering the above treatment function, Wang [20] found that a backward bifurcation takes place in an SIR epidemic model. In [30], J. C. Eckalbar and W. L. Eckalbar constructed an SIR epidemic model with a quadratic treatment function; that is, $T(I) = \max\{rI - gI^2, 0\},\ r, g > 0$. They found that the system has as many as four equilibria, with possible bistability, backward bifurcations, and limit cycles.

Recently, saturated treatment function has been widely applied in many epidemic models. In particular, in paper [23], Zhang and Liu took a continuous and differentiable saturated treatment function $T(I) = rI/(1 + \alpha I)$, where $r > 0$, $\alpha \ge 0$. r stands for the cure rate and α measures the extent of the effect of the infected being delayed for treatment. We can see that the treatment function $T(I) \sim rI$ when I is small enough, whereas $T(I) \sim r/\alpha$ when I is large enough. It is more realistic and it has the convenience of being continuous and differential compared to the previous ones. Furthermore, the authors in [23] found that $R_0 = 1$ is a critical threshold for disease eradication when this delayed effect for treatment is weak and a backward bifurcation will take place when this effect is strong. So, it is really important to adequately stress the interesting connection recently established between the choice of saturated treatment functions in epidemic models and the occurrence of backward bifurcation in the related system dynamics. In fact, recently, saturated-type treatment functions have been indicated as responsible for the occurrence of backward bifurcations for SIR [20, 23], for SIS [19, 21], and for SEIR [28, 29] models, supporting the general circumstance that saturated-type treatments can be one of the causes of backward bifurcations in epidemic models. In particular, in [29], such connection has also been shown and validated in a specific concrete disease-control setting.

In the real world, some infectious diseases do not confer immunity. Such infections do not have a recovered state and individuals become susceptible again after infection. This type of disease can be modeled by the SIS type. So the SIS epidemic model has been adopted by many mathematicians (see [8, 11, 18, 19, 21, 25]). And SIS models are appropriate for some bacterial agent diseases such as meningitis, plague, and venereal diseases and for protozoan agent diseases such as malaria and sleeping sickness.

Motivated by the above points, we will consider the following SIS model with bilinear incidence rate and saturated treatment function:

$$\begin{aligned} \frac{dS}{dt} &= A - dS - \lambda SI + \varepsilon I + \frac{rI}{1 + \alpha I}, \\ \frac{dI}{dt} &= \lambda SI - (d + \varepsilon + \mu) I - \frac{rI}{1 + \alpha I}, \end{aligned} \tag{4}$$

where S and I denote the numbers of susceptible and infective individuals, respectively. Positive constant A is the recruitment rate of the population. Positive constant d is the nature death rate of population. The bilinear incidence rate is λSI, where λ is positive. Positive constant ε is the natural recovery rate of infective individuals. Positive constant μ is the disease-related death rate. The saturated treatment function $h(I) \triangleq rI/(1 + \alpha I)$, where r is positive and α is nonnegative.

This paper focuses on the detailed dynamics analysis of the model (4). The local stability of these equilibria is investigated, which enables us to classify the types of model equilibria (e.g., attractor, saddle, or repeller). We show that the system has backward bifurcation and Bogdanov-Takens bifurcation (i.e., Hopf bifurcation, saddle-node bifurcation, and homoclinic bifurcation) under some certain conditions. Finally, the three bifurcation curves and the complicated global bifurcation phase portraits are derived by applying the Bogdanov-Takens normal form and the corresponding parameters which satisfy the conditions that ensure Bogdanov-Takens bifurcation exists.

The organization of this paper is as follows. In Section 2, we study the existence and local stability of equilibria and backward bifurcation. In Section 3, we investigate the global stability of the model. In Section 4, we give the supercritical and subcritical bifurcation under two different conditions in system (4). In Section 5, we show that the system (4) undergoes Bogdanov-Takens bifurcation under some certain conditions. In Section 6, some numerical simulations are displayed in detail. We close with a discussion in Section 7 on our mathematical results and epidemiological implications.

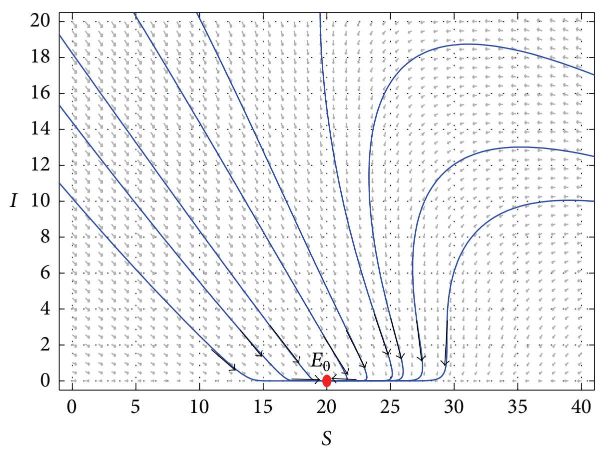

FIGURE 1: The disease-free equilibrium E_0 is locally asymptotically stable when $R_0 < 1$, with the parameter values $A = 2$, $d = 0.1$, $k = 0.1$, $\mu = 0.01$, $\lambda = 0.01$, $\alpha = 1$, $r = 2$, and $\varepsilon = 0.1$.

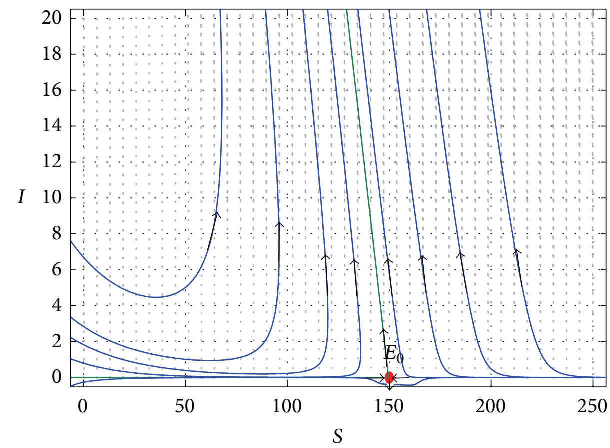

FIGURE 2: The disease-free equilibrium is unstable when $R_0 > 1$, with the parameter values $A = 15$, $d = 0.1$, $k = 0.1$, $\mu = 0.01$, $\lambda = 0.01$, $\alpha = 1$, $r = 0.8$, and $\varepsilon = 0.1$.

2. Equilibria and Backward Bifurcation

2.1. Disease-Free Equilibrium. Obviously, system (4) has a disease-free equilibrium $E_0 = (A/d, 0)$. The Jacobian matrix of (4) at E_0 is

$$M(E_0) = \begin{pmatrix} -d & \dfrac{-\lambda A}{d} + \varepsilon + r \\ 0 & \dfrac{\lambda A}{d} - (d + \varepsilon + \mu) - r \end{pmatrix}. \tag{5}$$

By using the next generation matrix of (4), we get the basic reproduction number $R_0 = \lambda A / d(d + \varepsilon + r + \mu)$. $M(E_0)$ has negative eigenvalues if $\lambda A/d - (d + \varepsilon + \mu) - r < 0$. Then we have the following result.

Theorem 1. *The disease-free equilibrium E_0 is locally asymptotically stable when $R_0 < 1$ (see Figure 1) and is unstable when $R_0 > 1$ (see Figure 2).*

2.2. Endemic Equilibria. An endemic equilibrium always satisfies

$$\begin{aligned} A - dS - \lambda SI + \varepsilon I + \frac{rI}{1 + \alpha I} &= 0, \\ \lambda SI - (d + \varepsilon + \mu) I - \frac{rI}{1 + \alpha I} &= 0. \end{aligned} \tag{6}$$

In view of $dS + (d + \mu)I = A$, we get $S = (A - (d + \mu)I)/d$ and substitute it into the second equation of (6). When $I \neq 0$, we obtain

$$\lambda \left[\frac{A - (d + \mu) I}{d} \right] - (d + \varepsilon + \mu) - \frac{r}{1 + \alpha I} = 0. \tag{7}$$

Then we have an equation of the form

$$aI^2 + bI + c = 0, \tag{8}$$

with

$$\begin{aligned} a &= \alpha\lambda (d + \mu), \\ b &= \lambda (d + \mu) + \alpha d (d + \varepsilon + \mu) - \alpha\lambda A, \\ c &= d (d + \varepsilon + r + \mu) - \lambda A. \end{aligned} \tag{9}$$

This equation may admit positive solution

$$I_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \qquad I_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{10}$$

Obviously, if $R_0 = 1$, then $c = 0$, if $R_0 > 1$, then $c < 0$, and if $R_0 < 1$, then $c > 0$. From (8), it is obvious that we have the following results.

