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Research Article

Mean Square Almost Periodic Solutions for Impulsive Stochastic Differential Equations with Delays

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We establish a result on existence and uniqueness on mean square almost periodic solutions for a class of impulsive stochastic differential equations with delays, which extends some earlier works reported in the literature.

1. Introduction

Impulsive effects widely exist in many evolution processes of real-life phenomena in which states are changed abruptly at certain moments of time, involving such areas as population dynamics and automatic control [1–3]. Because delay is ubiquitous in the dynamical system, impulsive differential equations with delays have received much interesting in recent years, intensively researched, some important results are obtained [4–9]. And almost periodic solutions for abstract impulsive differential equations and for impulsive neural networks with delay have been discussed by G. T. Stamov and I. M. Stamova [10], and Stamov and Alzabut [11].

However, besides delay and impulsive effects, stochastic effects likewise exist in real system. A lot of dynamic systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of components, changes in the interconnections of subsystems, sudden environment changes, and so on [12–14]. Moreover, differential descriptor systems also have abrupt changes [15, 16]. Recently, a large number of stability criteria of stochastic system with delays have been reported [17–19]. Almost periodic solutions to some functional integro-differential stochastic evolution equations and to some stochastic differential equations have been studied by Bezandry and Diagana [20], and Bezandry [21]. Huang and Yang investigated almost periodic solution for stochastic cellular neural networks with delays [22]. Because it is not easy to deal with the case of coexistence of impulsive, delay and stochastic effects in a dynamical system, there are few results about this problems [23–25]. To the best of our knowledge, there exists no result on the existence and uniqueness of mean square almost periodic solutions for impulsive stochastic differential equations with delays.

Motivated by the above discussions, the main aim of this paper is to study the mean square almost periodic solutions for impulsive stochastic differential equations with delays. By employing stochastic analysis, delay differential inequality technique and fixed points theorem, we obtain some criteria to ensure the existence and uniqueness of mean square almost periodic solutions.

The rest of this paper is organized as follows: in Section 2, we introduce a class of impulsive stochastic differential equations with delays, and the relating notations, definitions and lemmas which would be used later; in Section 3, a new sufficient condition is proposed to ensure the existence and uniqueness of mean square almost periodic solutions; in Section 4, an example is constructed to show the effectiveness of our results. Finally, a conclusion is given in Section 5.

2. Preliminaries

Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N} = \{1,2,3,\ldots\}$, and $\mathcal{B} = \{\{t_k\} : t_0 = 0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots, \lim_{k\to+\infty} t_k = +\infty\}$ be the set of all sequence unbounded and strictly increasing. For $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, let $\|x\|$ be any vector norm, and denote the induced matrix norm and the matrix measure, respectively, by

$$\|A\| = \sup_{x\neq 0}\frac{\|Ax\|}{\|x\|}, \qquad \mu(A) = \lim_{h\to 0^+}\frac{\|I+hA\|-1}{h}. \tag{2.1}$$

The norm and measure of vector and matrix are $\|x\| = \max_i |x_i|$, $\|A\| = \max_i \sum_{j=1}^{n} |a_{ij}|$, $\mu(A) = \max_i\{a_{ii} + \sum_{j\neq i}^{n} |a_{ij}|\}$.

Consider the following a class of Itô impulsive stochastic differential equations with delay

$$\begin{gathered} dx(t) = \left[Ax(t) + Bf(t,x(t)) + Cg(t,x(t-h)) + I(t)\right]dt + \sigma(t,x(t))d\omega(t), \quad t \geq 0,\ t \neq t_k, \\ \Delta x(t) = x(t_k) - x(t_k^-) = D_k x(t_k^-) + V_k\left(x(t_k^-)\right) + \beta_k, \quad t = t_k,\ k \in \mathbb{N}, \\ x(t) = \phi(t), \quad -h \leq t \leq 0, \end{gathered} \tag{2.2}$$

