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Research Article

Dynamics in a Delayed Neural Network Model of Two Neurons with Inertial Coupling

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A delayed neural network model of two neurons with inertial coupling is dealt with in this paper. The stability is investigated and Hopf bifurcation is demonstrated. Applying the normal form theory and the center manifold argument, we derive the explicit formulas for determining the properties of the bifurcating periodic solutions. An illustrative example is given to demonstrate the effectiveness of the obtained results.

1. Introduction

In recent years, a number of different classes of neural networks with or without delays, including Hopfield networks, cellular neural networks, Cohen-Grossberg neural networks, and bidirectional associate memory neural networks have been active research topic as [1], and substantial efforts have been made in neural network models, for example, Huang et al. [2] studied the global exponential stability and the existence of periodic solution of a class of cellular neural networks with delays, Guo and Huang [3] investigated the Hopf bifurcation natures of a ring of neurons with delays, Yan [4] analyzed the stability and bifurcation of a delayed tri-neuron network model, Hajihosseini et al. [5] made a discussion on the Hopf bifurcation of a delayed recurrent neural network in the frequency domain, and Liao et al. [6] did a theoretical and empirical investigation of a two-neuron system with distributed delays in the frequency domain. Agranovich et al. [7] considered the impulsive control of a hysteresis cellular neural network model. For more information, one can see [8–24]. In 1986 and 1987, Babcock and Westervelt [25, 26] had investigated the stability and dynamics of the following simple neural network model of two neurons with inertial coupling:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_3,\\
\frac{dx_2}{dt} &= x_4,\\
\frac{dx_3}{dt} &= -2\xi x_3 - x_1 + A_2\tanh(x_2),\\
\frac{dx_4}{dt} &= -2\xi x_4 - x_2 + A_1\tanh(x_1),
\end{aligned} \tag{1.1}$$

where x_i $(i = 1,2)$ is the input voltage of the ith neuron, x_j $(j = 3,4)$ denotes the output of the jth neuron, $\xi > 0$ is the damping factor, and A_i $(i = 1,2)$ is the overall gain of the neuron which determines the strength of the nonlinearity. For a more detailed interpretation of the parameters, one can see [25, 26]. In 1997, Lin and Li [27] made a detail discussion on the bifurcation direction of periodic solution for system (1.1).

From applications point of view, considering that there is a time delay (we assume that it is τ) in the response of the output voltages to changes in the input, that is, there exists a feedback delay of the input voltage of the ith neuron to the growth of the output of the jth neuron, then we modify system (1.1) as follows:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_3,\\
\frac{dx_2}{dt} &= x_4,\\
\frac{dx_3}{dt} &= -2\xi x_3 - x_1(t-\tau) + A_2\tanh(x_2(t-\tau)),\\
\frac{dx_4}{dt} &= -2\xi x_4 - x_2(t-\tau) + A_1\tanh(x_1(t-\tau)).
\end{aligned} \tag{1.2}$$

It is well known that the research on the Hopf bifurcation, especially on the stability of bifurcating periodic solutions and direction of Hopf bifurcation is very critical. When delays are incorporated into the network models, stability and Hopf bifurcation analysis become more difficult. To obtain a deep and clear understanding of dynamics of neural network model of two neurons with inertial coupling, we will make a discussion on system (1.2), that is, we study the stability, the local Hopf bifurcation for system (1.2).

The remainder of this paper is organized as follows. In Section 2, we investigate the stability of the equilibrium and the occurrence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results.

2. Stability of the Equilibrium and Local Hopf Bifurcations

In this section, we shall study the stability of the equilibrium and the existence of local Hopf bifurcations. For simplification, we only consider the zero equilibrium. One can check that if the following condition:

$$(H1) A_1 A_2 < 1 \tag{2.1}$$

holds, then (1.2) has a unique equilibrium $E(0,0,0,0)$. The linearization of (1.2) at $E(0,0,0,0)$ is given by

$$\begin{aligned}
\frac{dx_1}{dt} &= x_3, \\
\frac{dx_2}{dt} &= x_4, \\
\frac{dx_3}{dt} &= -2\xi x_3 - x_1(t-\tau) + A_2 x_2(t-\tau), \\
\frac{dx_4}{dt} &= -2\xi x_4 - x_2(t-\tau) + A_1 x_1(t-\tau),
\end{aligned} \tag{2.2}$$

whose characteristic equation takes the form of

$$\det\begin{pmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ e^{-\lambda\tau} & -A_2 e^{-\lambda\tau} & \lambda + 2\xi & 0 \\ -A_1 e^{-\lambda\tau} & e^{-\lambda\tau} & 0 & \lambda + 2\xi \end{pmatrix} = 0, \tag{2.3}$$

that is,

$$\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2 + \left(2\lambda^2 + 4\xi\lambda\right)e^{-\lambda\tau} + (1 - A_1 A_2)e^{-2\lambda\tau} = 0. \tag{2.4}$$

Multiplying $e^{\lambda\tau}$ on both sides of (2.4), it is easy to obtain

$$\left(\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2\right)e^{\lambda\tau} + 2\lambda^2 + 4\xi\lambda + (1 - A_1 A_2)e^{-\lambda\tau} = 0. \tag{2.5}$$

In order to investigate the distribution of roots of the transcendental equation (2.5), the following lemma is helpful.

