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Research Article

Fixed-Point Theorems for Multivalued Mappings in Modular Metric Spaces

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We give some initial properties of a subset of modular metric spaces and introduce some fixed-point theorems for multivalued mappings under the setting of contraction type. An appropriate example is as well provided. The stability of fixed points in our main theorems is also studied.

1. Introduction and Preliminaries

The field of metric fixed-point theory has been widely investigated since 1922, when Banach [1] had proved his contraction principle. We are going to recall this well-known theorem before we continue over on.

A self-mapping f on a metric space (X, d) is called a *contraction* if there exists $0 \leq k < 1$ such that

$$d(fx, fy) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$. The contraction principle simply stated that, if (X, d) is complete, such a mapping has a unique fixed point.

One of the most influenced generalizations of Banach's theorem is traced to Nadler [2]. In 1969, via Hausdorff's concept of a distance between two arbitrary sets, Nadler proved the contraction principle for multivalued mappings in complete metric spaces. Also, some authors extended Nadler's principle and established fixed-point theorems for multivalued mappings in metric spaces and other spaces (see [3–9]). One of the most interesting studies are the extensions of such principle in modular spaces and modular function spaces (see [10–12] and references therein).

Lately, in 2010, Chistyakov [13] introduced the notion of a modular metric space which is a new generalization of a metric space. We will give a short revisit to modular and modular metric spaces as follows.

Definition 1.1. Let X be a linear space over $\mathbb{R}$ with $\theta \in X$ as its zero element. A functional $\rho : X \to [0, +\infty]$ is said to be a *modular* on X if for any $x, y \in X$, the following conditions hold:

(i) $\rho(x) = 0$ if and only if $x = \theta$,

(ii) $\rho(x) = \rho(-x)$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

The linear subspace $X_\rho := \{x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$ is called a *modular space*.

Definition 1.2 (see [13]). Let X be a nonempty set. A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is said to be a *metric modular* on X if satisfying, for all $x, y, z \in X$, the following conditions hold:

(i) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$,

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$,

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$.

Suppose $x_\iota \in X$, the set $X_\omega(x_\iota) = \{x \in X : \lim_{\lambda \to +\infty} \omega_\lambda(x, x_\iota) = 0\}$ is called a *modular metric space* generated by x_ι and induced by ω. If its generator x_ι does not play any role in the situation, we will write X_ω instead of $X_\omega(x_\iota)$.

Observe that a metric modular ω on X is nonincreasing with respect to $\lambda > 0$. We can simply show this assertion by using the condition (iii) itself. For any $x, y \in X$ and $0 < \mu < \lambda$, we have

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y). \tag{1.2}$$

For each $x, y \in X$ and $\lambda > 0$, we set $\omega_{\lambda^+}(x, y) := \lim_{\epsilon \downarrow 0} \omega_{\lambda+\epsilon}(x, y)$ and $\omega_{\lambda^-}(x, y) := \lim_{\epsilon \downarrow 0} \omega_{\lambda-\epsilon}(x, y)$. Consequently, from (1.2), we have $\omega_{\lambda^+}(x, y) \leq \omega_\lambda(x, y) \leq \omega_{\lambda^-}(x, y)$.

If, for any $x, y \in X$, a metric modular ω on X possesses a finite value and $\omega_\lambda(x, y) = \omega_\mu(x, y)$ for all $\lambda, \mu > 0$, then $d(x, y) := \omega_\lambda(x, y)$ is a metric on X.

Recently, Mongkolkeha et al. [14] have introduced some notions and established some fixed-point results in modular metric spaces. We now state some notions and results in [14] in the following.

Definition 1.3 (see [14]). Let X_ω be a modular metric space.

(i) The sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_ω is said to be *convergent* if there exists $x \in X_\omega$ such that $\omega_\lambda(x_n, x) \to 0$, as $n \to \infty$ for all $\lambda > 0$.

(ii) The sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_ω is said to be a *Cauchy sequence* if $\omega_\lambda(x_m, x_n) \to 0$, as $m, n \to \infty$ for all $\lambda > 0$.

(iii) X_ω is said to be *complete* if every Cauchy sequence in X_ω converges.

(iv) A subset C of X_ω is said to be *closed* if the limit of a convergent sequence of C always belongs to C.

(v) A subset C of X_ω is said to be *bounded* if, for all $\lambda > 0$, $\phi_\lambda(C) = \sup\{\omega_\lambda(x,y) : x, y \in C\} < +\infty$.