Theorem 2. *The following results hold.*

(H_1) *Let $\alpha = 0$. Equation (8) is a linear equation with a unique solution $I = -c/b$. Then the system (4) has a unique endemic equilibrium when $R_0 > 1$ and has no endemic equilibrium when $R_0 \leq 1$.*

(H_2) *Let $\alpha > 0$. If $b > 0$, system (4) has a unique endemic equilibrium when $R_0 > 1$ and no endemic equilibrium when $R_0 \leq 1$.*

(H_3) *Let $\alpha > 0$. If $b < 0$, system (4) has a unique endemic equilibrium when $R_0 \geq 1$, no endemic equilibrium when $R_0 < R_0^*$, and two endemic equilibria E_1 and E_2 when $R_0^* \leq R_0 < 1$. When $R_0 = R_0^*$ and $E_1 = E_2$, one has $b^2 - 4ac = 0$ which is equivalent to*

$$\begin{aligned} R_0 &= \frac{4a\lambda A}{b^2 + 4a\lambda A} \\ &= \frac{4\lambda^2 \alpha (d + \mu) A}{[\lambda (d + \mu) + \alpha d (d + \varepsilon + \mu) - \alpha\lambda A]^2 + 4\lambda^2 \alpha (d + \mu) A}. \end{aligned} \tag{11}$$

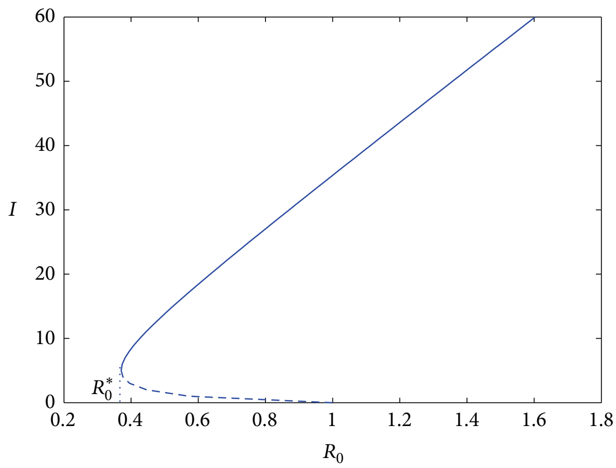

FIGURE 3: The figure of infective sizes at equilibria versus R_0 when $\alpha = 1$, $d = 0.1$, $\mu = 0.01$, $\lambda = 0.05$, $r = 2$, and $\varepsilon = 0.1$.

We define the right-hand side of (11) as

$$R_0^* \triangleq \frac{4\lambda^2\alpha(d+\mu)A}{[\lambda(d+\mu)+\alpha d(d+\varepsilon+\mu)-\alpha\lambda A]^2+4\lambda^2\alpha(d+\mu)A}. \tag{12}$$

Therefore, according to the qualitative approach recently proposed by [31] which is based on the analysis of the equilibria curve in the neighboring of the critical threshold $R_0 = 1$, we have the following theorem.

Theorem 3. *If $\alpha > 0$, $b < 0$, then system (4) has a backward bifurcation at $R_0 = 1$ (see Figure 3).*

In order to verify the bifurcation curve (the graph of I as a function of R_0) in Figure 3, we think of r as a variable with the other parameters as constant. Through implicit differentiation of (8) with respect to r, we get

$$(2aI+b)\frac{dI}{dr} = -d < 0. \tag{13}$$

From (13), we know that the sign of dI/dr is opposite to that of $2aI + b$. And from the definition of R_0 we know that R_0 decreases when r increases. It implies that the bifurcation curve has positive slope at equilibrium values with $2aI+b > 0$ and negative slope at equilibrium values with $2aI + b < 0$. If there is no backward bifurcation at $R_0 = 1$, then the unique endemic equilibrium for $R_0 > 1$ satisfies

$$2aI + b = \sqrt{b^2-4ac} > 0, \tag{14}$$

and the bifurcation curve has positive slope at all points where $I > 0$. If there is a backward bifurcation at $R_0 = 1$, then there is an interval on which there are two endemic equilibria given by

$$2aI + b = \pm\sqrt{b^2-4ac}. \tag{15}$$

The bifurcation curve has negative slope at the smaller one and positive slope at the larger one. Thus the bifurcation curve is as shown in Figure 3.

Under the conditions of Theorem 3, if a backward bifurcation takes place, we can see from Figure 3 that there is a critical value R_0^* at the turning point. In this case, the disease will not die out when $R_0 < 1$. However, the disease will die out when $R_0 < R_0^*$. Therefore, the critical value R_0^* can be taken as a new threshold for the control of the disease.

In the following, we give an explicit criterion of a backward bifurcation at $R_0 = 1$.

For convenience, we define

$$\alpha_0 := \frac{\lambda(d+\mu)}{dr}. \tag{16}$$

Corollary 4. *When $\alpha > \alpha_0$, system (4) has a backward bifurcation at $R_0 = 1$.*

Proof. When $R_0 \le 1 \Leftrightarrow c \ge 0$,

$$\lambda A \le d(d+\varepsilon+\mu+r). \tag{17}$$

The condition $b < 0$ is equivalent to

$$\lambda(d+\mu)+\alpha d(d+\varepsilon+\mu) < \alpha\lambda A. \tag{18}$$

From (17) and (18), we get $\lambda(d+\mu)+\alpha d(d+\varepsilon+\mu) < \alpha d(d+\varepsilon+r+\mu)$, which reduces to

$$\alpha > \frac{\lambda(d+\mu)}{dr} \triangleq \alpha_0. \tag{19}$$

It means that α is big enough to lead a backward bifurcation with two endemic equilibria when $R_0 < 1$. Therefore, the proof is complete. □

Next, we consider the local stability of the unique endemic equilibrium when $R_0 > 1$.

Theorem 5. *When $R_0 > 1$ and $0 \le \alpha < \lambda/r$, the unique endemic equilibrium E^* is locally asymptotically stable.*

Proof. Firstly, from Theorem 2, we can know that system (4) has a unique endemic equilibrium E^* when $R_0 > 1$. Moreover, the Jacobian matrix of system (4) is

$$M = \begin{pmatrix} -d-\lambda I & -\lambda S+\varepsilon+\dfrac{r}{1+\alpha I}-\dfrac{r\alpha I}{(1+\alpha I)^2} \\ \lambda I & \lambda S-(d+\varepsilon+\mu)-\dfrac{r}{1+\alpha I}+\dfrac{r\alpha I}{(1+\alpha I)^2} \end{pmatrix}. \tag{20}$$

From the second equation of (6), we have

$$-\lambda S+\varepsilon+\frac{r}{1+\alpha I} = -d-\mu. \tag{21}$$

From (21), the Jacobian matrix M reduces to

$$M = \begin{pmatrix} -d-\lambda I & -d-\mu-\dfrac{r\alpha I}{(1+\alpha I)^2} \\ \lambda I & \dfrac{r\alpha I}{(1+\alpha I)^2} \end{pmatrix}. \tag{22}$$

We obtain

$$\det(M) = \frac{I}{(1+\alpha I)^2}\left[\lambda(d+\mu)(1+\alpha I)^2 - r\alpha d\right]. \quad (23)$$

In fact, there is $\lambda(d+\mu)(1+\alpha I)^2 > \lambda d$. Since $R_0 > 1$ and $0 \le \alpha < \lambda/r$, we have

$$\lambda(d+\mu)(1+\alpha I)^2 - r\alpha d > \lambda d - r\alpha d > 0. \quad (24)$$

So we get $\det(M) > 0$. The trace of M is given by

$$\operatorname{tr}(M) = \frac{1}{(1+\alpha I)^2}\left[-(d+\lambda I)(1+\alpha I)^2 + r\alpha I\right]. \quad (25)$$

In the same way as the above calculation of (24), we have

$$-(d+\lambda I)(1+\alpha I)^2 + r\alpha I < -\lambda I + r\alpha I < 0. \quad (26)$$

So we get $\operatorname{tr}(M) < 0$. The proof is complete. □

Now we consider the case that there are two endemic equilibria E_1 and E_2; let M_i be the Jacobian matrix at E_i, $i = 1, 2$.