Lemma 2.1 (see [28]). *For the following transcendental equation:*

$$P\left(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m}\right) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)}$$
$$+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots \tag{2.6}$$
$$+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0,$$

as $(\tau_1, \tau_2, \tau_3, \dots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ *in the open right half-plane can change, and only a zero appears on or crosses the imaginary axis.*

For $\tau = 0$, (2.5) becomes

$$\lambda^4 + 4\xi\lambda^3 + \left(4\xi^2 + 2\right)\lambda^2 + 4\xi\lambda + 1 - A_1A_2 = 0. \tag{2.7}$$

In view of Routh-Hurwitz criteria, we know that all roots of (2.7) have a negative real part if the following condition:

$$(H2) 4\xi^2 + A_1A_2 > 0 \tag{2.8}$$

is fulfilled.

For $\omega > 0$, $i\omega$ is a root of (2.5) if and only if

$$\left(\omega^4 - 4\xi\omega^3 i - 4\xi^2\omega^2\right)(\cos\omega\tau + i\sin\omega\tau) - 2\omega^2 + 4\xi\omega i + (1 - A_1A_2)(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.9}$$

Separating the real and imaginary parts gives

$$\begin{aligned} &\left(\omega^4 - 4\xi^2\omega^2 + 1 - A_1A_2\right)\cos\omega\tau + 4\xi\omega^3\sin\omega\tau = 2\omega^2, \\ &\left(\omega^4 - 4\xi^2\omega^2 - 1 + A_1A_2\right)\sin\omega\tau - 4\xi\omega^3\cos\omega\tau = -4\xi\omega. \end{aligned} \tag{2.10}$$

Then, we obtain

$$\sin\omega\tau = \frac{4\xi\omega^5 + 16\xi^3\omega^3 - (1 - A_1A_2)}{\left(\omega^4 + 4\xi^2\omega^2\right)^2 - (1 - A_1A_2)^2}, \tag{2.11}$$

$$\cos\omega\tau = \frac{2\omega^6 + 8\xi^2\omega^4 - (1 - A_1A_2)}{\left(\omega^4 + 4\xi^2\omega^2\right)^2 - (1 - A_1A_2)^2}. \tag{2.12}$$

In view of $\sin^2\omega\tau + \cos^2\omega\tau = 1$, then, we have

$$\begin{aligned}&\left[4\xi\omega^5 + 16\xi^3\omega^3 - (1 - A_1A_2)\right]^2 + \left[2\omega^6 + 8\xi^2\omega^4 - (1 - A_1A_2)\right]^2\\ &\quad = \left[\left(\omega^4 + 4\xi^2\omega^2\right)^2 - (1 - A_1A_2)^2\right]^2,\end{aligned} \tag{2.13}$$

which is equivalent to

$$\omega^{16} + l_1\omega^{14} + l_2\omega^{12} + l_3\omega^{10} + l_4\omega^8 + l_5\omega^6 + l_6\omega^5 + l_7\omega^4 + l_8\omega^3 + l_9 = 0, \tag{2.14}$$

where

$$\begin{aligned}&l_1 = 16\xi^2,\ l_2 = 96\xi^4 - 4,\ l_3 = 256\xi^6 - 48\xi^2,\\ &l_4 = 256\xi^8 - 64\xi^4 - 128\xi^3 - 2(1 - A_1A_2)^2,\ l_5 = 2(1 - A_1A_2) - 16(1 - A_1A_2)^2\xi^2 - 256\xi^6,\\ &l_6 = 2\xi(1 - A_1A_2),\ l_7 = 16\xi^2(1 - A_1A_2) - 32\xi^4(1 - A_1A_2)^2,\\ &l_8 = 32\xi^3(1 - A_1A_2),\ l_9 = -[(1 - A_1A_2)^4 + (1 - A_1A_2)^2].\end{aligned}$$

Denote

$$h(\omega) = \omega^{16} + l_1\omega^{14} + l_2\omega^{12} + l_3\omega^{10} + l_4\omega^8 + l_5\omega^6 + l_6\omega^5 + l_7\omega^4 + l_8\omega^3 + l_9. \tag{2.15}$$

Since $l_9 < 0$ and $\lim_{\omega\to+\infty}h(\omega) = +\infty$, then we can conclude that (2.14) has at least one positive root. Without loss of generality, we assume that (2.14) has sixteen positive roots, denoted by ω_k $(k = 1, 2, 3, \ldots, 16)$. Then, by (2.12), we have

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left[\frac{2\omega^6 + 8\xi^2\omega^4 - (1 - A_1A_2)}{\left(\omega^4 + 4\xi^2\omega^2\right)^2 - (1 - A_1A_2)^2}\right] + 2j\pi\right\}, \tag{2.16}$$

where $k = 1, 2, 3, \ldots, 16;\ j = 0, 1, \ldots$, then $\pm i\omega_k$ are a pair of purely imaginary roots of (2.4) with $\tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3,\ldots,16\}}\left\{\tau_k^{(0)}\right\}. \tag{2.17}$$

The above analysis leads to the following result.

Lemma 2.2. *If* $(H1)$ *and* $(H2)$ *hold, then all roots of* (2.4) *have a negative real part when* $\tau \in [0, \tau_0)$ *and* (2.4) *admits a pair of purely imaginary roots* $\pm\omega_k$ *when* $\tau = \tau_k^{(j)}(k = 1, 2, 3, \ldots, 16;\ j = 0, 1, 2, \ldots)$.

