Research Article

Positive Mild Solutions of Periodic Boundary Value Problems for Fractional Evolution Equations

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The periodic boundary value problem is discussed for a class of fractional evolution equations. The existence and uniqueness results of mild solutions for the associated linear fractional evolution equations are established, and the spectral radius of resolvent operator is accurately estimated. With the aid of the estimation, the existence and uniqueness results of positive mild solutions are obtained by using the monotone iterative technique. As an application that illustrates the abstract results, an example is given.

1. Introduction

In this paper, we investigate the existence and uniqueness of positive mild solutions of the periodic boundary value problem (PBVP) for the fractional evolution equation in an ordered Banach space X

$$D^\alpha u(t) + Au(t) = f(t, u(t)), \quad t \in I,$$
$$u(0) = u(\omega),$$

(1.1)

where D^α is the Caputo fractional derivative of order $0 < \alpha < 1$, $I = [0, \omega]$, $-A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ of uniformly bounded linear operators on X, and $f : I \times X \to X$ is a continuous function.

The origin of fractional calculus goes back to Newton and Leibnitz in the seventieth century. We observe that fractional order can be complex in viewpoint of pure mathematics and there is much interest in developing the theoretical analysis and numerical methods to

fractional equations, because they have recently proved to be valuable in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, biology, and hydrogeology. For example space-fractional diffusion equations have been used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium [1, 2] or to model activator-inhibitor dynamics with anomalous diffusion [3].

Fractional evolution equations, which is field have abundant contents. Many differential equations can turn to semilinear fractional evolution equations in Banach spaces. For example, fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0, 1)$, namely,

$$\partial_t^\alpha u(y, t) = Au(y, t), \quad t \geq 0, y \in R, \tag{1.2}$$

we can take $A = \partial_y^{\beta_1}$, for $\beta_1 \in (0, 1]$, or $A = \partial_y + \partial_y^{\beta_2}$ for $\beta_2 \in (1, 2]$, where $\partial_t^\alpha, \partial_y^{\beta_1}, \partial_y^{\beta_2}$ are the fractional derivatives of order α, β_1, β_2, respectively. Recently, fractional evolution equations are attracting increasing interest, see El-Borai [4, 5], Zhou and Jiao [6, 7], Wang et al. [8, 9], Shu et al. [10] and Mu et al. [11, 12]. They established various criteria on the existence of solutions for some fractional evolution equations by using the Krasnoselskii fixed point theorem, the Leray-Schauder fixed point theorem, the contraction mapping principle, or the monotone iterative technique. However, no papers have studied the periodic boundary value problems for abstract fractional evolution equations (1.1), though the periodic boundary value problems for ordinary differential equations have been widely studied by many authors (see [13–18]).

In this paper, without the assumptions of lower and upper solutions, by using the monotone iterative technique, we obtain the existence and uniqueness of positive mild solutions for PBVP (1.1). Because in many practical problems such as the reaction diffusion equations, only the positive solution has the significance, we consider the positive mild solutions in this paper. The characteristics of positive operator semigroup play an important role in obtaining the existence of the positive mild solutions. Positive operator semigroup are widely appearing in heat conduction equations, the reaction diffusion equations, and so on (see [19]). It is worth noting that our assumptions are very natural and we have tested them in the practical context. In particular to build intuition and throw some light on the power of our results, we examine sufficient conditions for the existence and uniqueness of positive mild solutions for periodic boundary value problem for fractional parabolic partial differential equations (see Example 4.1).

We now turn to a summary of this work. Section 2 provides the definitions and preliminary results to be used in theorems stated and proved in the paper. In particular to facilitate access to the individual topics, the existence and uniqueness results of mild solutions for the associated linear fractional evolution equations are established and the spectral radius of resolvent operator is accurately estimated. In Section 3, we obtain very general results on the existence and uniqueness of positive mild solutions for PBVP (1.1), when the nonlinear term f satisfies some conditions related to the growth index of the operator semigroup $\{T(t)\}_{t \geq 0}$. The main method is the monotone iterative technique. In Section 4, we give also an example to illustrate the applications of the abstract results.

2. Preliminaries

Let us recall the following known definitions. For more details see [20–23].