Theorem 6. *The endemic equilibrium E_1 is a saddle whenever it exists.*

Proof. Since $I_1 = (-b - \sqrt{b^2 - 4ac})/2a$ and $\Delta = b^2 - 4ac$, we have $I_1 = (-b - \sqrt{\Delta})/2a$. Thus

$$\begin{aligned}\det(M_1) &= \frac{I_1}{(1+\alpha I_1)^2}\left[\lambda(d+\mu)(1+\alpha I_1)^2 - r\alpha d\right] \\ &\triangleq \frac{I_1}{(1+\alpha I_1)^2}\psi(I_1).\end{aligned} \quad (27)$$

From (H_3) of Theorem 2, we can know that if $\alpha > 0$, $b < 0$, and $R_0^* < R_0 < 1$, then E_1 exists. Hence, we can get $\psi(0) = \lambda(d+\mu) - r\alpha d < 0$ and $\psi'(I_1) = 2\lambda(d+\mu)(1+\alpha I_1)\alpha > 0$. It follows that there exists a unique $I^* > 0$ such that

$$\begin{aligned}&\psi(I_1) = 0, \quad \text{when } I_1 = I^*, \\ &\psi(I_1) < 0, \quad \text{when } 0 < I_1 < I^*, \\ &\psi(I_1) > 0, \quad \text{when } I_1 > I^*,\end{aligned} \quad (28)$$

where

$$I^* = \sqrt{\frac{rd}{\alpha\lambda(d+\mu)}} - \frac{1}{\alpha}. \quad (29)$$

On the other hand,

$$\begin{aligned}I_1 &= -\frac{b}{2a} - \frac{\sqrt{\Delta}}{2a} \\ &= \frac{\alpha\lambda A - \lambda(d+\mu) - \alpha d(d+\varepsilon+\mu)}{2\alpha\lambda(d+\mu)} - \frac{\sqrt{\Delta}}{2a} \\ &= \sqrt{\frac{rd}{\alpha\lambda(d+\mu)}} - \frac{1}{\alpha} + \frac{1}{\alpha} - \sqrt{\frac{rd}{\alpha\lambda(d+\mu)}} \\ &\quad + \frac{\alpha\lambda A - \lambda(d+\mu) - \alpha d(d+\varepsilon+\mu)}{2\alpha\lambda(d+\mu)} - \frac{\sqrt{\Delta}}{2a} \\ &= I^* + \left(\left[\lambda(d+\mu) + \alpha\lambda A - 2\sqrt{rd\alpha\lambda(d+\mu)}\right.\right. \\ &\qquad \left.\left. - \alpha d(d+\varepsilon+\mu)\right] - \sqrt{\Delta}\right) \times (2\alpha\lambda(d+\mu))^{-1},\end{aligned} \quad (30)$$

$$\begin{aligned}\Delta &= \{\lambda(d+\mu) - [\alpha\lambda A - \alpha d(d+\varepsilon+\mu)]\}^2 \\ &\quad - 4\alpha\lambda(d+\mu)[d(d+\varepsilon+r+\mu) - \lambda A] \\ &= \Big\{\lambda(d+\mu) + [\alpha\lambda A - \alpha d(d+\varepsilon+\mu)] \\ &\qquad - 2\sqrt{rd\alpha\lambda(d+\mu)}\Big\}^2 - 4rd\alpha\lambda(d+\mu) \\ &\quad - 4\lambda(d+\mu)[\alpha\lambda A - \alpha d(d+\varepsilon+\mu)] \\ &\quad + 4\sqrt{rd\alpha\lambda(d+\mu)}\{\lambda(d+\mu) + [\alpha\lambda A - \alpha d(d+\varepsilon+\mu)]\} \\ &\quad - 4\alpha\lambda(d+\mu)[-\lambda A + d(d+\varepsilon+r+\mu)] \\ &\triangleq \Big\{\lambda(d+\mu) + [\alpha\lambda A - \alpha d(d+\varepsilon+\mu)] \\ &\qquad - 2\sqrt{rd\alpha\lambda(d+\mu)}\Big\}^2 + P,\end{aligned} \quad (31)$$

where

$$\begin{aligned}P &= -8rd\alpha\lambda(d+\mu) + 4\sqrt{rd\alpha\lambda(d+\mu)} \\ &\quad \times \{\lambda(d+\mu) + [\alpha\lambda A - \alpha d(d+\varepsilon+\mu)]\}.\end{aligned} \quad (32)$$

In the following we will show that $P > 0$. Since $R_0^* < R_0$, we have

$$\begin{aligned}&\frac{4\alpha\lambda(d+\mu)}{[\lambda(d+\mu) + \alpha d(d+\varepsilon+\mu) - \alpha\lambda A]^2 + 4\alpha\lambda^2(d+\mu)A} \\ &\quad < \frac{1}{d(d+\varepsilon+r+\mu)};\end{aligned} \quad (33)$$

that is,

$$[\lambda(d+\mu)+\alpha d(d+\varepsilon+\mu)-\alpha\lambda A]^2 > 4\alpha\lambda(d+\mu)d(d+\varepsilon+r+\mu)-4\alpha\lambda^2(d+\mu)A. \quad (34)$$

Therefore,

$$\begin{aligned}\{\lambda(d+\mu)+[\alpha\lambda A-\alpha d(d+\varepsilon+\mu)]\}^2 &= \{\lambda(d+\mu)-[\alpha\lambda A-\alpha d(d+\varepsilon+\mu)]\}^2 \\ &\quad + 4\lambda(d+\mu)[\alpha\lambda A-\alpha d(d+\varepsilon+\mu)] \\ &> 4\alpha\lambda(d+\mu)d(d+\varepsilon+r+\mu)-4\alpha\lambda^2(d+\mu)A \\ &\quad + 4\lambda(d+\mu)[\alpha\lambda A-\alpha d(d+\varepsilon+\mu)] \\ &= 4rd\alpha\lambda(d+\mu).\end{aligned} \quad (35)$$

Obviously, we have $P > 0$. From (30), one has $I_1 < I^*$. So we get $\det(M_1) < 0$. Hence the endemic equilibrium E_1 is a saddle. The proof is complete. □

In order to explore the stability of the endemic equilibrium E_2, define

$$\begin{aligned} m_1 &:= \left(2d\alpha a^2+\lambda a^2-\varepsilon\alpha a^2-ac\lambda\alpha^2\right) \\ &\quad - b\left(2\lambda\alpha a+d\alpha^2 a-b\lambda\alpha^2\right), \\ m_2 &:= a^2 d - c\left(2\lambda\alpha a+d\alpha^2 a-b\lambda\alpha^2\right). \end{aligned} \quad (36)$$

Theorem 7. *If $\eta > 0$, then endemic equilibrium E_2 is locally asymptotically stable; if $\eta < 0$, then endemic equilibrium E_2 is unstable, where $\eta := 2am_2 + m_1(\sqrt{b^2-4ac}-b)$.*

Proof. Consider

$$\begin{aligned}\det(M_2) &= \frac{I_2}{(1+\alpha I_2)^2}\left[\lambda(d+\mu)(1+\alpha I_2)^2-\varepsilon\alpha d\right] \\ &= \frac{I_2}{(1+\alpha I_2)^2}\times\psi(I_2).\end{aligned} \quad (37)$$

By carrying out arguments similar to that of Theorem 6, we have $I_2 > I^*$. Therefore, $\det(M_2) > 0$. In addition, we have

$$\begin{aligned}\operatorname{tr}(M_2) &= -\frac{(d+\lambda I_2)(1+\alpha I_2)^2-\varepsilon\alpha I_2}{(1+\alpha I_2)^2} \\ &= -\frac{\lambda\alpha^2 I_2^3+\left(2\lambda\alpha+d\alpha^2\right)I_2^2+(2d\alpha+\lambda-\varepsilon\alpha)I_2+d}{(1+\alpha I_2)^2},\end{aligned} \quad (38)$$

and then $\operatorname{sgn}(\operatorname{tr}(M_2)) = -\operatorname{sgn}(G(I_2))$, where

$$G(x) = \lambda\alpha^2 x^3 + \left(2\lambda\alpha+d\alpha^2\right)x^2 + (2d\alpha+\lambda-\varepsilon\alpha)x + d. \quad (39)$$