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (2.5) near $\tau = \tau_k^{(j)}$, $\alpha(\tau_k^{(j)}) = 0$, and $\omega(\tau_k^{(j)}) = \omega_k$. Due to functional differential equation theory, for every $\tau_k^{(j)}$, $k = 1, 2, 3, \ldots, 16;\ j = 0, 1, 2, \ldots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k^{(j)}| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand side of (2.5) and taking derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(4\lambda^3 + 12\xi\lambda^2 + 8\xi^2\lambda)e^{\lambda\tau} + 4\lambda + 4\xi}{\lambda(1 - A_1A_2)e^{-\lambda\tau} - \lambda(\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2)e^{\lambda\tau}} - \frac{\tau}{\lambda}. \tag{2.18}$$

Noting that

$$\begin{gathered}\left[4\lambda^3 + 12\xi\lambda^2 + 8\xi^2\lambda)e^{\lambda\tau} + 4\lambda + 4\xi\right]_{\tau=\tau_k^{(j)}} = K_1 + K_2 i,\\ \left[\lambda(1 - A_1A_2)e^{-\lambda\tau} - \lambda\left(\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2\right)e^{\lambda\tau}\right]_{\tau=\tau_k^{(j)}} = P_1 + P_2 i,\end{gathered} \tag{2.19}$$

where

$$\begin{aligned}K_1 &= 4\xi - 12\xi\omega_k^2\cos\omega_k\tau_k^{(j)} - (18\xi^2\omega_k - 4\omega_k^3)\sin\omega_k\tau_k^{(j)},\\ K_2 &= 4\omega_k + (8\xi^2\omega_k - 4\omega_k^3)\cos\omega_k\tau_k^{(j)} - 12\xi\omega_k^2\sin\omega_k\tau_k^{(j)}.\\ P_1 &= 4\xi\omega_k^4\cos\omega_k\tau_k^{(j)} + [(1 - A_1A_2)\omega_k - \omega_k^5 - 4\xi^2\omega_k^3]\sin\omega_k\tau_k^{(j)}\\ P_2 &= [(1 - A_1A_2)\omega_k - \omega_k^5 - 4\xi^2\omega_k^3]\cos\omega_k\tau_k^{(j)} - 4\xi\omega_k^4\sin\omega_k\tau_k^{(j)},\end{aligned}$$

we derive

$$\begin{aligned}\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right]^{-1}_{\tau=\tau_k^{(j)}} &= \operatorname{Re}\left\{\frac{(4\lambda^3 + 12\xi\lambda^2 + 8\xi^2\lambda)e^{\lambda\tau} + 4\lambda + 4\xi}{\lambda(1 - A_1A_2)e^{-\lambda\tau} - \lambda(\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2)e^{\lambda\tau}}\right\}_{\tau=\tau_k^{(j)}}\\ &= \operatorname{Re}\left\{\frac{K_1 + K_2 i}{P_1 + P_2 i}\right\} = \frac{K_1P_1 - K_2P_2}{P_1^2 + P_2^2}.\end{aligned} \tag{2.20}$$

We assume that the following condition holds:

$$(H3)K_1P_1 \neq K_2P_2. \tag{2.21}$$

According to above analysis and the results of Kuang [29] and Hale [30], we have the following.

Theorem 2.3. *If $(H1)$ and $(H2)$ hold, then the equilibrium $E(0,0,0,0)$ of system* (1.2) *is asymptotically stable for $\tau \in [0, \tau_0)$. Under the conditions $(H1)$ and $(H2)$, if the condition $(H3)$ holds, then system* (1.2) *undergoes a Hopf bifurcation at the equilibrium $E(0,0,0,0)$ when $\tau = \tau_k^{(j)}$, $k = 1,2,3,\ldots,16$; $j = 0,1,2,\ldots$.*

3. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau = \tau_k^{(j)}$, $k = 1,2,3,\ldots,16$; $j = 0,1,2,\ldots$. In this section, we shall obtain the explicit formulae for determining the direction, stability, and periods of these periodic solutions bifurcating from the equilibrium $E(0,0,0,0)$ at these critical value of τ, by using techniques from normal form and center manifold theory [31]. Throughout this section, we always assume that system (1.2) undergoes Hopf bifurcation at the equilibrium $E(0,0,0,0)$ for $\tau = \tau_k^{(j)}$, $k = 1,2,3,\ldots,16$; $j = 0,1,2,\ldots$, and then $\pm i\omega_k$ are corresponding purely imaginary roots of the characteristic equation at the equilibrium $E(0,0,0,0)$.

For convenience, let $\overline{x}_i(t) = x_i(\tau t)$ $(i = 1,2,3,4)$ and $\tau = \tau_k^{(j)} + \mu$, where $\tau_k^{(j)}$ is defined by (2.16) and $\mu \in R$, drop the bar for the simplification of notations, then system (2.2) can be written as an FDE in $C = C([-1,0]), R^4)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{3.1}$$

where $u(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T \in C$ and $u_t(\theta) = u(t+\theta) = (x_1(t+\theta), x_2(t+\theta), x_3(t+\theta), x_4(t+\theta))^T \in C$, and $L_\mu : C \to R, F : R \times C \to R$ are given by

$$\begin{gathered}
L_\mu\phi = \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2\xi & 0 \\ 0 & 0 & 0 & -2\xi \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \\ \phi_4(0) \end{pmatrix} \\
+ \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & A_2 & 0 & 0 \\ A_1 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \\ \phi_4(-1) \end{pmatrix}, \\
f(\mu, \phi) = \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} 0 \\ 0 \\ A_2\phi_2^3(-1) + \text{ h.o.t.} \\ A_1\phi_1^3(-1) + \text{ h.o.t.} \end{pmatrix},
\end{gathered} \tag{3.2}$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta))^T \in C$.