where $x(t) = (x_1(t),\ldots,x_n(t))^T$ is the solution process, $A, B, C, D_k \in \mathbb{R}^{n\times n}$ are constant matrices, $f(t,x) = (f_1(t,x),\ldots,f_n(t,x))^T$, $g(t,x) = (g_1(t,x),\ldots,g_n(t,x))^T$, $I(t) = (I_1(t),\ldots,I_n(t))^T$, $\sigma(t,x) = (\sigma_{ij}(t,x))_{n\times n}$ is the diffusion coefficient matrix, $V_k(x) = (V_{1k}(x),\ldots,V_{nk}(x))^T$ is impulsive function, $h > 0$ is delay; $t_k \in \mathcal{B}$ is impulsive time, $\beta_k = (\beta_{1k},\ldots,\beta_{nk})^T$ is a constant vector, $\omega(t) = (\omega_1(t),\ldots,\omega_n(t))^T$ is an n-dimensional Brown motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $\omega(t)$, and denote by $\mathcal{F}$ the associated σ-algebra generated by $\omega(t)$ with the probability measure $\mathbb{P}$. Moreover, the initial conditions $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))^T \in PCB^b_{\mathcal{F}_0}([-h, 0], \mathbb{R}^n) \triangleq PCB^b_{\mathcal{F}_0}$. Denote by $PCB^b_{\mathcal{F}_0}$ the family of all bounded $\mathcal{F}_0$-measurable, $PC([-h, 0], \mathbb{R}^n)$-valued random variable ζ, satisfying $E\|\zeta\|^2 = E(\sup_{-h\leq\theta\leq 0}\|\zeta(\theta)\|^2) < +\infty$, where $PC([-h, 0], \mathbb{R}^n) = \{\zeta : [-h, 0] \to \mathbb{R}^n$ is continuous$\}$. E denotes the expectation of stochastic process.

Let $(\mathbb{H}, \|\cdot\|)$ be a Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Define $L^2(\mathbb{P}, \mathbb{H})$ to be the space of all $\mathbb{H}$-value random variable Y such that

$$E\|Y\|^2 = \int_\Omega \|Y\|^2 d\mathbb{P} < \infty. \tag{2.3}$$

It is then routine to check that $L^2(\mathbb{P}, \mathbb{H})$ is a Hilbert space when it is equipped with its natural norm $\|\cdot\|_2$ defined by

$$\|Y\|_2 = \left(\int_\Omega \|Y\|^2 d\mathbb{P}\right)^{1/2} < \infty, \tag{2.4}$$

for each $Y \in L^2(\mathbb{P}, \mathbb{H})$.

Definition 2.1 (see [25]). For any $\phi \in PCB^b_{\mathcal{F}_0}$, a function $x(t) : [-h, +\infty) \to L^2(\mathbb{P}, \mathbb{H})$ is said to be solution of system (2.2) on $[-h, +\infty)$ satisfying initial value condition, if the following conditions hold:

(i) $x(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}) \in [0, +\infty), k \in \mathbb{N}$;

(ii) for any $t_k \in [0, +\infty), k \in \mathbb{N}$, $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^+) = x(t_k)$;

(iii) $x(t)$ satisfies (2.2) for almost everywhere in $[-h, +\infty)$ and at impulsive points $t = t_k$ situated in $[0, +\infty), k \in \mathbb{N}$, may have discontinuity points of the first kind.

Obviously, the solution defined by definition 1 is piecewise continuous.

Definition 2.2 (see [26]). The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k, k \in \mathbb{N}, j \in \mathbb{N}, \{t_k\} \in \mathcal{B}$ is said to be uniformly almost periodic if for any $\varepsilon > 0$, there exists relatively dense set of ε-almost periods common for any sequences.

Definition 2.3. A piecewise continuous function $x(t) : [-h, +\infty) \to L^2(\mathbb{P}, \mathbb{H})$ with discontinuity points of first kind at $t = t_k$ is said to be mean square almost periodic, if

(i) the set of sequence $\{t_k^j\}$ is uniformly almost periodic;

(ii) for any $\varepsilon > 0$, there exists $\delta > 0$, such that if the points t' and t'' belong to one and the same interval of continuity of $x(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $E\|x(t') - x(t'')\|^2 < \varepsilon$;

(iii) for any $\varepsilon > 0$, there exists a relatively dense set T such that if $\tau \in T$, then $E\|x(t + \tau) - x(t)\|^2 < \varepsilon$ for all $t \in [-h, +\infty)$ satisfying the condition $|t - t_k| > \varepsilon, k \in \mathbb{N}$.

The collection of all functions $x(t) : [-h, +\infty) \to L^2(\mathbb{P}, \mathbb{H})$ with discontinuity points of the first kind at $t = t_k$ which are mean square almost periodic is denoted by

$AP\ ([-h,+\infty);L^2(\mathbb{P},\mathbb{H}))$, one can check that $AP([-h,+\infty);L^2(\mathbb{P},\mathbb{H}))$ is a Banach space when it is equipped with the norm:

$$\|x\|_\infty = \sup_{t\in\mathbb{R}}\left(E\|x(t)\|^2\right)^{1/2}. \tag{2.5}$$

Let $(B_1,\|\cdot\|_1)$ and $(B_2,\|\cdot\|_2)$ be Banach space and $L^2(\mathbb{P},B_1)$ and $L^2(\mathbb{P},B_2)$ be their corresponding L^2-space, respectively.