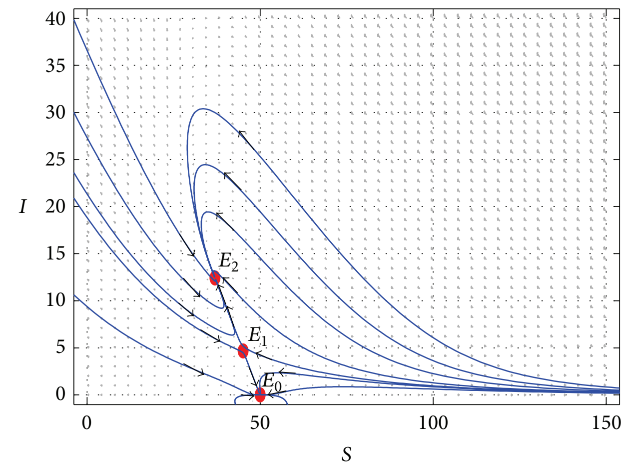

FIGURE 4: One region of disease persistence and one region of disease extinction when $A = 10$, $\alpha = 1$, $d = 0.1$, $\mu = 0.01$, $\lambda = 0.01$, $r = 0.1$, and $\varepsilon = 0.1$.

Using the expression of $m_1 = (2d\alpha a^2+\lambda a^2-\varepsilon\alpha a^2-ac\lambda\alpha^2) - b(2\lambda\alpha a+d\alpha^2 a-b\lambda\alpha^2)$ and $m_2 = a^2 d - c(2\lambda\alpha a+d\alpha^2 a-b\lambda\alpha^2)$, one has

$$G(I_2) = \left(aI_2^2+bI_2+c\right)\varphi_0 + \frac{m_1 I_2+m_2}{a^2}, \quad (40)$$

where φ_0 is a first degree polynomial of I_2. Since $aI_2^2+bI_2+c = 0$, $\operatorname{sgn}(\operatorname{tr}(M_2)) = -\operatorname{sgn}(G(I_2)) = -\operatorname{sgn}(m_1 I_2+m_2)$. From the expression of I_2, we have

$$\begin{aligned}&\operatorname{sgn}(m_1 I_2+m_2) \\ &= \operatorname{sgn}\left(2am_2+m_1\left(\sqrt{b^2-4ac}-b\right)\right) \triangleq \operatorname{sgn}(\eta).\end{aligned} \quad (41)$$

Thus, E_2 is locally asymptotically stable if $\eta > 0$ and E_2 is unstable if $\eta < 0$. The proof is complete. □

From the above discussion, we have the following conclusion. If two endemic equilibria E_1 and E_2 exist, the stable manifolds of the saddle E_1 split R_+^2 into two regions. The disease is persistent in the upper region and dies out in the lower region (see Figure 4).

3. Global Analysis

Firstly, we consider the global stability of the disease-free equilibrium E_0. Let $N = S + I$ be the total population size. Now we note that the equation for total population is given by $dN/dt = A - dS - (d+\mu)I \le A - dN$. It follows that $\lim_{t\to+\infty} N(t) \le A/d$. Let

$$\mathfrak{R} = \left\{(S,I)\in R_+^2 : S+I \le \frac{A}{d}\right\} \quad (42)$$

which is positive invariant with respect to system (4).

Theorem 8. *If $R_0 < R_0^*$, the disease-free equilibrium $E_0(A/d, 0)$ is globally asymptotically stable; that is, the disease dies out.*

Proof. Suppose $R_0 < R_0^*$. From the (H_3) of Theorem 2, we know that the model has no endemic equilibrium. From the corollary of Poincaré-Bendixson theorem [32], we know that there is no periodic orbit in $\mathfrak{R}$ as there is a disease-free equilibrium in $\mathfrak{R}$. Since $\mathfrak{R}$ is a bounded positively invariant region and E_0 is the only equilibrium in $\mathfrak{R}$, the local stability of E_0 implies that every solution initiating in $\mathfrak{R}$ approaches E_0. Thus, the disease-free equilibrium E_0 is globally asymptotically stable. The proof is complete. □

Now we analyze the global dynamics of the endemic equilibrium when $R_0 > 1$.

Theorem 9. *If $R_0 > 1$ and $0 \le \alpha < \lambda/r$, the system (4) has no limit cycles.*

Proof. We use Dulac theorem to exclude the limit cycle. Let

$$\begin{aligned} P(S,I) &= A - dS - \lambda SI + \varepsilon I + \frac{rI}{1+\alpha I}, \\ Q(S,I) &= \lambda SI - (d+\varepsilon+\mu) I - \frac{rI}{1+\alpha I}, \end{aligned} \tag{43}$$

and take the Dulac function

$$D = \frac{1}{I}. \tag{44}$$

According to $0 \le \alpha < \lambda/r$, we can get

$$\begin{aligned} &\frac{\partial(PD)}{\partial S} + \frac{\partial(QD)}{\partial I} \\ &= -\frac{d}{I} - \lambda + \frac{\alpha r}{(1+\alpha I)^2} \\ &< -\frac{d}{I} - \lambda + \frac{\lambda}{(1+\alpha I)^2} \\ &= \frac{1}{I(1+\alpha I)^2}\left\{-d(1+\alpha I)^2 - \lambda I\left[(1+\alpha I)^2 - 1\right]\right\} \\ &< 0. \end{aligned} \tag{45}$$

Hence, the system (4) has no limit cycles. The proof is complete. □

Therefore, we obtain the global result of the unique endemic equilibrium.

Theorem 10. *If $R_0 > 1$ and $0 \le \alpha < \lambda/r$, the unique endemic equilibrium E^* is globally asymptotically stable (see Figure 5).*

4. Hopf Bifurcation

In this section, we study the Hopf bifurcation of system (4). From the above discussion, we know that there is no closed orbit surrounding E_0 or E_1 because the S-axis is invariant with

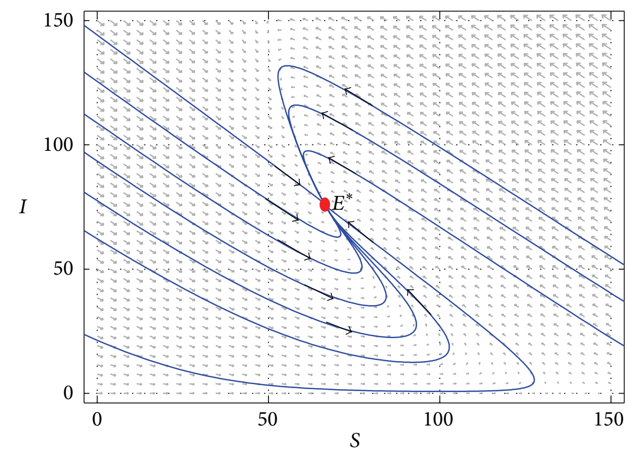

FIGURE 5: The unique endemic equilibrium E^* is globally asymptotically stable when $A = 15$, $\alpha = 0.01$, $d = 0.1$, $\mu = 0.01$, $\lambda = 0.01$, $r = 0.8$, and $\varepsilon = 0.1$.

respect to system (4) and E_1 is always a saddle. Therefore, Hopf bifurcation can only occur at E_2. Set

$$\begin{aligned} \sigma = &\left[-\lambda\alpha - \frac{c_1(c_2+2c_4)}{a_{12}}\right] D^4 \\ &+ \left[-\lambda\alpha a_{11}^2 - 2\lambda\alpha a_{11}a_{12} + \left(c_2 - \frac{2a_{11}c_1}{a_{12}}\right)\right. \\ &\times\left(a_{11}c_2 - \frac{a_{11}^2 c_1}{a_{12}}\right) \\ &-\left(\frac{2a_{11}^2 c_1}{a_{12}} - a_{11}c_2 + 2c_4 a_{11} - c_3 a_{12}\right) \\ &\left.\times\left(c_4 - \frac{a_{11}c_1}{a_{12}}\right)\right] D^2 \\ &+ a_{11}\left(\frac{a_{11}^2 c_1}{a_{12}} - a_{11}c_2 + c_4 a_{11} - c_3 a_{12}\right) \\ &\times\left(a_{12}c_3 - 2a_{11}c_4 - a_{11}c_2\right), \end{aligned} \tag{46}$$

where a_{ij} $(i,j = 1,2)$, c_k $(k = 1,2,3,4)$, and D are defined by (50) and (52).