Along this paper, we will use the following alternative notions of convergence and Cauchyness, which are equivalent to the notions given above.

Let X_ω be a modular metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ω.

(i) A point $x \in X_\omega$ is called a *limit* of $\{x_n\}_{n\in\mathbb{N}}$ if for each $\lambda, \epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n, x) < \epsilon$ for every $n \in \mathbb{N}$ with $n \geq n_0$. A sequence that has a limit is said to be *convergent* (or *converges* to x) and will be written as $\lim_{n\to+\infty} x_n = x$.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be a *Cauchy sequence* if, for each $\lambda, \epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n, x_m) < \epsilon$ for every $m, n \in \mathbb{N}$ with $m, n \geq n_0$.

Moreover, we observe that the limit of any sequence in X_ω is unique.

Definition 1.4 (see [14]). Let X_ω be a modular metric space. A self-mapping f on X_ω is said to be a *contraction* if there exists $0 \leq k < 1$ such that

$$\omega_\lambda(fx, fy) \leq k\omega_\lambda(x, y) \tag{1.3}$$

for all $x, y \in X_\omega$ and $\lambda > 0$.

Theorem 1.5 (see [14]). *Let X_ω be a complete modular metric space and f a contraction on X_ω. Then, the sequence $\{f^n x\}_{n\in\mathbb{N}}$ converges to the unique fixed point of f in X_ω for any initial $x \in X_\omega$.*

The purpose of this paper is to study some properties of a subset of modular metric spaces, establish and extend some fixed-point theorems of Mongkolkeha et al. [14] to multivalued mappings in modular metric spaces.

2. Some Properties of a Subset of Modular Metric Spaces

In this section, we study some properties of a subset of modular metric spaces, some of which will take advantages in the proof of our main theorems. Throughout this paper, let $\mathcal{CB}(X_\omega)$ denotes the set of all nonempty closed bounded subsets of X_ω and $\mathcal{C}(X)$ denotes the set of all nonempty closed subsets of X.

Let A be a non-empty subset of a modular metric space X_ω. For $x \in X_\omega$, we denotes $\omega_\lambda(x, A) := \inf_{y\in A}\omega_\lambda(x, y)$.

For $A, B \in \mathcal{CB}(X_\omega)$, define $\delta_\lambda(A, B) := \sup_{x\in A}\omega_\lambda(x, B)$ and the Hausdorff metric modular $\Omega_\lambda(A, B) := \max\{\delta_\lambda(A, B), \delta_\lambda(B, A)\}$. Notice that δ_λ is not symmetric.

Proposition 2.1. *Let X_ω be a modular metric space and $A, B, C \in \mathcal{CB}(X_\omega)$. Then, the following properties hold.*

(i) $\delta_\lambda(A, B) = 0$ *for all* $\lambda > 0 \Leftrightarrow A \subseteq B$.

(ii) $B \subseteq C \Rightarrow \delta_\lambda(A, C) \leq \delta_\lambda(A, B)$ *for all* $\lambda > 0$.

(iii) $\delta_\lambda(A \cup B, C) = \max\{\delta_\lambda(A, C), \delta_\lambda(B, C)\}$ *for all* $\lambda > 0$.

(iv) $\delta_{\lambda+\mu}(A, B) \leq \delta_\lambda(A, C) + \delta_\mu(C, B)$ *for all* $\lambda, \mu > 0$.

Proof. (i) By the definition of δ_λ, we have, for all $\lambda > 0$, that

$$\delta_\lambda(A,B) = 0 \Longleftrightarrow \sup_{x\in A} \omega_\lambda(x,B) = 0$$
$$\Longleftrightarrow \omega_\lambda(x,B) = 0, \quad \forall x \in A. \tag{2.1}$$

Since B is closed in X_ω, we get $\omega_\lambda(x,B) = 0$ for all $\lambda > 0 \Leftrightarrow x \in B$. That is, $\delta_\lambda(A,B) = 0$ for all $\lambda > 0 \Leftrightarrow A \subseteq B$.

(ii) It is obvious that $\omega_\lambda(x,C) \leq \omega_\lambda(x,B)$ for all $x \in X_\omega$ and $\lambda > 0$. Hence, $\delta_\lambda(A,C) \leq \delta_\lambda(A,B)$.