Definition 2.1. The fractional integral of order α with the lower limit zero for a function f is defined as:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \ \alpha > 0, \tag{2.1}$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann-Liouville derivative of order α with the lower limit zero for a function f can be written as:

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, \ n-1 < \alpha < n. \tag{2.2}$$

Definition 2.3. The Caputo fractional derivative of order α for a function f can be written as:

$$D^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \ n-1 < \alpha < n. \tag{2.3}$$

Remark 2.4. (i) If $f \in C^n[0, \infty)$, then

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, \ n-1 < \alpha < n. \tag{2.4}$$

(ii) The Caputo derivative of a constant is equal to zero.
(iii) If f is an abstract function with values in X, then the integrals and derivatives which appear in Definitions 2.1–2.3 are taken in Bochner's sense.

Throughout this paper, let X be an ordered Banach space with norm $\| \cdot \|$ and partial order $\leq$, whose positive cone $P = \{y \in X \mid y \geq \theta\}$ (θ is the zero element of X) is normal with normal constant N. Let $C(I, X)$ be the Banach space of all continuous X-value functions on interval I with norm $\|u\|_C = \max_{t \in I} \|u(t)\|$. Evidently, $C(I, X)$ is also an ordered Banach space with the partial $\leq$ reduced by the positive function cone $P_C = \{u \in C(I, X) \mid u(t) \geq \theta, t \in I\}$. P_C is also normal with the same constant N. For $u, v \in C(I, X)$, $u \leq v$ if $u(t) \leq v(t)$ for all $t \in I$. For $v, w \in C(I, X)$, denote the ordered interval $[v, w] = \{u \in C(I, X) \mid v \leq u \leq w\}$ in $C(I, X)$, and $[v(t), w(t)] = \{y \in X \mid v(t) \leq y \leq w(t)\}(t \in I)$ in X. Set $C^\alpha(I, X) = \{u \in C(I, X) \mid D^\alpha u$ exists and $D^\alpha u \in C(I, X)\}$. X_1 denotes the Banach space $D(A)$ with the graph norm $\| \cdot \|_1 = \| \cdot \| + \|A \cdot \|$. Suppose that $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $\{T(t)\}_{t \geq 0}$. This means there exists $M \geq 1$ such that

$$\|T(t)\| \leq M \quad t \geq 0. \tag{2.5}$$

Lemma 2.5 (see [4]). *If h satisfies a uniform Hölder condition, with exponent $\beta \in (0,1]$, then the unique solution of the linear initial value problem (LIVP) for the fractional evolution equation,*

$$D^\alpha u(t) + Au(t) = h(t), \quad t \in I,$$
$$u(0) = x_0 \in X, \tag{2.6}$$

is given by

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t-s)h(s)ds, \tag{2.7}$$

where

$$U(t) = \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha \theta)d\theta, \qquad V(t) = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) T(t^\alpha \theta)d\theta, \tag{2.8}$$

$\zeta_\alpha(\theta)$ *is a probability density function defined on* $(0,\infty)$.

Remark 2.6. (i) See [6, 7],

$$\zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\rho_\alpha\left(\theta^{-1/\alpha}\right),$$
$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{n=0}^\infty (-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty), \tag{2.9}$$

(ii) see [6, 24], $\zeta_\alpha(\theta) \geq 0$, $\theta \in (0,\infty)$, $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$, $\int_0^\infty \theta\zeta_\alpha(\theta)d\theta = 1/\Gamma(1+\alpha)$, $\int_0^\infty \theta^v \zeta_\alpha(\theta)d\theta = \Gamma(1+v)/\Gamma(1+\alpha v)$ for $v \in (-1,\infty)$,

(iii) see [4, 5], the Laplace transform of ζ_α is given by

$$\int_0^\infty e^{-p\theta}\zeta_\alpha(\theta)d\theta = \sum_{n=0}^\infty \frac{(-p)^n}{\Gamma(1+n\alpha)} = E_\alpha(-p), \tag{2.10}$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function (see [20]),

(iv) see [24] by (i) and (ii), we can obtain that for $p \geq 0$

$$\int_0^\infty e^{-p\theta}\theta\zeta_\alpha(\theta)d\theta = \frac{1}{\alpha}\sum_{n=0}^\infty \frac{(-p)^n}{\Gamma(\alpha(n+1))} = \frac{1}{\alpha}E_{\alpha,\alpha}(-p), \tag{2.11}$$

where $E_\alpha(\cdot)$, $E_{\alpha,\alpha}(\cdot)$ are the Mittag-Leffler functions.