Lemma 2.4 (see [20]). *Let $f:\mathbb{R}\times L^2(\mathbb{P},B_1)\to L^2(\mathbb{P},B_2),(t,x)\mapsto f(t,x)$ be mean square almost periodic in $t\in\mathbb{R}$ uniformly in $x\in K$, where $K\subset L^2(\mathbb{P},B_1)$ is compact. Suppose that there exists $L_f>0$ such that*

$$E\left\|f(t,x)-f(t,y)\right\|_2^2 \le L_f E\left\|x-y\right\|_1^2 \tag{2.6}$$

for all $x,y\in L^2(\mathbb{P},B_1)$ and for each $t\in\mathbb{R}$. Then for any mean square almost periodic function $\varphi(t):\mathbb{R}\to L^2(\mathbb{P},B_1)$, $f(t,\varphi(t))$ is mean square almost periodic.

In this paper, we always assume that:

(A1) $\det(I+D_k)\neq 0$ and the sequence $\{D_k\},k\in\mathbb{N}$, is almost periodic, where $I\in R^{n\times n}$ is the identity matrix;

(A2) the set of $\{t_k^j\}$ is uniformly almost periodic and $\theta=\inf_k\{t_k^1\}>0$.

Recall [2], consider the following linear system of system(2.2)

$$\begin{aligned}\dot{x}(t) &= Ax(t), \quad t\neq t_k,\\ \Delta x(t_k) &= D_k x(t_k^-), \quad k\in\mathbb{N},\end{aligned} \tag{2.7}$$

that if $U_k(t,s)$ is the Cauchy matrix for the system

$$\dot{x}(t) = Ax(t), \quad t_{k-1}\le t<t_k, \tag{2.8}$$

then the Cauchy matrix for the system (2.7) is in the form

$$W(t,s)=\begin{cases}U_k(t,s), & t_{k-1}\le s\le t<t_k,\\ U_{k+1}(t,t_k)(I+D_k)U_k(t_k,s), & t_{k-1}\le s<t_k\le t<t_{k+1},\\ U_{k+1}(t,t_k)\displaystyle\prod_{j=k}^{i+1}(I+D_k)U_j(t_j,t_{j+1})(I+D_i)U_i(t_i,s), & t_{i-1}\le s<t_i<t_k\le t<t_{k+1}.\end{cases} \tag{2.9}$$

As the special case of Lemma 1 in [10], we have the following lemma.

Lemma 2.5. *Assume that (A1), (A2) and the following condition hold. For the Cauchy matrix $W(t,s)$ of system* (2.7), *there exist positive constants M and λ such that*

$$\|W(t,s)\| \le Me^{-\lambda(t-s)}, \quad t \ge s,\ t,s \in \mathbb{R}. \tag{2.10}$$

Then for any $\varepsilon > 0,\ t \ge s,\ t,s \in \mathbb{R}, |t-t_k| > \varepsilon,\ |s-t_k| > \varepsilon, k \in \mathbb{N}$, there must be exist a relatively dense set T of ε-almost periodic of the matrix A and a positive constant Γ such that for $\tau \in T$, it follows:

$$\|W(t+\tau, s+\tau) - W(t,s)\| \le \varepsilon\Gamma e^{(-\lambda/2)(t-s)}. \tag{2.11}$$

Lemma 2.6 (see [6]). *Let $W(t,s)$ be the Cauchy matrix of the linear system* (2.7). *Given a constant $\eta \ge \|I + D_k\|$ for all $k \in \mathbb{N}$, if $\eta \ge 1$ and $\theta = \inf_k\{t_k^1\} > 0$, then*

$$\|W(t,s)\| \le \eta e^{(\mu(A)+(\ln\eta/\theta))(t-s)}, \quad t \ge s. \tag{2.12}$$

Introduce the following conditions:

(A3) The functions $f, g : \mathbb{R} \times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ are mean square almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \Theta$, where $\Theta \subset L^2(\mathbb{P},\mathbb{H})$ is compact, and $f(0,0) = g(0,0) = 0$. Moreover, there exist $L_f, L_g > 0$ such that

$$\begin{aligned} E\|f(t,x) - f(t,y)\|^2 &\le L_f E\|x-y\|^2, \\ E\|g(t,x) - g(t,y)\|^2 &\le L_g E\|x-y\|^2, \end{aligned} \tag{2.13}$$

for all stochastic processes $x, y \in L^2(\mathbb{P},\mathbb{H})$ and $t \in \mathbb{R}$.

(A4) The function $\sigma : \mathbb{R} \times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ is mean square almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \Theta'$, where $\Theta' \subset L^2(\mathbb{P},\mathbb{H})$ is compact, and $\sigma(0,0) = 0$. Moreover, there exists $L_\sigma > 0$ such that

$$E\|\sigma(t,x) - \sigma(t,y)\|^2 \le L_\sigma E\|x-y\|^2, \tag{2.14}$$

for all stochastic processes $x, y \in L^2(\mathbb{P},\mathbb{H})$ and $t \in \mathbb{R}$.