From the discussion in Section 2, we know that if $\mu = 0$, then system (3.1) undergoes a Hopf bifurcation at the equilibrium $E(0, 0, 0, 0)$ and the associated characteristic equation of system (3.1) has a pair of simple imaginary roots $\pm\omega_k\tau_k^{(j)}$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu)$, $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta) \quad \text{for } \phi \in C. \tag{3.3}$$

In fact, we can choose

$$\eta(\theta,\mu) = \left(\tau_k^{(j)}+\mu\right)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2\xi & 0 \\ 0 & 0 & 0 & -2\xi \end{pmatrix}\delta(\theta)$$
$$-\left(\tau_k^{(j)}+\mu\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & A_2 & 0 & 0 \\ A_1 & -1 & 0 & 0 \end{pmatrix}\delta(\theta+1), \tag{3.4}$$

where δ is the Dirac delta function.

For $\phi \in C([-1,0],R^4)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0, \end{cases}$$
$$R\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ f(\mu,\phi), & \theta = 0. \end{cases} \tag{3.5}$$

Then, (3.1) is equivalent to the following abstract differential equation:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{3.6}$$

where $u_t(\theta) = u(t+\theta)$, $\theta \in [-1,0]$. For $\psi \in C([0,1],(R^4)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{3.7}$$

For $\phi \in C([-1,0],R^4)$ and $\psi \in C([0,1],(R^4)^*)$, define the following bilinear form:

$$\langle \psi,\phi\rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.8}$$

where $\eta(\theta) = \eta(\theta, 0)$, and the $A = A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_k \tau_k^{(j)}$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_k \tau_k^{(j)}$ and $-i\omega_k \tau_k^{(j)}$, respectively. By direct computation, we can obtain

$$q(\theta) = (1, \alpha, \beta, \gamma)^T e^{i\omega_k \tau_k^{(j)} \theta}, \qquad q^*(s) = D(1, \alpha^*, \beta^*, \gamma^*) e^{i\omega_k \tau_k^{(j)} s}, \tag{3.9}$$

where

$$\alpha = \frac{i\omega_k(i\omega_k + 2\xi) + e^{-i\omega_k \tau_k^{(j)}}}{A_2 e^{-i\omega_k \tau_k^{(j)}}}, \qquad \beta = -i\omega_k, \qquad \gamma = \frac{i\omega_k e^{-i\omega_k \tau_k^{(j)}} - \omega_k^2(i\omega_k + 2\xi)}{A_2 e^{-i\omega_k \tau_k^{(j)}}},$$
$$\alpha^* = \frac{i\omega_k(2\xi - i\omega_k) - e^{-i\omega_k \tau_k^{(j)}}}{A_1 e^{-i\omega_k \tau_k^{(j)}}}, \qquad \beta^* = \frac{1}{2\xi - i\omega_k}, \qquad \gamma^* = \frac{\omega_k^2(i\omega_k + 2\xi) - i\omega_k e^{-i\omega_k \tau_k^{(j)}}}{A_2 e^{-i\omega_k \tau_k^{(j)}}},$$
$$D = \frac{1}{1 + \overline{\alpha}\alpha^* + \overline{\beta}\beta^* + \overline{\gamma}\gamma^* + \tau_k^{(j)} \left[\overline{\gamma}(A_1 - \alpha^*) - \overline{\beta}(A_2 \alpha^* + 1)\right] e^{i\omega_k \tau_k^{(j)}}}. \tag{3.10}$$

Furthermore, $\langle q^*(s), q(\theta) \rangle 1$ and $\langle q^*(s), \overline{q}(\theta) \rangle 0$.

Next, we use the same notations as those in Hassard et al. [31] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (3.1), when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t \rangle, \qquad W(t, \theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t) q(\theta)\}, \tag{3.11}$$

on the center manifold C_0, and we have

$$W(t, \theta) = W(z(t), \overline{z}(t), \theta), \tag{3.12}$$

where

$$W(z(t), \overline{z}(t), \theta) = W(z, \overline{z}) = W_{20} \frac{z^2}{2} + W_{11} z \overline{z} + W_{02} \frac{\overline{z}^2}{2} + \cdots, \tag{3.13}$$

and z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (3.1), we have

$$\dot{z}(t) = i\omega_k \tau_k^{(j)} z + \overline{q}^*(\theta) f\left(0, W(z, \overline{z}, \theta) + 2\,\mathrm{Re}\{z q(\theta)\}\right) \stackrel{\text{def}}{=} i\omega_k \tau_k^{(j)} z + \overline{q}^*(0) f_0. \tag{3.14}$$