(v) see [25] for $p < 0$, $0 < E_\alpha(p) < E_\alpha(0) = 1$,

(vi) see [10] if $\delta > 0$ and $t > 0$, then $-(1/\delta)(E_\alpha(-\delta t^\alpha))' = t^{\alpha-1}E_{\alpha,\alpha}(-\delta t^\alpha)$.

Remark 2.7. See [6, 8], the operators U and V, given by (2.8), have the following properties:

(i) For any fixed $t \geq 0$, $U(t)$ and $V(t)$ are linear and bounded operators, that is, for any $x \in X$,

$$\|U(t)x\| \leq M\|x\|, \qquad \|V(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\|, \qquad (2.12)$$

(ii) $\{U(t)\}_{t \geq 0}$ and $\{V(t)\}_{t \geq 0}$ are strongly continuous.

Definition 2.8. If $h \in C(I, X)$, by the mild solution of IVP (2.6), we mean that the function $u \in C(I, X)$ satisfying the integral (2.7).

We also introduce some basic theories of the operator semigroups. For an analytic semigroup $\{R(t)\}_{t \geq 0}$, there exist $M_1 > 0$ and $\delta \in \mathbb{R}$ such that (see [26])

$$\|R(t)\| \leq M_1 e^{\delta t}, \quad t \geq 0. \qquad (2.13)$$

Then

$$v_0 = \inf\left\{\delta \in \mathbb{R} \mid \text{there exist } M_1 > 0 \text{ such that } \|R(t)\| \leq M_1 e^{\delta t}, \forall t \geq 0\right\} \qquad (2.14)$$

is called the growth index of the semigroup $\{R(t)\}_{t \geq 0}$. Furthermore, v_0 can also be obtained by the following formula:

$$v_0 = \limsup_{t \to +\infty} \frac{\ln\|R(t)\|}{t}. \qquad (2.15)$$

Definition 2.9 (see [26]). A C_0-semigroup $\{T(t)\}_{t \geq 0}$ is called a compact semigroup if $T(t)$ is compact for $t > 0$.

Definition 2.10. An analytic semigroup $\{T(t)\}_{t \geq 0}$ is called positive if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.

Remark 2.11. For the applications of positive operators semigroup, we can see [27–31].

Definition 2.12. A bounded linear operator K on X is called to be positive if $Kx \geq \theta$ for all $x \geq \theta$.

Remark 2.13. By Remark 2.6(ii), we obtain that $U(t)$ and $V(t)$ are positive for $t \geq 0$ if $\{T(t)\}_{t \geq 0}$ is a positive semigroup.

Lemma 2.14. *Let X be an ordered Banach space, whose positive cone P is normal. If $\{T(t)\}_{t \geq 0}$ is an exponentially stable analytic semigroup, that is, $v_0 = \limsup_{t \to +\infty} (\ln\|T(t)\|/t) < 0$. Then the linear periodic boundary value problem (LPBVP),*

$$D^{\alpha}u(t) + Au(t) = h(t), \quad t \in I,$$
$$u(0) = u(\omega), \qquad (2.16)$$

has a unique mild solution

$$u(t) := (Qh)(t) = U(t)B(h) + \int_0^t (t - s)^{\alpha - 1} V(t - s)h(s)ds, \tag{2.17}$$

where $U(t)$ and $V(t)$ are given by (2.8),

$$B(h) = (I - U(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha - 1} V(\omega - s)h(s)ds, \tag{2.18}$$

$Q : C(I, X) \rightarrow C(I, X)$ *is a bounded linear operator, and the spectral radius $r(Q) \leq 1/|v_0|$.*