(iii) Observe that, if $B \subseteq C$, then

$$\delta_\lambda(A \cup B, C) = \sup_{x\in A\cup B} \omega_\lambda(x,C) = \max\left\{\sup_{x\in A} \omega_\lambda(x,C),\ \sup_{x\in B} \omega_\lambda(x,C)\right\}. \tag{2.2}$$

(iv) Let $a \in A$, $b \in B$, and $c \in C$. Then,

$$\omega_{\lambda+\mu}(a,b) \leq \omega_\lambda(a,c) + \omega_\mu(c,b), \tag{2.3}$$

which implies that

$$\omega_{\lambda+\mu}(a,B) \leq \omega_\lambda(a,c) + \omega_\mu(c,B)$$
$$\leq \omega_\lambda(a,c) + \delta_\mu(C,B). \tag{2.4}$$

Since $c \in C$ is arbitrary, we have

$$\omega_{\lambda+\mu}(a,B) \leq \omega_\lambda(a,C) + \delta_\mu(C,B). \tag{2.5}$$

Similarly, since $a \in A$ is arbitrary, we can deduce that

$$\delta_{\lambda+\mu}(A,B) \leq \delta_\lambda(A,C) + \delta_\mu(C,B). \tag{2.6}$$

□

Proposition 2.2. *Let X_ω be a modular metric space. Then,*

$$\Omega_\lambda(A\cup B, C\cup D) \leq \max\{\Omega_\lambda(A,C), \Omega_\lambda(B,D)\} \tag{2.7}$$

for all $A,B,C,D \in \mathcal{CB}(X_\omega)$.

Proof. Suppose $\lambda > 0$ is arbitrary. For $a \in A$ and $b \in B$, we have $\omega_\lambda(a, C \cup D) \leq \omega_\lambda(a, C)$ and $\omega_\lambda(b, C \cup D) \leq \omega_\lambda(b, D)$. Hence, we get

$$\begin{aligned} \delta_\lambda(A \cup B, C \cup D) &= \max\{\delta_\lambda(A, C \cup D), \delta_\lambda(B, C \cup D)\} \\ &\leq \max\{\delta_\lambda(A, C), \delta_\lambda(B, D)\} \\ &\leq \max\{\Omega_\lambda(A, C), \Omega_\lambda(B, D)\}. \end{aligned} \tag{2.8}$$

Similarly, we have

$$\delta_\lambda(C \cup D, A \cup B) \leq \max\{\Omega_\lambda(A, C), \Omega_\lambda(B, D)\}. \tag{2.9}$$

Hence, we have

$$\begin{aligned} \Omega_\lambda(A \cup B, C \cup D) &= \max\{\delta_\lambda(A \cup B, C \cup D), \delta_\lambda(C \cup D, A \cup B)\} \\ &\leq \max\{\Omega_\lambda(A, C), \Omega_\lambda(B, D)\}. \end{aligned} \tag{2.10}$$

□

Proposition 2.3. *Let X_ω be a modular metric space generated by x_ι. Then, $\mathcal{CB}(X_\omega)$ is a modular metric space generated by $\{x_\iota\}$ and is induced by Ω.*

Proof. For $\{x_\iota\}, A \in \mathcal{CB}(X_\omega)$, we have

$$\begin{aligned} \Omega_\lambda(A, \{x_\iota\}) &= \max\left\{\sup_{x \in A} \omega_\lambda(x, \{x_\iota\}), \sup_{x \in \{x_\iota\}} \omega_\lambda(x, A)\right\} \\ &= \max\left\{\sup_{x \in A} \omega_\lambda(x, x_\iota), \inf_{x \in A} \omega_\lambda(x_\iota, x)\right\} \\ &= \sup_{x \in A} \omega_\lambda(x, x_\iota). \end{aligned} \tag{2.11}$$

Since $x \in A \subseteq X_\omega$ and $\lim_{\lambda \to +\infty} \omega_\lambda(x, x_\iota) = 0$, we have $\lim_{\lambda \to +\infty} \Omega_\lambda(A, \{x_\iota\}) = 0$.

By the definition of Ω and Proposition 2.1, it is clear that $\Omega_\lambda(A, B) = \Omega_\lambda(B, A) \geq 0$ for all $\lambda > 0$ and $\Omega_\lambda(A, B) = 0$ for all $\lambda > 0$ if and only if $A = B$.