That is,

$$\dot{z}(t) = i\omega_k \tau_k^{(j)} z + g(z, \overline{z}), \tag{3.15}$$

where

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.16}$$

Hence, we have

$$\begin{aligned}
g(z, \overline{z}) &= \overline{q}^*(0) f_0(z, \overline{z}) = \overline{q}^*(0) f(0, u_t) \\
&= \tau_k^{(j)} \overline{D}\left(1, \overline{\alpha}^*, \overline{\beta}^*, \overline{\gamma}^*\right) \begin{pmatrix} 0 \\ 0 \\ A_2 x_{2t}^3(-1) + \text{h.o.t.} \\ A_1 x_{1t}^3(-1) + \text{h.o.t.} \end{pmatrix} \\
&= \overline{D}\tau_k^{(j)} \left[3\beta^* e^{-i\omega_k \tau_k^{(j)}} + 3\gamma^* \alpha^2 \overline{\alpha} e^{-2i\omega_k \tau_k^{(j)}}\right] z^2\overline{z} + \text{h.o.t},
\end{aligned} \tag{3.17}$$

and we obtain

$$g_{20} = g_{11} = g_{02} = 0, \tag{3.18}$$

$$g_{21} = 2\overline{D}\tau_k^{(j)} \left[3\beta^* e^{-i\omega_k \tau_k^{(j)}} + 3\gamma^* \alpha^2 \overline{\alpha} e^{-2i\omega_k \tau_k^{(j)}}\right]. \tag{3.19}$$

Thus, we derive the following values:

$$\begin{aligned}
c_1(0) &= \frac{i}{2\omega_k \tau_k^{(j)}} \left(g_{20} g_{11} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3} \right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\left\{\lambda'\left(\tau_k^{(j)}\right)\right\}}, \\
\beta_2 &= 2\operatorname{Re}(c_1(0)), \\
T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\left\{\lambda'\left(\tau_k^{(j)}\right)\right\}}{\omega_k \tau_k^{(j)}},
\end{aligned} \tag{3.20}$$

which determine the quantities of bifurcation periodic solutions of (3.1) on the center manifold at the critical value $\tau = \tau_k^{(j)}$, $(k = 1, 2, 3, \ldots, 16;\ j = 0, 1, 2, 3, \ldots)$. Summarizing the results obtained above leads to the following theorem.

Theorem 3.1. *The periodic solution is forward (backward) if $\mu_2 > 0$ ($\mu_2 < 0$). The bifurcating periodic solutions on the center manifold are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$). The periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

4. Numerical Examples

In this section, we present some numerical results of system (1.2) to verify the analytical predictions obtained in the previous section. Let us consider the following system:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_3, \\
\frac{dx_2}{dt} &= x_4, \\
\frac{dx_3}{dt} &= -0.8x_3 - x_1(t-\tau) + 0.2\tanh(x_2(t-\tau)), \\
\frac{dx_4}{dt} &= -0.8x_4 - x_2(t-\tau) + 0.4\tanh(x_1(t-\tau)),
\end{aligned} \tag{4.1}$$

which has an equilibrium $E_0(0,0,0,0)$ and satisfies the conditions indicated in Theorem 2.3. The equilibrium $E_0(0,0,0,0)$ is asymptotically stable for $\tau = 0$. Using the software MATLAB (here take $j = 0$ for example), we derive $\omega_0 \approx 4.1208$, $\tau_0 \approx 0.6801$, $\lambda'(\tau_0) \approx 0.3307 - 3.1524i$, $g_{21} \approx -1.4203 - 4.5518i$. Thus by algorithm (3.20) derived in Section 3, we have $c_1(0) \approx -0.7102 - 2.1609i$, $\mu_2 \approx -2.1476$, $\beta_2 \approx -1.4202$, $T_2 \approx 3.1867$. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the equilibrium $E_0(0,0,0,0)$ is stable when $\tau < \tau_0 \approx 0.6801$. Figures 1(a)–1(j) show that the equilibrium $E_0(0,0,0,0)$ is asymptotically stable when $\tau = 0.65 < \tau_0 \approx 0.6801$. It is observed from Figures 1(a)–1(j) that the input voltage of the i ($i = 1,2$)th neuron and the output of the j ($j = 3,4$)th neuron converge to their steady states in finite time. If we gradually increase the value of τ and keep other parameters fixed, when τ passes through the critical value $\tau_0 \approx 0.6801$, the equilibrium $E_0(0,0,0,0)$ loses its stability and a Hopf bifurcation occurs, that is, the input voltage of the i ($i = 1,2$)th neuron the output of the j ($j = 3,4$)th neuron will keep an oscillary mode near the equilibrium $E_0(0,0,0,0)$. Due to $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0 \approx 0.6801$, and these bifurcating periodic solutions from $E_0(0,0,0,0)$ at $\tau_0 \approx 0.6801$ are stable. Figures 1(j)–2(d) suggest that Hopf bifurcation occurs from the equilibrium $E_0(0,0,0,0)$ when $\tau = 0.8 > \tau_0 \approx 0.6801$.

5. Conclusions

In this paper, we have studied the bifurcation natures of a delayed neural network model of two neurons with inertial coupling. Regarding delay as the bifurcation parameter and analyzing the characteristic equation of the linearized system of the original system at the equilibrium $E_0(0,0,0,0)$, we proposed the conditions to define the parameters for