Proof. For any $v \in (0, |v_0|)$, by there exists M_1 such that

$$\|T(t)\| \leq M_1 e^{-vt}, \quad t \geq 0. \tag{2.19}$$

In X, give the equivalent norm $|\cdot|$ by

$$|x| = \sup_{t \geq 0} \|e^{vt} T(t)x\|, \tag{2.20}$$

then $\|x\| \leq |x| \leq M_1 \|x\|$. By $|T(t)|$ we denote the norm of $T(t)$ in $(X, |\cdot|)$, then for $t \geq 0$,

$$\begin{aligned}
|T(t)x| &= \sup_{s \geq 0} \|e^{vs} T(s)T(t)x\| \\
&= e^{-vt} \sup_{s \geq 0} \|e^{v(s+t)} T(s + t)x\| \\
&= e^{-vt} \sup_{\eta \geq t} \|e^{v\eta} T(\eta)x\| \\
&\leq e^{-vt} |x|.
\end{aligned} \tag{2.21}$$

Thus, $|T(t)| \leq e^{-vt}$. Then by Remark 2.6,

$$\begin{aligned}
|U(t)| &= \left| \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha \theta)d\theta \right| \\
&\leq \int_0^\infty \zeta_\alpha(\theta) e^{-vt^\alpha \theta} d\theta \\
&= E_\alpha(-vt^\alpha) < 1.
\end{aligned} \tag{2.22}$$

Therefore, $I - U(\omega)$ has bounded inverse operator and

$$(I - U(\omega))^{-1} = \sum_{n=0}^{\infty} (U(\omega))^n, \tag{2.23}$$

$$\left| (I - U(\omega))^{-1} \right| \leq \frac{1}{1 - E_\alpha(-\nu\omega^\alpha)}. \tag{2.24}$$

Set

$$x_0 = (I - U(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha-1} V(\omega - s) h(s) ds, \tag{2.25}$$

then

$$u(t) = U(t)x_0 + \int_0^t (t - s)^{\alpha-1} V(t - s) h(s) ds \tag{2.26}$$

is the unique mild solution of LIVP (2.6) satisfing $u(0) = u(\omega)$. So set

$$B(h) = (I - U(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha-1} V(\omega - s) h(s) ds,$$

$$(Qh)(t) = U(t)B(h) + \int_0^t (t - s)^{\alpha-1} V(t - s) h(s) ds, \tag{2.27}$$

then $u := Qh$ is the unique mild solution of LPBVP (2.16). By Remark 2.7, $Q : C(I, X) \to C(I, X)$ is a bounded linear operator. Furthermore, by Remark 2.6, we obtain that

$$|V(t)| = \left| \alpha \int_0^\infty \theta \zeta_\alpha(\theta) T(t^\alpha \theta) d\theta \right|$$

$$\leq \alpha \int_0^\infty \theta \zeta_\alpha(\theta) e^{-\nu t^\alpha \theta} d\theta \tag{2.28}$$

$$= E_{\alpha,\alpha}(-\nu t^\alpha).$$

By (2.24), (2.28) and Remark 2.6, for $t \geq 0$ we have that

$$|(Qh)(t)| \leq \left| U(t)(I - U(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha-1} V(\omega - s) h(s) ds \right|$$

$$+ \left| \int_0^t (t - s)^{\alpha-1} V(t - s) h(s) ds \right|$$

$$\leq \frac{E_\alpha(-\nu t^\alpha)}{1 - E_\alpha(-\nu\omega^\alpha)} \left| \int_0^\omega (\omega - s)^{\alpha-1} V(\omega - s) ds \right| |h|_C$$

$$+ \left| \int_0^t (t-s)^{\alpha-1} V(t-s) ds \right| |h|_C$$

$$= \left[\frac{E_\alpha(-\nu t^\alpha)}{1 - E_\alpha(-\nu \omega^\alpha)} \frac{1}{\nu} E_\alpha(-\nu(\omega-s)^\alpha) \Big|_0^\omega + \frac{1}{\nu} E_\alpha(-\nu(t-s)^\alpha) \Big|_0^t \right] |h|_C$$

$$= \frac{|h|_C}{\nu},$$

$$(2.29)$$

where $|\cdot|_C = \max_{t \in I} |\cdot(t)|$. Thus, $|Qh|_C \le |h|_C/\nu$. Then $|Q| \le 1/\nu$ and the spectral radius $r(Q) \le 1/\nu$. By the randomicity of $\nu \in (0, |\nu_0|)$, we obtain that $r(Q) \le 1/|\nu_0|$. □

Remark 2.15. For sufficient conditions of exponentially stable operator semigroups, one can see [32].