Theorem 11. *System (4) undergoes a Hopf bifurcation if $\eta = 0$. Moreover, if $\sigma < 0$, there is a family of stable period orbits of system (4) as η decreases from 0; that is, a supercritical Hopf bifurcation occurs; if $\sigma > 0$, there is a family of unstable period orbits of system (4) as η increases from 0; that is, a subcritical Hopf bifurcation occurs.*

Proof. The proof of Theorem 7 shows that $\mathrm{tr}(M_2) = 0$ if and only if $\eta = 0$, and $\det(M_2) > 0$ when E_2 exists. Therefore, the eigenvalues of M_2 are a pair of pure imaginary roots if and only if $\eta = 0$. The direct calculations show that

$$\left.\frac{d(\mathrm{tr}(M_2))}{d\eta}\right|_{\eta=0} = -\frac{1}{2a^3(1+\alpha I_2)^2} < 0. \tag{47}$$

By Theorem 3.4.2 in [24], $\eta = 0$ is the Hopf bifurcation point for system (4).

To be concise in notations, rescale (4) by $\tau = t/(1+\alpha I)$. For simplicity, we still use t instead of τ. Then we obtain

$$\begin{aligned}\frac{dS}{dt} &= (A - dS - \lambda SI + \varepsilon I)(1+\alpha I) + rI,\\ \frac{dI}{dt} &= \lambda SI(1+\alpha I) - (d+\varepsilon+\mu) I (1+\alpha I) - rI.\end{aligned} \tag{48}$$

Let $x = S - S_2$ and $y = I - I_2$; then (48) becomes

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x + a_{12}y + c_1 y^2 + c_2 xy - \lambda\alpha xy^2,\\ \frac{dy}{dt} &= a_{21}x + a_{22}y + c_3 xy + c_4 y^2 + \lambda\alpha xy^2,\end{aligned} \tag{49}$$

where

$$\begin{aligned}a_{11} &= -(d+\lambda I_2)(1+\alpha I_2),\\ a_{12} &= (-\lambda S_2 + r)(1+\alpha I_2)\\ &\quad + \alpha(A - dS_2 - \lambda S_2 I_2 + rI_2) + \varepsilon,\\ a_{21} &= \lambda I_2 (1+\alpha I_2),\\ a_{22} &= [\lambda S_2 - (d+r+\mu)](1+\alpha I_2)\\ &\quad + \alpha[\lambda S_2 I_2 - (d+r+\mu) I_2] - \varepsilon,\\ c_1 &= \alpha(-\lambda S_2 + r),\\ c_2 &= \alpha(-d - \lambda I_2) - \lambda(1+\alpha I_2),\\ c_3 &= \lambda(1+\alpha I_2) + \lambda I_2 \alpha,\\ c_4 &= [\lambda S_2 - (d+r+\mu)]\alpha.\end{aligned} \tag{50}$$

Let E^* denote the origin of x-y plane. Since $E_2 = (S_2, I_2)$ satisfies (6), we obtain

$$\begin{aligned}&\det(M(E^*))\\ &= -(d+\lambda I_2)\alpha\varepsilon I_2 + (d+\mu)\left(1+\alpha I_2^2\right)^2 \lambda I_2 + \alpha\lambda\varepsilon I_2^2\\ &= I_2 \times \psi(I_2).\end{aligned} \tag{51}$$

From the proof of Theorem 7, it follows that $\psi(I_2)$ is always positive. It is easy to verify that $a_{11} + a_{22} = 0$ if and only if $\eta = 0$. Set

$$D = \sqrt{\det(M(E^*))} \tag{52}$$

and let $u = -x$ and $v = (a_{11}/D)x + (a_{12}/D)y$; then the normal form of (48) for Hopf bifurcation reads

$$\begin{aligned}\frac{du}{dt} &= -Dv + f(u,v),\\ \frac{dv}{dt} &= Du + g(u,v),\end{aligned} \tag{53}$$

where

$$\begin{aligned}f(u,v) &= \left(\frac{a_{11}}{a_{12}}c_2 - \frac{a_{11}^2}{a_{12}^2}c_1\right)u^2 - \frac{D^2 c_1}{a_{12}^2}v^2\\ &\quad + \left(\frac{Dc_2}{a_{12}} - 2\frac{Da_{11}c_1}{a_{12}^2}\right)uv - \frac{\lambda\alpha a_{11}^2}{a_{12}^2}u^3\\ &\quad - 2\frac{Da_{11}\lambda\alpha}{a_{12}^2}u^2 v - \frac{D^2\lambda\alpha}{a_{12}^2}uv^2,\\ g(u,v) &= \frac{a_{11}}{D}\left(\frac{a_{11}^2}{a_{12}^2}c_1 - \frac{a_{11}}{a_{12}}c_2 + \frac{a_{11}}{a_{12}}c_4 - c_3\right)u^2\\ &\quad + \left(\frac{Da_{11}c_1}{a_{12}^2} + \frac{Dc_4}{a_{12}}\right)v^2\\ &\quad + \left(2\frac{a_{11}^2}{a_{12}^2}c_1 - \frac{a_{11}}{a_{12}}c_2 + 2\frac{a_{11}}{a_{12}}c_4 - c_3\right)uv\\ &\quad + \frac{\lambda\alpha a_{11}^2}{a_{12}D}\left(\frac{1}{a_{12}} - 1\right)u^3 + \frac{2\lambda\alpha a_{11}}{a_{12}}\left(\frac{a_{11}}{a_{12}} - 1\right)u^2 v\\ &\quad + \frac{D\lambda\alpha}{a_{12}}\left(\frac{a_{11}}{a_{12}} - 1\right)uv^2.\end{aligned} \tag{54}$$

Now, we evaluate the first Lyapunov coefficient Γ of system (4) as follows:

$$\begin{aligned}\Gamma &= \frac{1}{16}[f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}] + \frac{1}{16D}\\ &\quad \times [f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}],\end{aligned} \tag{55}$$

where f_{uv} denotes $(\partial^2 f/\partial u \partial v)(0,0)$ and so forth. Then

$$\begin{aligned}\Gamma &= \frac{1}{8a_{12}^2 D^2}\left\{\left[-\lambda\alpha - \frac{c_1(c_2 + 2c_4)}{a_{12}}\right]D^4\right.\\ &\quad + \left[-\lambda\alpha a_{11}^2 - 2\lambda\alpha a_{11}a_{12} + \left(c_2 - \frac{2a_{11}c_1}{a_{12}}\right)\right.\\ &\quad \times \left(a_{11}c_2 - \frac{a_{11}^2 c_1}{a_{12}}\right)\\ &\quad - \left(\frac{2a_{11}^2 c_1}{a_{12}} - a_{11}c_2 + 2c_4 a_{11} - c_3 a_{12}\right)\\ &\quad \left.\times \left(c_4 - \frac{a_{11}c_1}{a_{12}}\right)\right]D^2\\ &\quad + a_{11}\left(\frac{a_{11}^2 c_1}{a_{12}} - a_{11}c_2 + c_4 a_{11} - c_3 a_{12}\right)\\ &\quad \left.\times (a_{12}c_3 - 2a_{11}c_4 - a_{11}c_2)\right\}\\ &= \frac{\sigma}{8a_{12}^2 D^2}.\end{aligned} \tag{56}$$

Obviously, the sign of Γ is determined by σ. By Theorem 3.4.2 and (3.4.11) in [24], the rest of the claims in Theorem 11 are valid. The proof is complete. □

5. Bogdanov-Takens Bifurcations

The purpose of this section is to study the Bogdanov-Takens bifurcation of (4). Now, we assume the following:

(A_1) $b^2 - 4ac = 0$,

(A_2) $2ad^2 + b\lambda\mu = 0$.