Again, by Proposition 2.1, we have

$$\begin{aligned} \Omega_{\lambda+\mu}(A, B) &= \max\{\delta_{\lambda+\mu}(A, B), \delta_{\lambda+\mu}(B, A)\} \\ &\leq \max\{\delta_\lambda(A, C) + \delta_\mu(C, B), \delta_\mu(B, C) + \delta_\lambda(C, A)\} \\ &\leq \max\{\delta_\lambda(A, C), \delta_\lambda(C, A)\} + \max\{\delta_\mu(B, C), \delta_\mu(C, B)\} \\ &= \Omega_\lambda(A, C) + \Omega_\mu(C, B) \end{aligned} \tag{2.12}$$

for all $\lambda, \mu > 0$. Therefore, $\mathcal{CB}(X_\omega)$ is a modular metric space generated by $\{x_\iota\}$ and is induced by Ω. □

Remark 2.4. Note that the metric modular Ω depends on ω, so the completeness of X_ω implies the completeness of $\mathcal{CB}(X_\omega)$.

Now, we are arriving at the most important lemma used in our proof of main theorems.

Lemma 2.5. *Let $A, B \in \mathcal{CB}(X_\omega)$ and $a \in A$. Then, for $\epsilon > 0$, there exists a point $b_\epsilon \in B$ such that $\omega_\lambda(a, b_\epsilon) \leq \Omega_\lambda(A, B) + \epsilon$.*

Proof. Let $a \in A$, $\epsilon, \lambda > 0$ be arbitrary. Since $\omega_\lambda(a, B) = \inf_{b \in B} \omega_\lambda(a, b)$, we claim that $\omega_\lambda(a, B) + \epsilon$ is not a lower bound of the set $\{\omega_\lambda(a, b) : b \in B\}$. Therefore, there exists $b_\epsilon \in B$ for which $\omega_\lambda(a, b_\epsilon) \leq \omega_\lambda(a, B) + \epsilon$ and hence $\omega_\lambda(a, b_\epsilon) \leq \Omega_\lambda(A, B) + \epsilon$. □

3. Fixed-Point Theorems for Multivalued Mappings

In this section, we extend the result by Mongkolkeha et al. [14] under the multivalued setting and hereby obtain some corollaries. Beforehand, we will give the notion of a multivalued ω-contraction in modular metric spaces.

Definition 3.1. Let X_ω be a modular metric space. A multivalued mapping $F : X_\omega \to \mathcal{CB}(X_\omega)$ is said to be a *multivalued ω-contraction* if there exists $0 \leq k < 1$ such that

$$\Omega_\lambda(Fx, Fy) \leq k\omega_\lambda(x, y) \tag{3.1}$$

for all $x, y \in X_\omega$ and $\lambda > 0$. In this case, the least number k which satisfies the inequality (3.1) is said to be the *contraction constant*.

Remark 3.2. For a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω, it is obvious that, if $\lim_{n\to+\infty} x_n = x$ and F is a multivalued ω-contraction on X_ω, then $\lim_{n\to+\infty} Fx_n = Fx$.

Theorem 3.3. *Let X_ω be a complete modular metric space and F a multivalued ω-contraction on X_ω with contraction constant k. Then, F has a fixed point in X_ω.*

Proof. Let $x_0 \in X_\omega$ be arbitrary and $x_1 \in Fx_0$. By Lemma 2.5, there exists $x_2 \in Fx_1$ such that

$$\omega_\lambda(x_1, x_2) \leq \Omega_\lambda(Fx_0, Fx_1) + k. \tag{3.2}$$

Similarly, by this procedure, we define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω such that $x_n \in Fx_{n-1}$ and

$$\omega_\lambda(x_n, x_{n+1}) \leq \Omega_\lambda(Fx_{n-1}, Fx_n) + k^n \tag{3.3}$$

for all $n \in \mathbb{N}$. Hence, by the multivalued ω-contractivity, we have

$$\begin{aligned} \omega_\lambda(x_n, x_{n+1}) &\leq \Omega_\lambda(Fx_{n-1}, Fx_n) + k^n \\ &\leq k\omega_\lambda(x_{n-1}, x_n) + k^n \\ &\leq k\left[k\omega_\lambda(x_{n-2}, x_{n-1}) + k^{n-1}\right] + k^n \\ &\leq k^2\omega_\lambda(x_{n-2}, x_{n-1}) + 2k^n. \end{aligned} \tag{3.4}$$

Thus, by induction, we deduce that

$$\omega_\lambda(x_n, x_{n+1}) \le k^n \omega_\lambda(x_0, x_1) + nk^n. \tag{3.5}$$