Remark 2.16. If $\{T(t)\}_{t \ge 0}$ is a positive and exponentially stable analytic semigroup generated by $-A$, by Remark 2.13, then the resolvent operator $Q : C(I, X) \to C(I, X)$ is also a positive bounded linear operator.

Remark 2.17. For the applications of Lemma 2.14, it is important to estimate the growth index of $\{T(t)\}_{t \ge 0}$. If $T(t)$ is continuous in the uniform operator topology for $t > 0$, it is well known that ν_0 can be obtained by $\sigma(A)$: the spectrum of A (see [33])

$$\nu_0 = -\inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}. \tag{2.30}$$

We know that $T(t)$ is continuous in the uniform operator topology for $t > 0$ if $T(t)$ is a compact semigroup, see [26]. Assume that P is a regeneration cone, $\{T(t)\}_{t \ge 0}$ is a compact and positive analytic semigroup. Then by the characteristic of positive semigroups (see [31]), for sufficiently large $\lambda_0 > -\inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$, we have that $\lambda_0 I + A$ has positive bounded inverse operator $(\lambda_0 I + A)^{-1}$. Since $\sigma(A) \ne \emptyset$, the spectral radius $r((\lambda_0 I + A)^{-1}) = 1/\operatorname{dist}(-\lambda_0, \sigma(A)) > 0$. By the Krein-Rutmann theorem (see [34, 35]), A has the first eigenvalue λ_1, which has a positive eigenfunction x_1, and

$$\lambda_1 = \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}, \tag{2.31}$$

that is, $\nu_0 = -\lambda_1$.

Corollary 2.18. *Let X be an ordered Banach space, whose positive cone P is a regeneration cone. If $\{T(t)\}_{t \ge 0}$ is a compact and positive analytic semigroup, and its first eigenvalue of A is*

$$\lambda_1 = \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\} > 0, \tag{2.32}$$

then LPBVP (2.16) has a unique mild solution $u := Qh$, $Q : C(I, X) \to C(I, X)$ is a bounded linear operator, and the spectral radius $r(Q) = 1/\lambda_1$

Proof. By (2.32), we know that the growth index of $\{T(t)\}_{t \ge 0}$ is $\nu_0 = -\lambda_1 < 0$, that is, $\{T(t)\}_{t \ge 0}$ is exponentially stable. By Lemma 2.14, $Q : C(I, X) \to C(I, X)$ is a bounded linear operator,

and the spectral radius $r(Q) \leq 1/\lambda_1$. On the other hand, since λ_1 has a positive eigenfunction x_1, in LPBVP (3.17) we set $h(t) = x_1$, then x_1/λ_1 is the corresponding mild solution. By the definition of the operator Q, $Q(x_1) = x_1/\lambda_1$, that is, $1/\lambda_1$ is an eigenvalue of Q. Then $r(Q) \geq 1/\lambda_1$. Thus, $r(Q) = 1/\lambda_1$. □

3. Main Results

Theorem 3.1. *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N. If $\{T(t)\}_{t\geq0}$ is a positive analytic semigroup, $f(t,\theta) \geq \theta$ for all $t \in I$, and the following conditions are satisfied.*

(H$_1$) *For any $R > 0$, there exists $C = C(R) > 0$ such that*

$$f(t,x_2) - f(t,x_1) \geq -C(x_2 - x_1), \tag{3.1}$$

for any $t \in I$, $\theta \leq x_1 \leq x_2$, $\|x_1\|, \|x_2\| \leq R$.

(H$_2$) *There exists $L < -v_0$ (v_0 is the growth index of $\{T(t)\}_{t\geq0}$), such that*

$$f(t,x_2) - f(t,x_1) \leq L(x_2 - x_1), \tag{3.2}$$

for any $t \in I$, $\theta \leq x_1 \leq x_2$.

Then PBVP (1.1) has a unique positive mild solution.