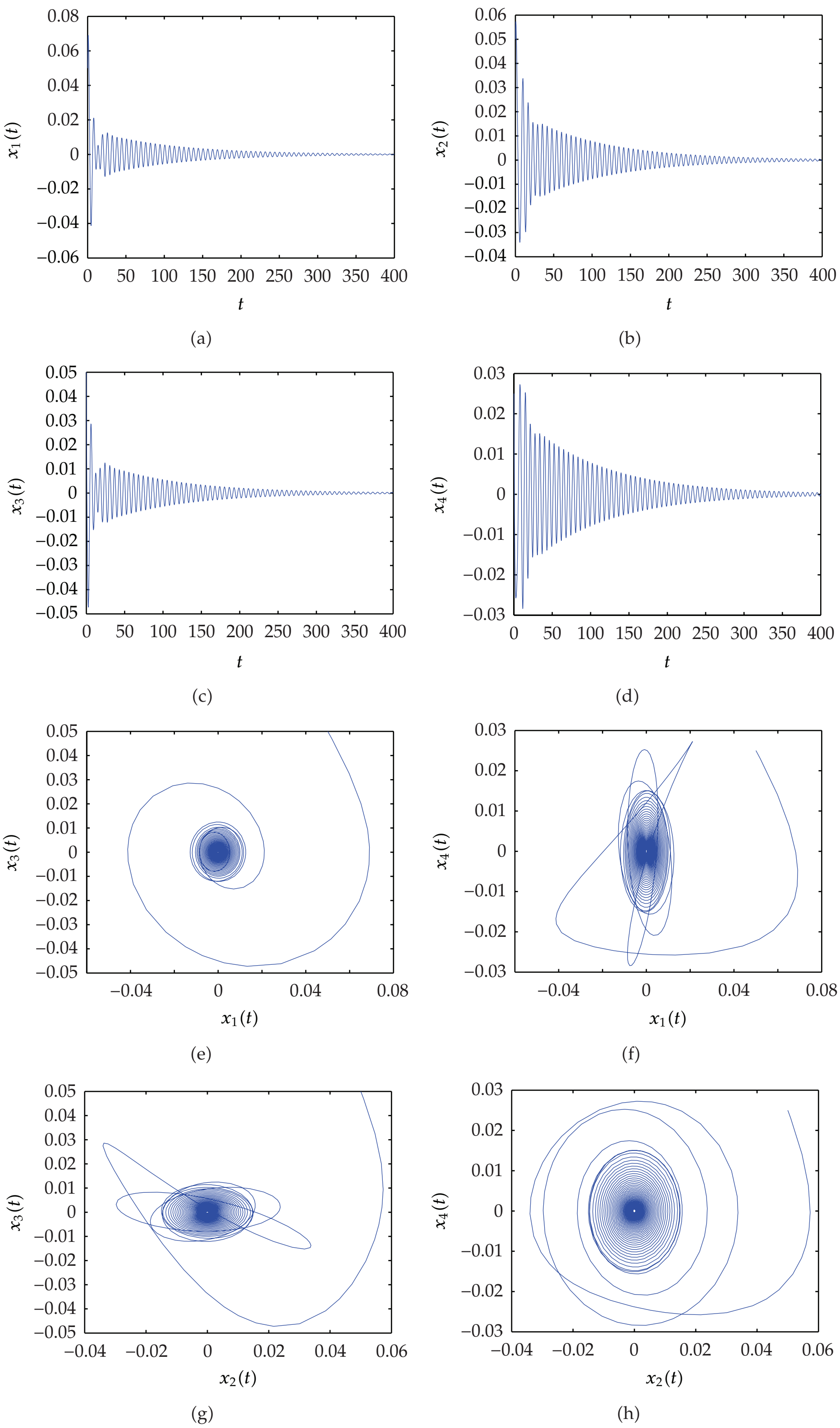

Figure 1: Continued.