Then (6) admits a unique positive equilibrium $E^* = (S^*, I^*)$, where

$$I^* = -\frac{b}{2a}, \qquad S^* = \frac{A - (d+\mu) I^*}{d}. \tag{57}$$

The Jacobian matrix of system (4) at E^* is

$$M^* = \begin{pmatrix} -d - \lambda I^* & -\lambda S^* + \varepsilon + \dfrac{r}{(1+\alpha I^*)^2} \\ \lambda I^* & \lambda S^* - (d + \varepsilon + \mu) - \dfrac{r}{(1+\alpha I^*)^2} \end{pmatrix}. \tag{58}$$

By (58), we have

$$\begin{aligned} \det(M^*) &= \frac{I^*}{(1+\alpha I^*)^2}\left[(d+\mu)\lambda(1+\alpha I^*)^2 - rd\alpha\right] \\ &= 0, \end{aligned} \tag{59}$$

because of

$$\begin{aligned} (1+\alpha I^*)^2 &= \frac{4a^2 - 4a\alpha b + \alpha^2 b^2}{4a^2} \\ &= \frac{4a^2 - 4a\alpha b + \alpha^2 4ac}{4a^2} \\ &= \frac{a - \alpha b + \alpha^2 c}{a} \\ &= \frac{\alpha^2 rd}{a}. \end{aligned} \tag{60}$$

Furthermore, (A_2) implies that

$$\operatorname{tr}(M^*) = 0. \tag{61}$$

Thus, (A_1) and (A_2) imply that the Jacobian matrix has a zero eigenvalue with multiplicity 2. This suggests that (4) may admit a Bogdanov-Takens singularity.

Theorem 12. *Suppose that (A_1), (A_2), and $2b_1 + b_4 \neq 0$ hold. Then the endemic equilibrium $E^* = (S^*, I^*)$ of (4) is a cusp of codimension 2; that is, it is a Bogdanov-Takens singularity. Here, b_1 and b_4 are defined by (65).*

Proof. Using the transformation of $x = I - I^*$ and $y = S - S^*$, system (4) becomes

$$\begin{aligned} \frac{dx}{dt} &= a_1 x + a_2 y + \lambda xy + \hat{a}_{11} x^2 - P_1(x), \\ \frac{dy}{dt} &= -\frac{a_1^2}{a_2} x - a_1 y - \lambda xy - \hat{a}_{11} x^2 + P_1(x), \end{aligned} \tag{62}$$

where $P_1(x)$ is a smooth function of x at least of the third order and

$$\begin{aligned} a_1 &= \lambda S^* - (d + \varepsilon + \mu) - \frac{r}{(1+\alpha I^*)^2} > 0, \\ a_2 &= \lambda I^* > 0, \\ \hat{a}_{11} &= \frac{r\alpha}{(1+\alpha I^*)^3} > 0. \end{aligned} \tag{63}$$

Set $X = x$, $Y = a_1 x + a_2 y$. Then (62) is transformed into

$$\begin{aligned} \frac{dX}{dt} &= Y + b_1 X^2 + b_2 XY + Q_1(X), \\ \frac{dY}{dt} &= b_3 X^2 + b_4 XY + Q_2(X), \end{aligned} \tag{64}$$

where $Q_i(X)$ are smooth functions of X at least of the third order and

$$\begin{aligned} b_1 &= \hat{a}_{11} - \frac{\lambda a_1}{a_2}, \\ b_2 &= \frac{\lambda}{a_2}, \\ b_3 &= a_1 \hat{a}_{11} - \frac{\lambda a_1^2}{a_2} + a_1 \lambda - a_2 \hat{a}_{11}, \\ b_4 &= \frac{\lambda a_1}{a_2} - \lambda. \end{aligned} \tag{65}$$

In order to obtain the canonical normal form, we perform the transformation of variables by

$$u = X - \frac{b_2}{2} X^2, \qquad v = Y + b_1 X^2. \tag{66}$$

Then, we obtain

$$\begin{aligned} \frac{du}{dt} &= v + R_1(u), \\ \frac{dv}{dt} &= b_3 u^2 + (2b_1 + b_4) uv + R_2(u), \end{aligned} \tag{67}$$

where $R_i(u)$ are smooth functions of u at least of the third order. Note that $b_3 < 0$ and $2b_1 + b_4 \neq 0$. It follows from [33–35] that (4) admits a Bogdanov-Takens bifurcation. □

In the following, we will find the versal unfolding in terms of the original parameters in (4). In this way, we will know the approximate saddle-node, Hopf, and homoclinic bifurcation curves. We choose A and d as bifurcation parameters. Fix $\lambda = \lambda_0$, $\varepsilon = \varepsilon_0$, $r = r_0$, $\alpha = \alpha_0$, and $\mu = \mu_0$. Let $A = A_0 + \theta_1$ and $d = d_0 + \theta_2$, where θ_1 and θ_2 are parameters which vary in a small neighborhood of the origin.

Suppose that $A = A_0$, $d = d_0$, $\lambda = \lambda_0$, $\varepsilon = \varepsilon_0$, $r = r_0$, $\alpha = \alpha_0$, and $\mu = \mu_0$ satisfy (A_1) and (A_2). Consider the following system:

$$\begin{aligned} \frac{dI}{dt} &= \lambda_0 SI - (d_0 + \theta_2 + \varepsilon_0 + \mu_0) I - \frac{r_0 I}{1 + \alpha_0 I}, \\ \frac{dS}{dt} &= A_0 + \theta_1 - (d_0 + \theta_2) S - \lambda_0 SI + \varepsilon_0 I + \frac{r_0 I}{1 + \alpha_0 I}. \end{aligned} \tag{68}$$

By the transformations of $x = I - I^*$ and $y = S - S^*$, system (68) becomes

$$\begin{aligned} \frac{dx}{dt} &= -\theta_2 I^* + \hat{c}_1 x + \hat{c}_2 y + c_{11} x^2 + \lambda_0 xy - w_1(x), \\ \frac{dy}{dt} &= (\theta_1 - \theta_2 S^*) + \hat{c}_3 x + \hat{c}_4 y - c_{11} x^2 - \lambda_0 xy + w_1(x), \end{aligned} \tag{69}$$

where $w_1(x)$ is a smooth function of x at least of the third order and

$$\begin{aligned} \hat{c}_1 &= \lambda_0 S^* - (d_0 + \theta_2 + \varepsilon_0 + \mu_0) - \frac{r_0}{(1 + \alpha_0 I^*)^2}, \\ \hat{c}_2 &= \lambda_0 I^*, \\ \hat{c}_3 &= -\lambda_0 S^* + \varepsilon_0 + \frac{r_0}{(1 + \alpha_0 I^*)^2}, \\ \hat{c}_4 &= -(d_0 + \theta_2) - \lambda_0 I^*, \\ c_{11} &= \frac{r_0 \alpha_0}{(1 + \alpha_0 I^*)^3}. \end{aligned} \tag{70}$$

Making the change of variables $X = x$, $Y = -\theta_2 I^* + \hat{c}_1 x + \hat{c}_2 y + c_{11} x^2 + \lambda_0 xy - w_1(x)$ and rewriting X, Y as x and y, respectively, we have

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= e_0 + e_1 x + e_2 y + e_{11} x^2 + e_{12} xy + e_{22} y^2 + w_2(x, y, \theta), \end{aligned} \tag{71}$$

where $\theta = (\theta_1, \theta_2)$, $w_2(x, y, \theta)$ is a smooth function of x, y, θ_1, and θ_2 at least of the third order, and

$$\begin{aligned} e_0 &= \hat{c}_2 (\theta_1 - \theta_2 S^*) + \hat{c}_4 \theta_2 I^*, \\ e_1 &= \lambda_0 (\theta_1 - \theta_2 S^*) + \hat{c}_2 \hat{c}_3 - \hat{c}_1 \hat{c}_4 - \lambda_0 \theta_2 I^*, \\ e_2 &= \hat{c}_1 + \hat{c}_4 + \lambda_0 \frac{\theta_2 I^*}{\hat{c}_2}, \\ e_{11} &= \lambda_0 \hat{c}_3 - \hat{c}_4 c_{11} - c_{11} \hat{c}_2 + \hat{c}_1 \lambda_0, \\ e_{12} &= -\lambda_0 + 2c_{11} + \lambda_0 \frac{-\hat{c}_1 \hat{c}_2 - \lambda_0 \theta_2 I^*}{\hat{c}_2^2}, \\ e_{22} &= \frac{\lambda_0}{\hat{c}_2}. \end{aligned} \tag{72}$$