Notice that $\sum_{n\in\mathbb{N}} k^n < +\infty$ and $\sum_{n\in\mathbb{N}} nk^n < +\infty$. Now, since

$$\sum_{n\in\mathbb{N}} \omega_\lambda(x_n, x_{n+1}) \le \omega_\lambda(x_0, x_1) \sum_{n\in\mathbb{N}} k^n + \sum_{n\in\mathbb{N}} nk^n < +\infty, \tag{3.6}$$

for all $\lambda > 0$. Without loss of generality, suppose $m, n \in \mathbb{N}$ and $m > n$. Observe that, for arbitrary $\epsilon > 0$,

$$\begin{aligned} \omega_\lambda(x_n, x_m) &\le \omega_{\lambda/(m-n)}(x_n, x_{n+1}) + \omega_{\lambda/(m-n)}(x_{n+1}n, x_{n+2}) + \cdots + \omega_{\lambda/(m-n)}(x_{m-1}, x_m) \\ &\le \omega_{\lambda/m}(x_n, \ x_{n+1}) + \omega_{\lambda/m}(x_{n+1}n, x_{n+2}) + \cdots + \omega_{\lambda/m}(x_{m-1}, x_m) \\ &\le \sum_{n=n_*}^{+\infty} \omega_{\lambda/m}(x_n, x_{n+1}) \\ &< \epsilon \end{aligned} \tag{3.7}$$

for all $m > n \ge n_*$ for some $n_* \in \mathbb{N}$, and hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Then, the completeness of X_ω implies that $\lim_{n\to+\infty} x_n = x$ for some $x \in X_\omega$. Consequently, the sequence $\{Fx_n\}_{n\in\mathbb{N}}$ converges to Fx, that is, $\lim_{n\to+\infty} \Omega_\lambda(Fx_n, Fx) = 0$ for all $\lambda > 0$. Since $x_n \in Fx_{n-1}$, we have

$$0 \le \omega_\lambda(x_{n+1}, Fx) \le \delta_\lambda(Fx_n, Fx) \le \Omega_\lambda(Fx_n, Fx) \tag{3.8}$$

which implies that $\omega_\lambda(x, Fx) = 0$. Since Fx is closed, it follows that $x \in Fx$. □

Example 3.4. Let $X = [0,1]$, $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ defined by $\omega_\lambda(x, y) := (1/(1+\lambda))|x - y|$. Clearly, $X_\omega = [0,1]$ for any generator $x_\iota \in X$. Now, we define a multivalued mapping $F : X_\omega \to \mathcal{CB}(X_\omega)$ given by

$$Fx = \left\{0, \ \frac{x+1}{2}\right\}. \tag{3.9}$$

We have $\Omega_\lambda(Fx, Fy) = (1/2(1+\lambda)\ )|x - y| \le (1/2)\omega_\lambda(x, y)$. Therefore, F is a multivalued ω-contraction with contraction constant $k = 1/2$, and we have that 0 and 1 are fixed points of F.

Remark 3.5. Note that our result does not assure the uniqueness of a fixed point, as illustrated in the above example.

We next present the local version of Theorem 3.3.

Theorem 3.6. *Let X_ω be a complete modular metric space,*

$$\mathcal{B}_\omega(x_0,\gamma) := \{x \in X_\omega : \omega_\lambda(x,x_0) \le \gamma,\ \forall\lambda > 0\}, \tag{3.10}$$

and $F : \mathcal{B}_\omega(x_0,\gamma) \to \mathcal{CB}(X_\omega)$. Suppose there exists $0 \le k < 1$ for which

$$\Omega_\lambda(Fx,Fy) \le k\omega_\lambda(x,y) \tag{3.11}$$

for all $x,y \in \mathcal{B}_\omega(x_0,\gamma)$, $\lambda > 0$ and

$$\Omega_\lambda(Fx_0,\{x_0\}) \le (1-k)\gamma \tag{3.12}$$

for all $\lambda > 0$. Then, F has a fixed point in $\mathcal{B}_\omega(x_0,\gamma)$.

Proof. To prove this theorem, we only need to show that $\mathcal{B}_\omega(x_0,\gamma)$ is complete and $Fx \subseteq \mathcal{B}_\omega(x_0,\gamma)$. To show that $\mathcal{B}_\omega(x_0,\gamma)$ is complete, suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}_\omega(x_0,\gamma)$. Since X_ω is complete, $\lim_{n\to+\infty}\omega_\lambda(x_n,x) = 0$ for some $x \in X_\omega$ for all $\lambda > 0$. Since, for each $n \in \mathbb{N}$, $x_n \in \mathcal{B}_\omega(x_0,\gamma)$, we get

$$\begin{aligned}\omega_\lambda(x_0,x) &\le \omega_{\lambda/2}(x_0,x_n) + \omega_{\lambda/2}(x_n,x)\\ &\le \gamma + \omega_{\lambda/2}(x_n,x).\end{aligned} \tag{3.13}$$

As $n \to +\infty$, we have $\omega_\lambda(x_0,x) \le \gamma$. Therefore, $\mathcal{B}_\omega(x_0,\gamma)$ is complete.