(A5) The function $I_i(t) : \mathbb{R} \to \mathbb{R}$ is almost periodic in the sense of Bohr, $\{\beta_k\}_{k\in\mathbb{N}}$ is almost periodic sequence and there exists a constant $\gamma_0 > 0$, such that

$$\max\left\{\max_k|\beta_k|, \sup_t\|I(t)\|\right\} \le \gamma_0. \tag{2.15}$$

(A6) The sequence of functions $V_k(x) : L^2(\mathbb{P},\mathbb{H}) \rightarrow L^2(\mathbb{P},\mathbb{H})$ is mean square almost periodic uniformly with respect to $x \in \Theta''$, where $\Theta'' \subset L^2(\mathbb{P},\mathbb{H})$ is compact. Moreover, there exists $L_V > 0$ such that

$$E\left\|V_k(x) - V_k(y)\right\|^2 \leq L_V E\left\|x - y\right\|^2 \tag{2.16}$$

for all stochastic processes $x, y \in L^2(\mathbb{P},\mathbb{H})$.

Lemma 2.7 (see [26]). *If conditions (A1)–(A6) are satisfied, then for each $\varepsilon > 0$, there exists ε_1, $0 < \varepsilon_1 < \varepsilon$ and relatively dense sets T of real numbers and Q of integral numbers, such that*

(i) $E\|f(t+\tau,y) - f(t,y)\|^2 < \varepsilon$, $E\|g(t+\tau,y) - g(t,y)\|^2 < \varepsilon$, $t \in \mathbb{R}$, $\tau \in T$, $|t - t_k| > \varepsilon$, $k \in \mathbb{N}$, $y \in L^2(\mathbb{P},\mathbb{H})$;

(ii) $E\|\sigma(t+\tau,y) - \sigma(t,y)\|^2 < \varepsilon$, $t \in \mathbb{R}$, $\tau \in T$, $|t - t_k| > \varepsilon$, $k \in \mathbb{N}$, $y \in L^2(\mathbb{P},\mathbb{H})$;

(iii) $\|I(t+\tau) - I(t)\|^2 < \varepsilon$, $t \in \mathbb{R}$, $\tau \in T$, $|t - t_k| > \varepsilon$;

(iv) $E\|V_{k+q}(y) - V_k(y)\|^2 < \varepsilon$, $q \in Q$, $k \in \mathbb{N}$;

(v) $\|\beta_{k+q} - \beta_k\|^2 < \varepsilon$, $q \in Q$, $k \in \mathbb{N}$;

(vi) $\|t_{k+q} - \tau\|^2 < \varepsilon_1$, $q \in Q$, $\tau \in T$, $k \in \mathbb{N}$.

Lemma 2.8 (see [26]). *Let condition (A2) holds. Then for each $p > 0$, there exists a positive integer N such that on each interval of length p, there are no more than N elements of the sequence $\{t_k\}$, that is,*

$$i(s,t) \leq N(t-s) + N, \tag{2.17}$$

where $i(s,t)$ is the number of points t_k in the interval (s,t).

3. Main Results

Theorem 3.1. *Assume that (A1)–(A6) hold, then there exists a unique mean square almost periodic solution of system* (2.2) *if the following conditions are satisfied: There exists a constant $\eta \geq 1$, such that $\|I + D_k\| \leq \eta$, $k \in \mathbb{N}$ and*

$$\mu(A) + \frac{\ln \eta}{\theta} \triangleq -\lambda < 0. \tag{3.1}$$

Furthermore,

$$\rho = 6\eta^2 \left[\frac{2}{\lambda^2} \left(\|B\|^2 L_f^2 + \|C\|^2 L_g^2 \right) + \frac{N^2}{\left(1 - e^{-\lambda}\right)^2} L_V^2 + \frac{L_\sigma^2}{2\lambda} \right] < 1. \tag{3.2}$$

Proof. Let $D = \{\varphi(t) \in L^2(\mathbb{P},\mathbb{H}) : \varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))^T\} \subset AP([-h,+\infty); L^2(\mathbb{P},\mathbb{H}))$ satisfying the equality $E\|\varphi\|^2 < \overline{K}$, where $\overline{K} = 2\eta^2\gamma_0^2((1/\lambda) + (N/(1 - e^{-\lambda})))^2 > 0$.

Set

$$x(t) = W(t,0)\phi_0 + \int_0^t W(t,s)\left[Bf(s,x(s)) + Cg(s,x(s-h)) + I(s)\right]ds$$
$$+ \sum_{0\le t_k<t} W(t,t_k)\left[V_k(x(t_k)) + \beta_k\right] + \int_0^t W(t,s)\sigma(s,x(s))d\omega(s), \quad t \ge 0. \tag{3.3}$$

where $\phi_0 = x(0)$, it is easy to see that $x(t)$ given by (3.3) is the solution of system (2.2) according to [2] and Lemma 2.2 in [27].