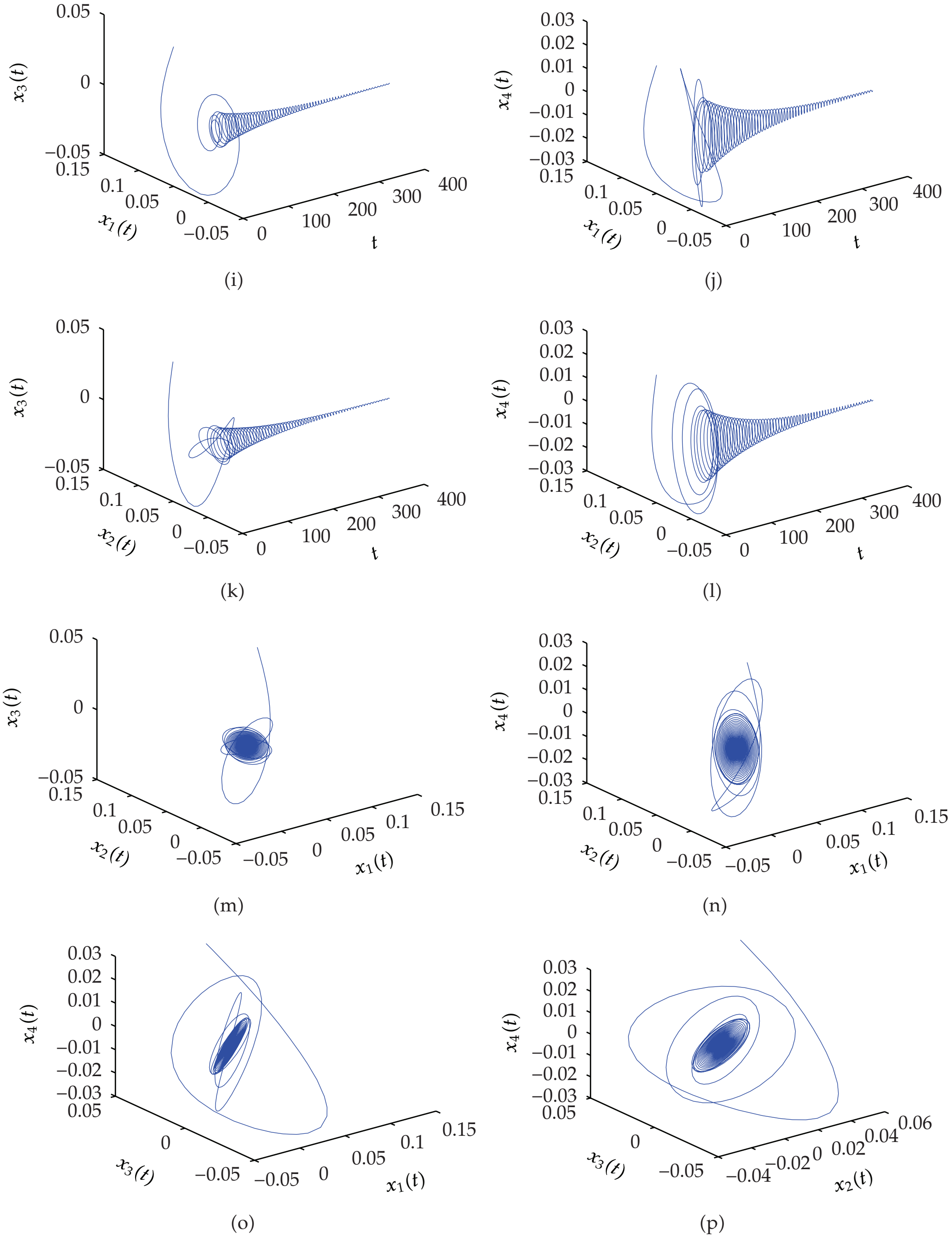

Figure 1: The time histories and phase trajectories of system (4.1) with $\tau = 0.65 < \tau_0 \approx 0.6801$ and the initial value (0.05, 0.05, 0.05, 0.025). The equilibrium $E_0(0, 0, 0, 0)$ is asymptotically stable.

the occurrence of Hopf bifurcation and the oscillatory solutions of the models equations. It is shown that if conditions $(H1)$ and $(H2)$ hold, the equilibrium $E_0(0,0,0,0)$ of system (1.2) is asymptotically stable for all $\tau \in [0,\tau_0)$. Under conditions $(H1)$ and $(H2)$, if condition $(H3)$ is satisfied, as the delay τ increases and crosses a threshold value $\tau_k^{(j)}$, the equilibrium loses its stability and the delayed network model of two