Now, we prove the latter. For any $x \in \mathcal{B}_\omega(x_0,\gamma)$, let $y \in Fx$. Observe that, for all $\lambda > 0$,

$$\begin{aligned}\omega_\lambda(y,x_0) &= \delta_\lambda(\{y\},\{x_0\})\\ &\le \delta_{\lambda/3}(\{y\},Fx) + \delta_{\lambda/3}(Fx,Fx_0) + \delta_{\lambda/3}(Fx_0,\{x_0\})\\ &\le \Omega_{\lambda/3}(Fx,Fx_0) + \Omega_{\lambda/3}(Fx_0,\{x_0\})\\ &\le k\omega_\lambda(x,x_0) + (1-k)\gamma\\ &\le \gamma.\end{aligned} \tag{3.14}$$

This implies that $Fx \subseteq \mathcal{B}_\omega(x_0,\gamma)$ for all $x \in \mathcal{B}_\omega(x_0,\gamma)$. Applying Theorem 3.3 to complete the proof. □

In the following theorem, we prove the existence of fixed points for a mapping introduced in 1969 by Kannan [15] in view of multivalued mappings in modular metric spaces.

Theorem 3.7. *Let X_ω be a complete modular metric space and $F : X_\omega \to \mathcal{CB}(X_\omega)$ a multivalued mapping such that there exists $0 \le k < 1/2$ such that*

$$\Omega_\lambda(Fx,Fy) \le k\left[\omega_{2\lambda}(x,Fx) + \omega_{2\lambda}(y,Fy)\right] \tag{3.15}$$

for all $x,y \in X_\omega$ and $\lambda > 0$. Then, F has a fixed point in X_ω.

Proof. Let the sequence $\{x_n\}_{n\in\mathbb{N}}$ be constructed as in the proof of Theorem 3.3, so we get, for all $\lambda > 0$,

$$\omega_\lambda(x_n, x_{n+1}) \le \Omega_\lambda(Fx_{n-1}, Fx_n) + k^n \tag{3.16}$$

for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}\omega_\lambda(x_n, x_{n+1}) &\le \Omega_\lambda(Fx_{n-1}, Fx_n) + k^n \\ &\le k[\omega_{2\lambda}(x_{n-1}, Fx_{n-1}) + \omega_{2\lambda}(x_n, Fx_n)] + k^n \\ &\le k[\omega_\lambda(x_{n-1}, Fx_{n-1}) + \omega_\lambda(x_n, Fx_n)] + k^n \\ &\le k[\omega_\lambda(x_{n-1}, Fx_{n-1}) + \omega_\lambda(x_n, x_{n+1})] + k^n.\end{aligned} \tag{3.17}$$

Further, set $\xi := k/(1-k) < 1$, we obtain

$$\begin{aligned}\omega_\lambda(x_n, x_{n+1}) &\le \xi\omega_\lambda(x_{n-1}, x_n) + \frac{k^n}{1-k} \\ &\le \xi^2\omega_\lambda(x_{n-2}, x_{n-1}) + \frac{k^n}{(1-k)^2} + \frac{k^n}{(1-k)} \\ &\le \xi^2\omega_\lambda(x_{n-2}, x_{n-1}) + 2\frac{k^n}{(1-k)^2} \\ &\ \ \vdots \\ &\le \xi^n\omega_\lambda(x_0, x_1) + n\xi^n.\end{aligned} \tag{3.18}$$

As in the proof of Theorem 3.3, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. The completeness of X_ω implies that $\lim_{n\to+\infty} x_n = x$ for some $x \in X_\omega$.