Next, introduce a new time variable τ by $dt = (1 - (\lambda_0/\hat{c}_2)x)d\tau$. Rewriting τ as t, we obtain

$$\begin{aligned} \frac{dx}{dt} &= y \left(1 - \frac{\lambda_0}{\hat{c}_2} x\right), \\ \frac{dy}{dt} &= \left(1 - \frac{\lambda_0}{\hat{c}_2} x\right) \\ &\quad \times \left(e_0 + e_1 x + e_2 y + e_{11} x^2 \right. \\ &\qquad \left. + e_{12} xy + e_{22} y^2 + w_2(x, y, \theta)\right). \end{aligned} \tag{73}$$

Let $X = x$, $Y = y(1 - (\lambda_0/\hat{c}_2)x)$ and rename X and Y as x and y; we have

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= e_0 + g_1 x + e_2 y + g_{11} x^2 + g_{12} xy + w_3(x, y, \theta), \end{aligned} \tag{74}$$

where $\theta = (\theta_1, \theta_2)$, $w_3(x, y, \theta)$ is a smooth function of x, y, θ_1, and θ_2 at least of the third order, and

$$\begin{aligned} g_1 &= -2e_0 \frac{\lambda_0}{\hat{c}_2} + e_1, \\ g_{11} &= e_{11} - 2\frac{e_1 \lambda_0}{\hat{c}_2} + \frac{e_0 \lambda_0^2}{\hat{c}_2^2}, \\ g_{12} &= e_{12} - \frac{e_2 \lambda_0}{\hat{c}_2}. \end{aligned} \tag{75}$$

Now, we assume that $g_{11} \neq 0$ and $g_{12} \neq 0$ when λ_i are small. Set $x = X - e_2/g_{12}$ and rewrite X as x; we can get

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= f_0 + f_1 x + g_{11} x^2 + g_{12} xy + w_4(x, y, \theta), \end{aligned} \tag{76}$$

where $\theta = (\theta_1, \theta_2)$, $w_4(x, y, \theta)$ is a smooth function of x, y, θ_1, and θ_2 at least of the third order, and

$$\begin{aligned} f_0 &= e_0 - \frac{g_1 e_2}{g_{12}} + \frac{g_{11} e_2^2}{g_{12}^2}, \\ f_1 &= g_1 - \frac{2 g_{11} e_2}{g_{12}}. \end{aligned} \tag{77}$$

Making the final change of variables by $X = g_{12}^2 x/g_{11}$, $Y = g_{12}^3 y/g_{11}^2$, and $\tau = g_{11}t/g_{12}$ and then denoting them again by x, y, and t, respectively, we obtain

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = \tau_1(\theta_1,\theta_2) + \tau_2(\theta_1,\theta_2)x + x^2 + xy + w_5(x,y,\theta), \tag{78}$$

where $\theta = (\theta_1,\theta_2)$ and $w_5(x,y,\theta)$ is a smooth function of x, y, θ_1, and θ_2 at least of the third order.

We substitute values of $\alpha_0 = 1$, $\lambda_0 = 1/4$, $d_0 = 1/10$, $\varepsilon_0 = 1/10$, $\mu_0 = 1/100$, $A_0 = 537/500$, and $r_0 = 55/8$ for the above system and these values satisfy conditions (A_1) and (A_2). And we obtain the following equations:

$$\begin{aligned}\tau_1(\theta_1,\theta_2) &= \frac{f_0 g_{12}^4}{g_{11}^3}\\ &= -\frac{237291605}{85184}\theta_1 + \frac{25485118377}{851840}\theta_2\\ &\quad + O\left(|\theta_1,\theta_2|^2\right),\end{aligned} \tag{79}$$

$$\begin{aligned}\tau_2(\theta_1,\theta_2) &= \frac{f_1 g_{12}^2}{g_{11}^2}\\ &= -\frac{172225}{1936}\theta_1 + \frac{3983253}{3872}\theta_2 + O\left(|\theta_1,\theta_2|^2\right),\end{aligned}$$

where $g_{11} = -11/500 - (1/16)\theta_1 + (341/800)\theta_2 + (1/4)\theta_2^2 < 0$ and $g_{12} = -83/200 + (1/4)\theta_2 < 0$ in a small neighborhood of $(\theta_1,\theta_2) = (0,0)$. Let

$$J = \begin{pmatrix} \dfrac{\partial\tau_1}{\partial\theta_1} & \dfrac{\partial\tau_1}{\partial\theta_2} \\ \dfrac{\partial\tau_2}{\partial\theta_1} & \dfrac{\partial\tau_2}{\partial\theta_2} \end{pmatrix}. \tag{80}$$

And after simple calculation we obtain that

$$\det(J)|_{\lambda=0} = -\frac{16839398748825}{82458112} \neq 0. \tag{81}$$

Thus, τ_1 and τ_2 are regular maps in a small neighborhood of $(\theta_1,\theta_2) = (0,0)$. By the Bogdanov and Takens bifurcation theorems [36], we obtain the following conclusion.

Theorem 13. *Suppose that A_0, d_0, λ_0, ε_0, r_0, α_0, and μ_0 satisfy (A_1), (A_2), $g_{11} \neq 0$, and $g_{12} \neq 0$ when θ_i are small. Then (4) admits the following bifurcation behavior.*

(1) *There is a saddle-node bifurcation curve* $SN = \{(\theta_1,\theta_2) : 4f_0g_{11} = f_1^2 + o(|(\theta_1,\theta_2)|^2),\ f_1 \neq 0\} = \{(\theta_1,\theta_2) : (5907/6250)\theta_2 - (11/125)\theta_1 + (2313/400)\theta_1\theta_2 - (5/16)\theta_1^2 - (1028637/40000)\theta_2^2 + o(|(\theta_1,\theta_2)|^2) = 0,\ (47991/16600)\theta_2 - (1/4)\theta_1 + (25/83)\theta_1\theta_2 - (13645/13778)\theta_2^2 + o(|(\theta_1,\theta_2)|^2) \neq 0\}$.

(2) *There is a Hopf bifurcation curve* $H = \{(\theta_1,\theta_2) : f_0 + o(|(\theta_1,\theta_2)|^2) = 0,\ f_1 < 0\} = \{(\theta_1,\theta_2) : -(537/50)\theta_2 + \theta_1 + (50/83)\theta_1\theta_2 - (74667/6889)\theta_2^2 + o(|(\theta_1,\theta_2)|^2) = 0,\ (47991/16600)\theta_2 - (1/4)\theta_1 + (25/83)\theta_1\theta_2 - (13645/13778)\theta_2^2 + o(|(\theta_1,\theta_2)|^2) < 0\}$.

(3) *There is a homoclinic bifurcation curve* $HL = \{(\theta_1,\theta_2) : 25g_{11}f_0 + 6f_1^2 = o(|(\theta_1,\theta_2)|^2),\ f_1 < 0\} = \{(\theta_1,\theta_2) : (5907/1000)\theta_2 - (11/20)\theta_1 + (611979/33200)\theta_1\theta_2 - (19/16)\theta_1^2 - (16075834839/275560000)\theta_2^2 + o(|(\theta_1,\theta_2)|^2) = 0,\ (47991/16600)\theta_2 - (1/4)\theta_1 + (25/83)\theta_1\theta_2 - (13645/13778)\theta_2^2 + o(|(\theta_1,\theta_2)|^2) < 0\}$.

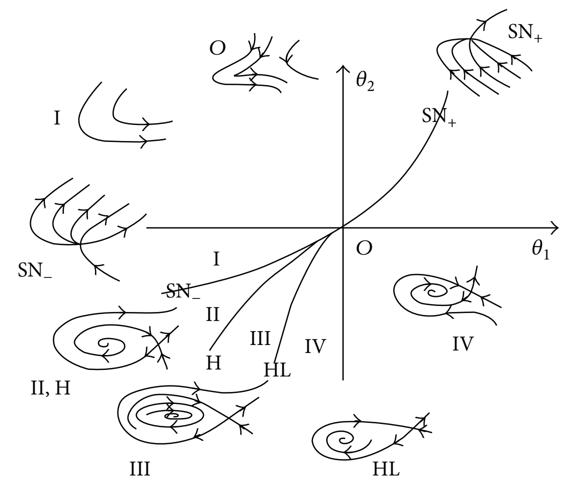

Figure 6: The bifurcation set and the corresponding phase portraits for system (68).

6. Numerical Simulations

When $\alpha = 1$, $\lambda = 1/4$, $d = 1/10$, $\varepsilon = 1/10$, $\mu = 1/100$, $A = 537/500$, and $r = 55/8$, $\sigma > 0$. By applying PPLANE8 and Photoshop software, the (θ_1, θ_2)-plane near the origin is divided into 4 regions by these bifurcation curves as shown in Figure 6. Fix $\theta_1 < 0$ and decrease θ_2 from 0; our conclusions are summarized as follows.