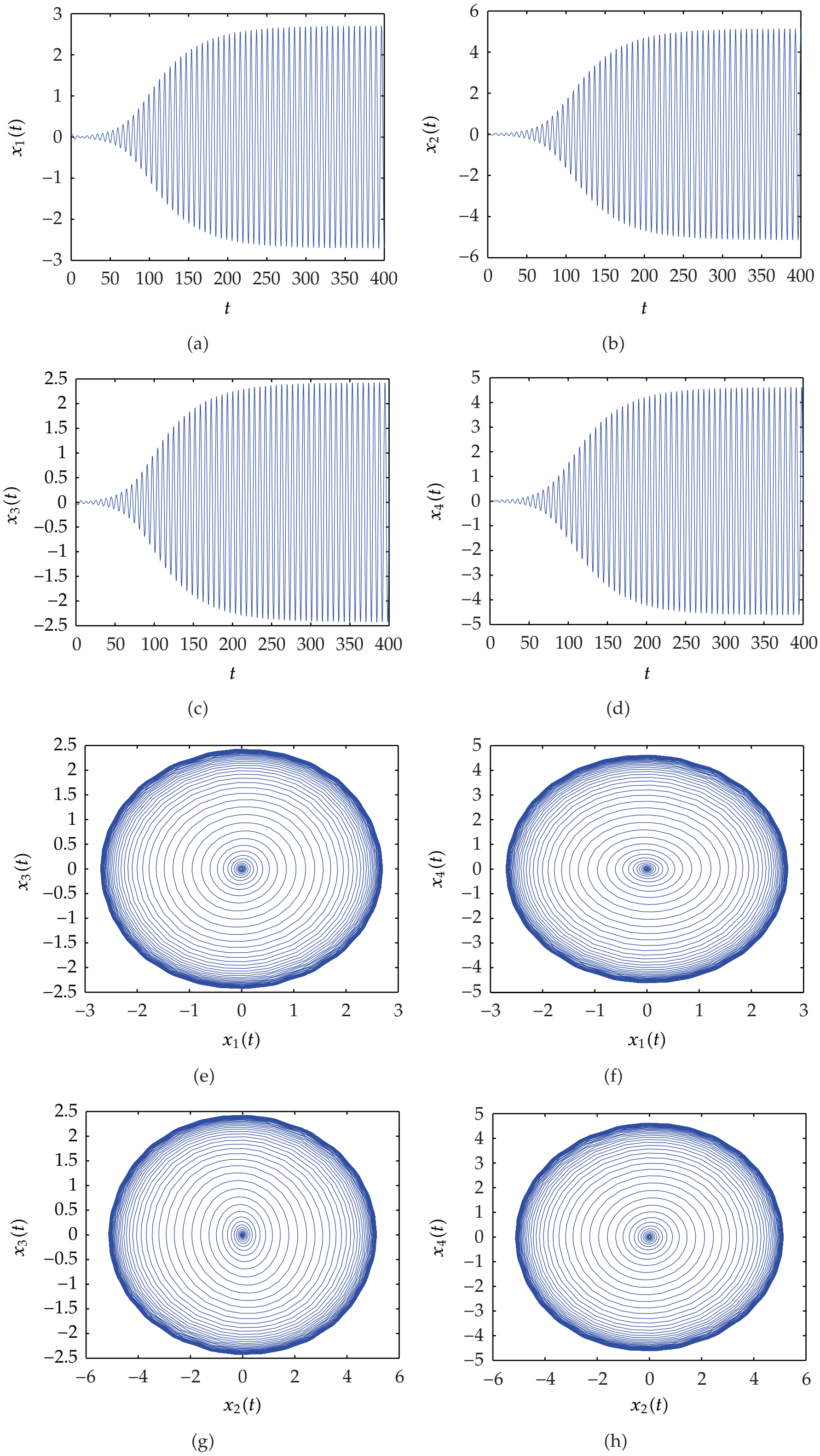

Figure 2: Continued.

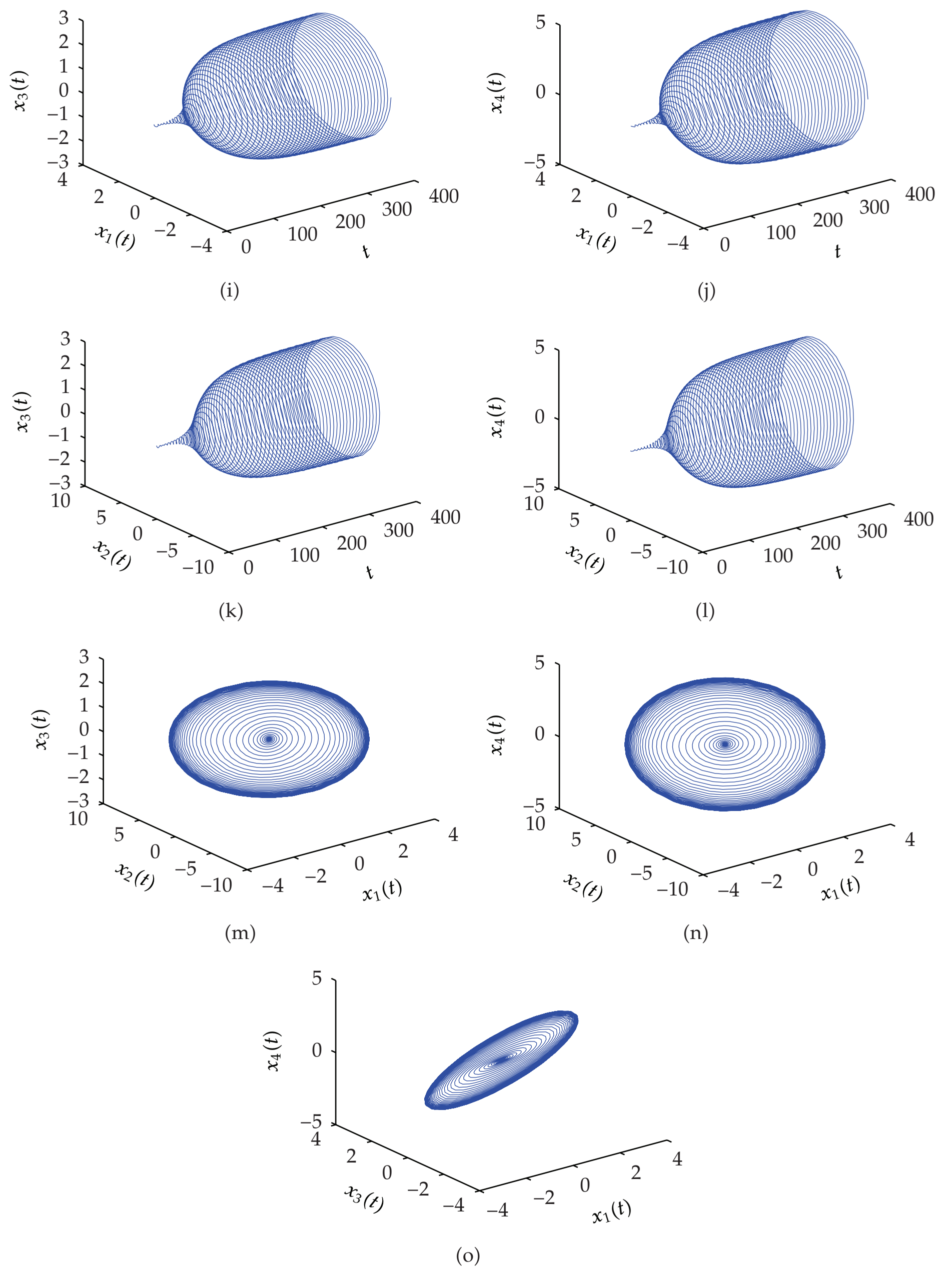

Figure 2: The time histories and phase trajectories of system (4.1) with $\tau = 0.8 > \tau_0 \approx 0.6801$ and the initial value $(0.05, 0.05, 0.05, 0.025)$. Hopf bifurcation occurs from the equilibrium $E_0(0, 0, 0, 0)$.

neurons with inertial coupling enters into a Hopf bifurcation. In addition, using the normal form method and center manifold theorem, explicit formulaes for determining the properties of periodic solutions are worked out. Simulations are included to verify the theoretical findings. The obtained findings are useful in applications of network control.

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