Now, we show that x is a fixed point of F. Observe that

$$\begin{aligned}\omega_\lambda(x, Fx) &= \delta_\lambda(\{x\}, Fx) \\ &\le \delta_{\lambda/2}(\{x\}, Fx_n) + \delta_{\lambda/2}(Fx_n, Fx) \\ &= \omega_{\lambda/2}(x, Fx_n) + \delta_{\lambda/2}(Fx_n, Fx) \\ &\le \omega_{\lambda/2}(x, x_{n+1}) + \Omega_{\lambda/2}(Fx_n, Fx) \\ &\le \omega_{\lambda/2}(x, x_{n+1}) + k[\omega_\lambda(x_n, Fx_n) + \omega_\lambda(x, Fx)].\end{aligned} \tag{3.19}$$

Again, we have that

$$\omega_\lambda(x, Fx) \le \frac{1}{1-k}\omega_{\lambda/2}(x, x_{n+1}) + \frac{k}{1-k}\omega_\lambda(x_n, Fx_n). \tag{3.20}$$

As $n \to +\infty$, we have $\omega_\lambda(x, Fx) = 0$. Since Fx is closed, we have $x \in Fx$. Therefore, x is a fixed point of F in X_ω. □

4. Stability of Fixed Points

In this section, we discuss some stability of fixed points in Theorems 3.3 and 3.7. In this context, $\text{Fix}(F)$ will denote the set of all fixed points of a self-mapping F on X_ω.

Theorem 4.1. *Let X_ω be a complete modular metric space, and let $F, G : X_\omega \to \mathcal{CB}(X_\omega)$ be two multivalued ω-contractions having the same contraction constant k. If, for any $A, B \in \mathcal{CB}(X_\omega)$, $\lim_{\lambda\downarrow 0}\Omega_\lambda(A,B) = \zeta_{(A,B)} < +\infty$, then $\Omega_\lambda(\text{Fix}(F), \text{Fix}(G)) \le (1-k)^{-1}\sup_{x\in X_\omega}\zeta_{(Fx,Gx)}$.*

Proof. Suppose $\lambda > 0$, by Theorem 3.3, we can conclude that $\text{Fix}(F) \ne \emptyset \ne \text{Fix}(G)$. Let $\epsilon > 0$ be arbitrary, and let $\gamma > 0$ be such that $\gamma\sum_{n\in\mathbb{N}} nk^n < 1$. For $x_0 \in \text{Fix}(F)$, choose $x_1 \in Gx_0$ such that

$$\omega_\lambda(x_0, x_1) \le \Omega_\lambda(Fx_0, Gx_0) + \epsilon. \tag{4.1}$$

By the multivalued ω-contractivity, it is possible to choose $x_2 \in Gx_1$ such that

$$\omega_\lambda(x_1, x_2) \le k\omega_\lambda(x_0, x_1) + \frac{\gamma\epsilon k}{1-k}. \tag{4.2}$$

Now, define a sequence $\{x_n\}_{n\in\mathbb{N}}$ inductively by $x_n \in Gx_{n-1}$ and

$$\omega_\lambda(x_n, x_{n+1}) \le k\omega_\lambda(x_{n-1}, x_n) + \frac{\gamma\epsilon k^n}{1-k}. \tag{4.3}$$

Set $\eta := \gamma\epsilon/(1-k)$, it follows that

$$\begin{aligned}\omega_\lambda(x_n, x_{n+1}) &\le k\omega_\lambda(x_{n-1}, x_n) + \eta k^n \\ &\le k^2\omega_\lambda(x_{n-1}, x_{n-1}) + 2\eta k^n.\end{aligned} \tag{4.4}$$

Inductively, we have that

$$\omega_\lambda(x_n, x_{n+1}) \le k^n\omega_\lambda(x_0, x_1) + n\eta k^n. \tag{4.5}$$

Notice that $\sum_{n\in\mathbb{N}} k^n < +\infty$ and $\sum_{n\in\mathbb{N}} nk^n < +\infty$. Now, since

$$\sum_{n\in\mathbb{N}}\omega_\lambda(x_n, x_{n+1}) \le \omega_\lambda(x_0, x_1)\sum_{n\in\mathbb{N}}k^n + \eta\sum_{n\in\mathbb{N}}nk^n < +\infty, \tag{4.6}$$

we can say that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. The completeness of X_ω implies that $\lim_{n\to+\infty}x_n = x$ for some $x \in X_\omega$. Since $\lim_{n\to+\infty}\Omega_\lambda(Gx_n, Gx) = 0$ and $x_n \in Gx_{n-1}$, we get $x \in \text{Fix}(G)$. Now, observe that