By Lemma 2.6 and the conditions of Theorem, we have

$$\|W(t,s)\| \le \eta e^{-\lambda(t-s)}, \quad t \ge s,\ t,s \in \mathbb{R}. \tag{3.4}$$

For $z(t) \in D$, we define the operator L in the following way

$$(Lz)(t) = \int_0^t W(t,s)\left[Bf(s,z(s)) + Cg(s,z(s-h)) + I(s)\right]ds$$
$$+ \sum_{0\le t_k<t} W(t,t_k)\left[V_k(z(t_k)) + \beta_k\right] + \int_0^t W(t,s)\sigma(s,z(s))d\omega(s). \tag{3.5}$$

Define subset $D^* \subset D$, $D^* = \{z \in D : E\|z - z_0\|^2 \le \rho\overline{K}/(1-\rho)\}$, and $z_0 = \int_0^t W(t,s) I(s)ds + \sum_{0\le t_k<t} W(t,t_k)\beta_k$.

We have

$$E\|z_0\|^2 \le 2E\left\|\int_0^t W(t,s)I(s)ds\right\|^2 + 2E\left\|\sum_{0\le t_k<t} W(t,t_k)\beta_k\right\|^2$$
$$\le 2\left[\int_0^t \eta e^{-\lambda(t-s)}\sup_s\|I(s)\|ds\right]^2 + 2\left[\sum_{0\le t_k<t} \eta e^{-\lambda(t-t_k)}\max_k|\beta_k|\right]^2$$
$$\le 2\eta^2\gamma_0^2\left(\frac{1}{\lambda} + \frac{N}{1-e^{-\lambda}}\right)^2 = \overline{K}. \tag{3.6}$$

Then for $\forall z \in D^*$, from the definition of D^* and (3.6), since $(a+b)^2 \le 2a^2 + 2b^2$, we have

$$E\|z\|^2 = E\|(z - z_0) + z_0\|^2 \le 2E\left(\|z - z_0\|^2 + \|z_0\|^2\right)$$
$$\le 2\left(\frac{\rho\overline{K}}{1-\rho} + \overline{K}\right) = \frac{2\overline{K}}{1-\rho}. \tag{3.7}$$

For $\forall z \in D^*$, we have

$$\|Lz - z_0\| = \left\| \int_0^t W(t,s)\left[Bf(s,z(s)) + Cg(s,z(s-h))\right]ds + \sum_{0\le t_k<t} W(t,t_k)V_k(z(t_k)) + \int_0^t W(t,s)\sigma(s,z(s))d\omega(s) \right\|. \tag{3.8}$$

Since $(a+b+c)^2 \le 3a^2 + 3b^2 + 3c^2$, it follows

$$E\|Lz - z_0\|^2 \le 3E\left(\int_0^t \|W(t,s)\| \left\|Bf(s,z(s)) + Cg(s,z(s-h))\right\| ds\right)^2 + 3E\left(\left\|\sum_{0\le t_k<t} W(t,t_k)V_k(z(t_k))\right\|\right)^2 + 3E\left(\int_0^t \|W(t,s)\sigma(s,z(s))d\omega(s)\|\right)^2. \tag{3.9}$$

For first term of the right-hand side, using (3.7), (A3) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} &E\left(\int_0^t \|W(t,s)\| \left\|Bf(s,z(s)) + Cg(s,z(s-h))\right\| ds\right)^2 \\ &\le \eta^2\left(\int_0^t e^{-\lambda(t-s)}ds\right)\cdot\left(\int_0^t e^{-\lambda(t-s)}\cdot E\left\|Bf(s,z(s)) + Cg(s,z(s-h))\right\|^2 ds\right) \\ &\le \eta^2\left(\int_0^t e^{-\lambda(t-s)}ds\right)\cdot\left[\int_0^t e^{-\lambda(t-s)}\cdot\left(2\|B\|^2L_f^2E\|z(s)\|^2 + 2\|C\|^2L_g^2E\|z(s-h)\|^2\right)ds\right] \\ &\le \eta^2\cdot\frac{2\overline{K}}{1-\rho}\left[\frac{2}{\lambda^2}\left(\|B\|^2L_f^2 + \|C\|^2L_g^2\right)\right]. \end{aligned} \tag{3.10}$$