Proof. Let $h_0(t) = f(t,\theta)$, then $h_0 \in C(I,X)$, $h_0 \geq \theta$. Consider LPBVP

$$D^\alpha u(t) + (A - LI)u(t) = h_0(t), \quad t \in I,$$
$$u(0) = u(\omega). \tag{3.3}$$

$-(A-LI)$ generates a positive analytic semigroup $e^{Lt}T(t)$, whose growth index is $L+v_0 < 0$. By Lemma 2.14 and Remark 2.16, LPBVP (3.3) has a unique mild solution $w_0 \in C(I,X)$ and $w_0 \geq \theta$.

Set $R_0 = N\|w_0\|+1$, $C = C(R_0)$ is the corresponding constant in (H$_1$). We may suppose $C > \max\{v_0, -L\}$, otherwise substitute $C + |v_0| + |L|$ for C, (H$_1$) is also satisfied. Then we consider LPBVP

$$D^\alpha u(t) + (A + CI)u(t) = h(t), \quad t \in I,$$
$$u(0) = u(\omega). \tag{3.4}$$

$-(A + CI)$ generates a positive analytic semigroup $T_1(t) = e^{-Ct}T(t)$, whose growth index is $-C + v_0 < 0$. By Lemma 2.14 and Remark 2.16, for $h \in C(I,X)$ LPBVP (3.4) has a unique mild solution $u := Q_1 h$, $Q_1 : C(I,X) \rightarrow C(I,X)$ is a positive bounded linear operator and the spectral radius $r(Q_1) \leq 1/(C - v_0)$.

Set $F(u) = f(t, u) + Cu$, then $F : C(I, X) \to C(I, X)$ is continuous, $F(\theta) = h_0 \ge \theta$. By (H_1), F is an increasing operator on $[\theta, w_0]$. Set $v_0 = \theta$, we can define the sequences

$$v_n = Q_1 \cdot F(v_{n-1}), \qquad w_n = Q_1 \cdot F(w_{n-1}), \quad n = 1, 2, \ldots. \tag{3.5}$$

By (3.4), we have that

$$w_0 = Q_1(h_0 + Lw_0 + Cw_0). \tag{3.6}$$

In (H_2), we set $x_1 = \theta$, $x_2 = w_0(t)$, then

$$f(t, w_0) \le h_0(t) + Lw_0(t), \tag{3.7}$$

$$\theta \le F(\theta) \le F(w_0) \le h_0 + Lw_0 + Cw_0. \tag{3.8}$$

By (3.6) and (3.8), the definition and the positivity of Q_1, we have that

$$Q_1\theta = \theta = v_0 \le v_1 \le w_1 \le w_0. \tag{3.9}$$

Since $Q_1 \cdot F$ is an increasing operator on $[\theta, w_0]$, in view of (3.5), we have that

$$\theta \le v_1 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{3.10}$$

Therefore, we obtain that

$$\begin{aligned}
\theta \le w_n - v_n &= Q_1(F(w_{n-1}) - F(v_{n-1})) \\
&= Q_1\big(f(\cdot, w_{n-1}) - f(\cdot, v_{n-1}) + C(w_{n-1} - v_{n-1})\big) \\
&\le (C + L)Q_1(w_{n-1} - v_{n-1}).
\end{aligned} \tag{3.11}$$

By induction,

$$\theta \le w_n - v_n \le (C + L)^n Q_1^n(w_0 - v_0) = (C + L)^n Q_1^n(w_0). \tag{3.12}$$

In view of the normality of the cone P, we have that

$$\|w_n - v_n\|_C \le N(C + L)^n \|Q_1^n(w_0)\|_C \le N(C + L)^n \|Q_1^n\|_C \|w_0\|_C. \tag{3.13}$$

On the other hand, since $0 < C + L < C - v_0$, for some $\varepsilon > 0$, we have that $C + L + \varepsilon < C - v_0$. By the Gelfand formula, $\lim_{n \to \infty} \sqrt[n]{\|Q_1^n\|_C} = r(Q_1) \le 1/(C - v_0)$. Then there exist N_0, for $n \ge N_0$, we have that $\|Q_1^n\|_C \le 1/(C + L + \varepsilon)^n$. By (3.13), we have that

$$\|w_n - v_n\|_C \le N\|w_0\|_C \left(\frac{C + L}{C + L + \varepsilon}\right)^n \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{3.14}$$