(a) When (θ_1, θ_2) lies in region I, there is no positive equilibrium, which implies that the positive orbits of (4) meet the positive S-axis in finite time, and therefore the disease disappears.

(b) There is a saddle-node point when (θ_1, θ_2) lies on curve SN.

(c) When (θ_1, θ_2) crosses SN into region II, two positive equilibria which are an unstable focus and a saddle appear.

(d) When the parameters lie on the curve H, there is also a saddle and an unstable focus. An unstable limit cycle appears with the parameters crossing H into III.

(e) A homoclinic cycle appears as the parameters passing III into HL through the homoclinic bifurcation.

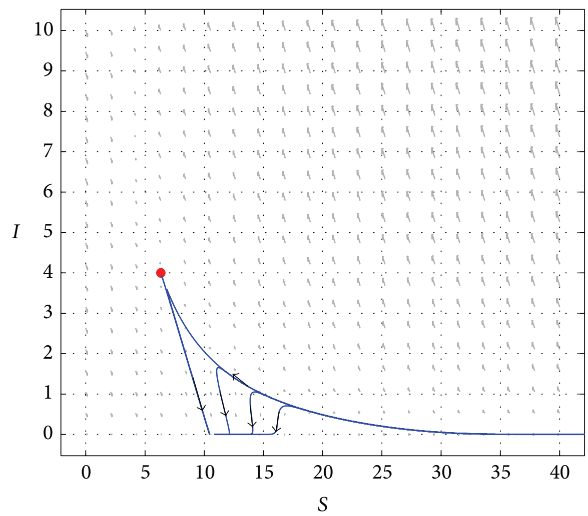

Figure 7: When $(\theta_1, \theta_2) = (0, 0)$, the unique positive equilibrium is a cusp of codimension 2.

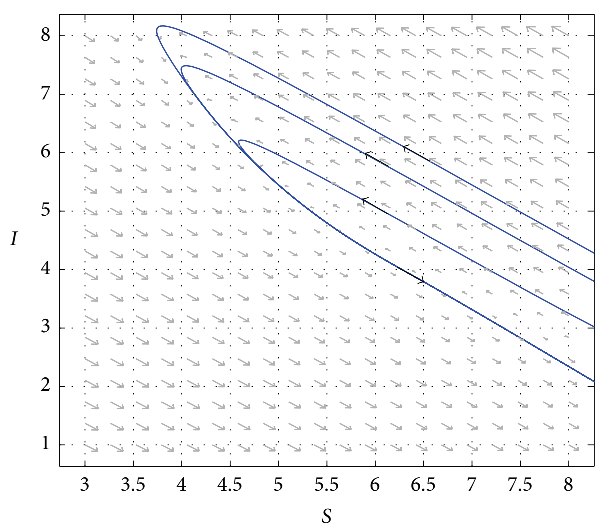

Figure 8: When $(\theta_1, \theta_2) = (-0.1, -0.0093)$ lies in region I, there is no positive equilibrium.

(f) The homoclinic loop breaks with the parameter crossing HL into IV, and a saddle and a stable focus appear.

Furthermore, using PPLANE8, some numerical simulations of system (68) are depicted in Figures 7–10. When $(\theta_1, \theta_2) = (0, 0)$, that is $(A_0, d_0) = (537/500, 1/10)$, there is a unique positive equilibrium $(S^*, I^*) = (317/50, 4)$, which is a cusp of codimension 2 (Figure 7). When $(\theta_1, \theta_2) = (-0.1, -0.0093)$ lies in region I, there is no positive equilibrium (Figure 8). When $(\theta_1, \theta_2) = (-0.02, -0.001863798)$, there is a homoclinic loop (Figure 9). When $(\theta_1, \theta_2) = (-0.02, -0.0020)$ lies in region IV, the homoclinic loop is broken, and there is a stable focus and a saddle (Figure 10).

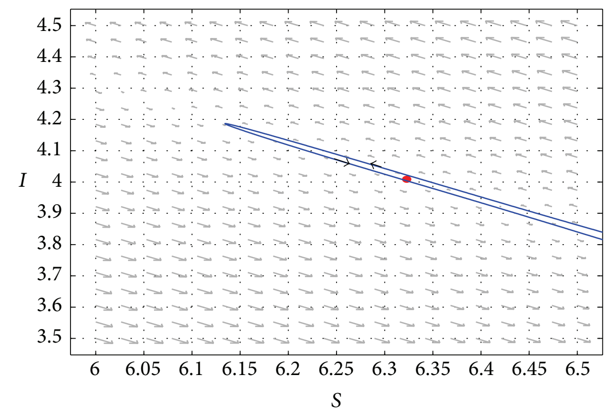

Figure 9: When $(\theta_1, \theta_2) = (-0.02, -0.001863798)$, there is a homoclinic loop.

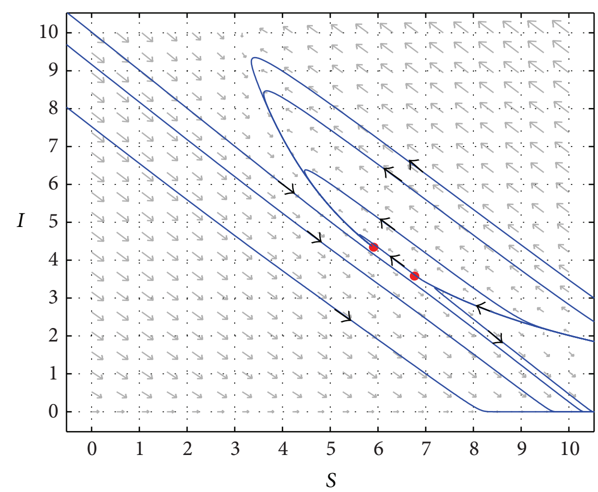

Figure 10: When $(\theta_1, \theta_2) = (-0.02, -0.0020)$ lies in region IV, there is a stable focus and a saddle.

7. Discussion

In this paper, we focus on the bifurcation analysis of an SIS epidemic model with bilinear incidence rate and saturated treatment. Generally speaking, in many epidemic models, the basic reproduction number, which is the key concept in epidemiology, can be decreased below unity to eradicate the disease. However, in our model, the basic reproduction number below 1 is not enough to eradicate the disease. According to our analysis in this paper, we demonstrate not only the global stability of the disease-free equilibrium when $R_0 < R_0^*$ and the local asymptotic stability of the endemic equilibrium E_2 when $R_0^* < R_0 < 1$, but also the global stability of the unique endemic equilibrium E^* when $R_0 > 1$. Moreover, it has been shown in Corollary 4 that backward bifurcations occur if the effect of the infected being delayed for treatment is strong. That is to say, we should get prompt treatment for patients. Through Figure 4, we can see that there is a region such that the disease will persist if the initial position lies in the region and disappear if the initial position lies outside this region. So, in order to restrain the spread of the disease, governments should take timely measures to control the initial infected individuals in a lower level. In addition, from Theorem 8, we can see that the disease will die out if R_0 is small enough. Therefore, from the definition of R_0, one knows that we can decrease the incidence rate λ and increase the cure rate r so as to eradicate the diseases or to make them controlled in a lower endemic steady state.

The stability analysis of the model equilibria enables us to completely analyze their local bifurcation behavior, such as Hopf, saddle-node, and Bogdanov-Takens bifurcation. By computing the first Lyapunov coefficient, we can determine that the Hopf bifurcation is a supercritical Hopf bifurcation or a subcritical Hopf bifurcation. We also show that, under assumptions (A_1) and (A_2), the model undertakes Bogdanov-Takens bifurcation; that is, there are saddle-node bifurcation, Hopf bifurcation, and homoclinic bifurcation in the system. The normal norm of the Bogdanov-Takens bifurcation is derived in Section 5 which is very helpful to obtain the three kinds of bifurcation curves. By analytical techniques, A and d are chosen as bifurcation parameters and other parameters are fixed, and we easily get a clear picture about the rich dynamics behaviors of our model. Through studying the bifurcations of the SIS epidemic model, we are suggesting that, in order to eradicate the disease, more medical facilities as well as medical professionals are needed and the medical standard needs to be improved as well.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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