As to the second term, using (3.7), (A6) and Cauchy-Schwarz inequality, we can write

$$\begin{aligned} &E\left(\left\|\sum_{0\le t_k<t} W(t,t_k)V_k(z(t_k))\right\|\right)^2 \\ &\le \eta^2\left(\sum_{0\le t_k<t} e^{-\lambda(t-t_k)}\right)\cdot\left(\sum_{0\le t_k<t} e^{-\lambda(t-t_k)}E\|V_k(z(t_k))\|^2\right) \\ &\le \eta^2\left(\sum_{0\le t_k<t} e^{-\lambda(t-t_k)}\right)\cdot\left(\sum_{0\le t_k<t} e^{-\lambda(t-t_k)}L_V^2E\|z(t_k)\|^2\right) \\ &\le \eta^2\cdot\frac{2\overline{K}}{1-\rho}\left[L_V^2\cdot\frac{N^2}{(1-e^{-\lambda})^2}\right]. \end{aligned} \tag{3.11}$$

As far as last term is concerned, using (3.7), (A4), and the Itô isometry theorem, we obtain

$$E\left(\int_0^t \|W(t,s)\sigma(s,z(s))d\omega(s)\|\right)^2 \leq \int_0^t \|W(t,s)\|^2 E\|\sigma(s,z(s))\|^2 ds \leq \eta^2 \int_0^t e^{-2\lambda(t-s)} L_\sigma^2 E\|z(s)\|^2 ds \leq \eta^2 \cdot \frac{2\overline{K}}{1-\rho} \cdot \frac{L_\sigma^2}{2\lambda}. \tag{3.12}$$

Thus, by combining (3.9)–(3.12), it follows that

$$E\|Lz - z_0\|^2 \leq 3\eta^2 \cdot \frac{2\overline{K}}{1-\rho}\left[\frac{2}{\lambda^2}\left(\|B\|^2 L_f^2 + \|C\|^2 L_g^2\right) + \frac{N^2}{(1-e^{-\lambda})^2} L_V^2 + \frac{L_\sigma^2}{2\lambda}\right] = \frac{\rho\overline{K}}{1-\rho}. \tag{3.13}$$

By Lemmas 2.5 and 2.6, one can obtain

$$\|W(t+\tau, s+\tau) - W(t,s)\| \leq \varepsilon \Gamma e^{-(\lambda/2)(t-s)}. \tag{3.14}$$

Let $\tau \in T, q \in Q$, where the sets T and Q are determined in Lemma 2.7, and we assume that $0 < \varepsilon < 1$, then

$$\begin{aligned}
&\|Lz(t+\tau) - Lz(t)\| \\
&\quad = \Bigg\| \int_0^t [W(t+\tau, s+\tau) - W(t,s)]\left[Bf(s+\tau, z(s+\tau)) + Cg(s+\tau, z(s+\tau-h)) + I(s+\tau)\right] ds \\
&\qquad + \int_0^t W(t,s)\left\{\left[Bf(s+\tau, z(s+\tau)) + Cg(s+\tau, z(s+\tau-h)) + I(s+\tau)\right]\right. \\
&\qquad\qquad \left. -\left[Bf(s,z(s)) + Cg(s,z(s-h)) + I(s)\right] ds\right\} \\
&\qquad + \sum_{0\leq t_k<t} \left[W(t+\tau, t_{k+q}) - W(t,t_k)\right]\left[V_{k+q}(z(t_{k+q})) + \beta_{k+q}\right] \\
&\qquad + \sum_{0\leq t_k<t} W(t,t_k)\left[V_{k+q}(z(t_{k+q})) - V_k(z(t_k)) + \beta_{k+q} - \beta_k\right] \\
&\qquad + \int_0^t [W(t+\tau, s+\tau) - W(t,s)][\sigma(s+\tau, z(s+\tau))] d\omega(s) \\
&\qquad + \int_0^t W(t,s)[\sigma(s+\tau, z(s+\tau)) - \sigma(s,z(s))] d\omega(s) \Bigg\|.
\end{aligned} \tag{3.15}$$

Therefore, we have

$$
\begin{aligned}
&E\|Lz(t+\tau)-Lz(t)\|^2\\
&\quad\le 3E\bigg\|\int_0^t [W(t+\tau,s+\tau)-W(t,s)]\big[Bf(s+\tau,z(s+\tau))+Cg(s+\tau,z(s+\tau-h))+I(s+\tau)\big]ds\\
&\qquad+\int_0^t W(t,s)\big\{\big[Bf(s+\tau,z(s+\tau))+Cg(s+\tau,z(s+\tau-h))+I(s+\tau)\big]\\
&\qquad\qquad-\big[Bf(s,z(s))+Cg(s,z(s+-h))+I(s)\big]ds\big\}\bigg\|^2\\
&\qquad+3E\bigg\|\sum_{0\le t_k<t}[W(t+\tau,t_{k+q})-W(t,t_k)]\big[V_{k+q}(z(t_{k+q}))+\beta_{k+q}\big]\\
&\qquad\quad+\sum_{0\le t_k<t}W(t,t_k)\big[V_{k+q}(z(t_{k+q}))-V_k(z(t_k))+\beta_{k+q}-\beta_k\big]\bigg\|^2\\
&\qquad+3E\bigg\|\int_0^t [W(t+\tau,s+\tau)-W(t,s)][\sigma(s+\tau,z(s+\tau))]d\omega(s)\\
&\qquad\quad+\int_0^t W(t,s)[\sigma(s+\tau,z(s+\tau))-\sigma(s,z(s))]d\omega(s)\bigg\|^2.
\end{aligned}
\tag{3.16}
$$