By (3.10) and (3.14), similarly to the nested interval method, we can prove that there exists a unique $u^* \in \bigcap_{n=1}^{\infty}[v_n, w_n]$, such that

$$\lim_{n \to \infty} v_n = \lim_{n \to \infty} w_n = u^*. \tag{3.15}$$

By the continuity of the operator $Q_1 \cdot F$ and (3.5), we have that

$$u^* = Q_1 \cdot F(u^*). \tag{3.16}$$

By the definition of Q_1 and (3.10), we know that u^* is a positive mild solution of (3.4) when $h(t) = f(t, u^*(t)) + Cu^*(t)$. Then u^* is the positive mild solution of PBVP (1.1).

In the following, we prove that the uniqueness. If u_1, u_2 are the positive mild solutions of PBVP (1.1). Substitute u_1 and u_2 for w_0, respectively, then $w_n = Q_1 \cdot F(u_i) = u_i$ ($i = 1, 2$). By (3.14), we have that

$$\|u_i - v_n\|_C \longrightarrow 0, \quad (n \longrightarrow \infty, i = 1, 2). \tag{3.17}$$

Thus, $u_1 = u_2 = \lim_{n \to \infty} v_n$, PBVP (1.1) has a unique positive mild solution. □

Corollary 3.2. *Let X be an ordered Banach space, whose positive cone P is a regeneration cone. If $\{T(t)\}_{t \geq 0}$ is a compact and positive analytic semigroup, $f(t, \theta) \geq \theta$ for for all $t \in I$, f satisfies (H_1) and the following condition:*

$(H_2)'$ *There exist $L < \lambda_1$, where λ_1 is the first eigenvalue of A, such that*

$$f(t, x_2) - f(t, x_1) \leq L(x_2 - x_1), \tag{3.18}$$

for any $t \in I$, $\theta \leq x_1 \leq x_2$.

Then PBVP (1.1) has a unique positive mild solution.

Remark 3.3. In Corollary 3.2, since λ_1 is the first eigenvalue of A, the condition "$L < \lambda_1$" in $(H_2)'$ cannot be extended to "$L < \lambda_1$". Otherwise, PBVP (1.1) does not always have a mild solution. For example, $f(t, x) = \lambda_1 x$.

4. Examples

Example 4.1. Consider the following periodic boundary value problem for fractional parabolic partial differential equations in X:

$$\partial_t^\alpha u - \Delta u = f(t, u(t, x)), \quad (t, x) \in I \times \Omega,$$

$$u \mid \partial\Omega = 0, \tag{4.1}$$

$$u(0, x) = u(\omega, x), \quad x \in \Omega,$$

where ∂_t^α is the Caputo fractional partial derivative of order $0 < \alpha < 1$, $I = [0, \omega]$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, Δ is the Laplace operator, $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Let $X = L^2(\Omega)$, $P = \{v \mid v \in L^2(\Omega), v(x) \geq 0 \text{ a.e. } x \in \Omega\}$. Then X is an Banach space with the partial order "$\leq$" reduced by the normal cone P. Define the operator A as follows:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad Au = -\Delta u. \tag{4.2}$$

Then $-A$ generates an operator semigroup $\{T(t)\}_{t \geq 0}$ which is compact, analytic, and uniformly bounded. By the maximum principle, we can find that $\{T(t)\}_{t \geq 0}$ is a positive semigroup. Denote $u(t)(x) = u(t, x)$, $f(t, u(t))(x) = f(t, u(t, x))$, then the system (4.1) can be reformulated as the problem (1.1) in X.

Theorem 4.2. *Assume that $f(t, 0) \geq 0$ for $t \in I$, the partial derivative $f_u'(t, u)$ is continuous on any bounded domain and $\sup f_u'(t, u) < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ under the condition $u \mid \partial\Omega = 0$. Then the problem (4.1) has a unique positive mild solution.*

Proof. It is easy to see that (H_1) and $(H_2)'$ are satisfied. By Corollary 3.2, the problem (4.1) has a unique positive mild solution. □

Acknowledgments

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