We first evaluate the first term of the right hand side

$$
\begin{aligned}
&E\bigg\|\int_0^t [W(t+\tau,s+\tau)-W(t,s)]\big[Bf(s+\tau,z(s+\tau))+Cg(s+\tau,z(s+\tau-h))+I(s+\tau)\big]ds\\
&\quad+\int_0^t W(t,s)\big\{\big[Bf(s+\tau,z(s+\tau))+Cg(s+\tau,z(s+\tau-h))+I(s+\tau)\big]\\
&\qquad-\big[Bf(s,z(s))+Cg(s,z(s+-h))+I(s)\big]ds\big\}\bigg\|^2\\
&\le 2E\bigg[\int_0^t\|W(t+\tau,s+\tau)-W(t,s)\|\\
&\qquad\times\big\|Bf(s+\tau,z(s+\tau))+Cg(s+\tau,z(s+\tau-h))+I(s+\tau)\big\|\bigg]^2ds\\
&\quad+2E\bigg[\int_0^t\|W(t,s)\|\Big\|B\big(f(s+\tau,z(s+\tau))-f(s,z(s))\big)\\
&\qquad+C\big(g(s+\tau,z(s+\tau-h))-g(s,z(s-h))\big)(I(s+\tau)-I(s))\bigg]\Big\|^2ds\\
&\le c_1\varepsilon,
\end{aligned}
\tag{3.17}
$$

where $c_1=(96\eta^2/\lambda^2)[\|B\|^2L_f^2\cdot((\overline{K}/(1-\rho))+1)+\|C\|^2L_g^2\cdot((\overline{K}/(1-\rho))+1)+\gamma_0^2+1]$.

For the second term, we can estimate that

$$
\begin{aligned}
&E\Bigg\| \sum_{0\le t_k<t} \left[W(t+\tau,t_{k+q}) - W(t,t_k)\right]\left[V_{k+q}\left(z\left(t_{k+q}\right)\right) + \beta_{k+q}\right] \\
&\qquad + \sum_{0\le t_k<t} W(t,t_k)\left[V_{k+q}\left(z\left(t_{k+q}\right)\right) - V_k(z(t_k)) + \beta_{k+q} - \beta_k\right]\Bigg\|^2 \\
&\qquad \le 2E\left\| \sum_{0\le t_k<t} \left[W(t+\tau,t_{k+q}) - W(t,t_k)\right]\left[V_{k+q}(z(t_{k+q})) + \beta_{k+q}\right]\right\|^2 \\
&\qquad + 2E\left\| \sum_{0\le t_k<t} W(t,t_k)[V_{k+q}(z(t_{k+q})) - V_k(z(t_k)) + \beta_{k+q} - \beta_k]\right\|^2 \\
&\qquad \le c_2\varepsilon,
\end{aligned}
\tag{3.18}
$$

where $c_2 = (8\eta^2 N^2/(1-e^{-\lambda}))[L_V^2 \cdot ((\overline{K}/(1-\rho)) + 1) + \gamma_0^2 + 1]$.

For the last term, using (A4) and Itô isometry identity, we have

$$
\begin{aligned}
&E\Bigg\| \int_0^t [W(t+\tau,s+\tau) - W(t,s)][\sigma(s+\tau,z(s+\tau))]d\omega(s) \\
&\quad + \int_0^t W(t,s)[\sigma(s+\tau,z(s+\tau)) - \sigma(s,z(s)]d\omega(s)\Bigg\|^2 \\
&\le 2E\left\| \int_0^t [W(t+\tau,s+\tau) - W(t,s),\sigma(s+\tau,z(s+\tau))]d\omega(s)\right\|^2 \\
&\qquad + 2E\left\| \int_0^t W(t,s)[\sigma(s+\tau,z(s+\tau)) - \sigma(s,z(s))]d\omega(s)\right\|^2 \\
&\le 2E\int_0^t \|W(t+\tau,s+\tau) - W(t,s)\|^2 \|\sigma(s+\tau,z(s+\tau))\|^2 ds \\
&\qquad + 2E\int_0^t \|W(t,s)\|^2 \|\sigma(s+\tau,z(s+\tau)) - \sigma(s,z(s))\|^2 ds \\
&\le c_3\varepsilon,
\end{aligned}
\tag{3.19}
$$

where $c_3 = (2/\lambda)[\Gamma^2 L_\sigma^2(\overline{K}/(1-\rho)) + 1]$.

Combining (3.17), (3.18) and (3.19), it follows that

$$
E\|Lz(t+\tau) - Lz(t)\|^2 \le c_0\varepsilon,
\tag{3.20}
$$

where $c_0 = 3(c_1 + c_2 + c_3)$.

So, $Lz \in D^*$, that is L is self-mapping from D^* to D^* by (3.13) and (3.20).

Secondly, we will show L is contracting operator in D^*.

For $\forall x, y \in D^*$,

$$\begin{aligned}\|Lx - Ly\| = \bigg\| \int_0^t & W(t,s)B\left[f(s,x(s)) - f(s,y(s))\right] \\ &+ C\left[g(s,x(s-h)) - g(s,y(s-h))\right]ds \\ &+ \sum_{0\le t_k<t} W(t,t_k)\left[V_k(x(t_k)) - V_k(y(t_k))\right] \\ &+ \int_0^t W(t,s)\left[\sigma(s,x(s)) - \sigma(s,y(s))\right]d\omega(s) \bigg\|.\end{aligned} \tag{3.21}$$

By a minor modification of the proof of (3.13), we can obtain

$$\begin{aligned}&E\|Lx - Ly\|^2 \\ &\quad \le 6\eta^2\left[\frac{2}{\lambda^2}\left(\|B\|^2L_f^2 + \|C\|^2L_g^2\right) + \frac{N^2}{(1-e^{-\lambda})^2}L_V^2 + \frac{L_\sigma^2}{2\lambda}\right]\sup_t E\|x(t) - y(t)\|^2 \\ &\quad = \rho\|x - y\|_\infty^2,\end{aligned} \tag{3.22}$$

and therefore, $\|Lx - Ly\|_\infty \le \rho\|x - y\|_\infty$, it follows that L is contracting operator in D^*, so there exists a unique mean square almost periodic solution of (2.2) by the fixed points theorem. □

4. Example

Consider the following impulsive stochastic differential equation with delay

$$\begin{aligned} dx_i(t) &= \left[a_ix_i(t) + \sum_{j=1}^2 b_{ij}f_j(x_j(t)) + \sum_{j=1}^2 c_{ij}g_j(x_j(t-0.1)) + I_i(t)\right]dt \\ &\quad + 0.5x_i(t)d\omega_i(t), \quad t \ge 0,\ t \ne t_k, \\ \Delta x(t) &= x(t_k) - x(t_k^-) = D_kx(t_k^-) + V_k(x(t_k^-)) + \beta_k, \quad t = t_k,\ k \in \mathbb{N}, \\ x(t) &= \phi(t), \qquad -h \le t \le 0, \end{aligned} \tag{4.1}$$

where $t_k = k,\ k \in \mathbb{N},\ f(x(t)) = [\sin x_1(t), \sin x_2(t)]^T,\ g(x(t-0.1)) = [\cos x_1(t-0.1), \cos x_2(t-0.1)]^T, V_{ik} = [0.01\sin x_1(t), 0.01\cos x_2(t)]^T,\ \beta_k = 0.1,\ I(t) = [0.1, 0.1]^T,\ \gamma_0 = 0.1$, for convenience, we can choose

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad D_k = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}. \tag{4.2}$$

Then $\mu(A) = -2,\ \|I + D_k\| = 1/2, \|B\| = 0.1,\ \|C\| = 0.2,\ L_f = L_g = 1, L_V = 0.01, L_\sigma = 0.5$. Choose $\theta = \inf_k\{t_k^1\} = 0.01, \eta = 1, N = 6$. By simple calculation, we have $\lambda = -(\mu(A) + (\ln\eta/\theta)) = 2, \rho \doteq 0.8139 < 1, \overline{K} \doteq 1.107, (\rho\overline{K}/(1-\rho)) \doteq 4.841$.

Let $D^* = \{z \in D : E\|z - z_0\|^2 \leq 4.841\}$, so, by Theorem 3.1, system (4.1) has a unique mean square almost periodic solution in D^*.

Remark 4.1. Since there exist no results for almost periodic solutions for impulsive stochastic differential equations with delays, one can easily see that all the results in [10, 11, 20–22, 28] and the references therein cannot be applicable to system (4.1). This implies that the results of this paper are essentially new.

5. Conclusion

In this paper, a class of Itô impulsive stochastic differential equations with delays has been investigated. We conquer the difficulty of coexistence of impulsive, delay and stochastic factors in a dynamic system, and give a result for the existence and uniqueness of mean square almost periodic solutions. The results in this paper extend some earlier works reported in the literature. Moreover, our results have important applications in almost periodic oscillatory stochastic delayed neural networks with impulsive control.

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