$$\begin{aligned}\omega_\lambda(x_0, x) &\le \omega_{\lambda/2n}(x_0, x_1) + \omega_{\lambda/2n}(x_1, x_2) + \cdots + \omega_{\lambda/2n}(x_{n-1}, x_n) + \omega_{\lambda/2}(x_n, x) \\ &\le \sum_{m\in\mathbb{N}}\omega_{\lambda/2n}(x_{m-1}, x_m) + \omega_{\lambda/2}(x_n, x) \\ &\le \omega_{\lambda/2n}(x_0, x_1)\sum_{m\in\mathbb{N}}k^{m-1} + \eta\sum_{m\in\mathbb{N}}(m-1)k^{m-1} + \omega_{\lambda/2}(x_n, x) \\ &\le (1-k)^{-1}\left[\omega_{\lambda/2n}(x_0, x_1) + \epsilon\right] + \omega_{\lambda/2}(x_n, x) \\ &\le (1-k)^{-1}\left[\Omega_{\lambda/2n}(Fx_0, Gx_0) + 2\epsilon\right] + \omega_{\lambda/2}(x_n, x).\end{aligned} \tag{4.7}$$

Since $\omega_\lambda(x_0, x) \geq \omega_\lambda(x_0, \mathrm{Fix}(G))$ and together with (4.18), we have, as $n \to +\infty$, that

$$\begin{aligned}\delta_\lambda(\mathrm{Fix}(F), \mathrm{Fix}(G)) &\leq (1-k)^{-1}\left[\sup_{x\in \mathrm{Fix}(F)} \zeta_{(Fx,Gx)} + 2\epsilon\right] \\ &\leq (1-k)^{-1}\left[\sup_{x\in X_\omega} \zeta_{(Fx,Gx)} + 2\epsilon\right].\end{aligned} \tag{4.8}$$

Similarly, we have

$$\delta_\lambda(\mathrm{Fix}(G), \mathrm{Fix}(F)) \leq (1-k)^{-1}\left[\sup_{x\in X_\omega} \zeta_{(Fx,Gx)} + 2\epsilon\right]. \tag{4.9}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof. □

Corollary 4.2. *Let X_ω be a complete modular metric space and $F_n : X_\omega \to \mathcal{CB}(X_\omega)$, for $n \in \mathbb{N}$, multivalued ω-contractions having the same contraction constant k, and for any $A, B \in \mathcal{CB}(X_\omega)$, $\lim_{\lambda\downarrow 0}\Omega_\lambda(A,B) = \zeta_{(A,B)} < +\infty$. If $\lim_{n\to+\infty}\zeta_{(F_n x, Fx)} = 0$ uniformly for $x \in X_\omega$, then $\lim_{n\to+\infty}\Omega_\lambda(\mathrm{Fix}(F_n), \mathrm{Fix}(F)) = 0$.*

Proof. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to+\infty}\zeta_{(F_n x, Fx)} = 0$ uniformly for $x \in X_\omega$ and $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in X_\omega} \zeta_{(F_n x, Fx)} < (1-k)\epsilon \tag{4.10}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$. By Theorem 4.1, we have

$$\Omega_\lambda(\mathrm{Fix}(F_n),\ \mathrm{Fix}(F)) < \epsilon \tag{4.11}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$ and $\lambda > 0$. □

Likewise, we can deduce a stability theorem for fixed points in Theorem 3.7.

Theorem 4.3. *Let X_ω be a complete modular metric space, and let $F, G : X_\omega \to \mathcal{CB}(X_\omega)$ be two multivalued mappings such that there exists $0 \leq k < 1$ such that*

$$\begin{aligned}\Omega_\lambda(Fx, Fy) &\leq k\left[\omega_{2\lambda}(x, Fx) + \omega_{2\lambda}(y, Fy)\right], \\ \Omega_\lambda(Gx, Gy) &\leq k\left[\omega_{2\lambda}(x, Gx) + \omega_{2\lambda}(y, Gy)\right],\end{aligned} \tag{4.12}$$

for all $x, y \in X_\omega$ and $\lambda > 0$. If, for any $A, B \in \mathcal{CB}(X_\omega)$, $\lim_{\lambda\downarrow 0}\Omega_\lambda(A,B) = \zeta_{(A,B)} < +\infty$, then $\Omega_\lambda(\mathrm{Fix}(F), \mathrm{Fix}(G)) \leq (1-k)^{-1}\sup_{x\in X_\omega}\zeta_{(Fx,Gx)}$.

Proof. Suppose $\lambda > 0$, by Theorem 3.7, we can conclude that $\mathrm{Fix}(F) \neq \emptyset \neq \mathrm{Fix}(G)$. Let $\epsilon > 0$ be arbitrary, and let $\gamma > 0$ be such that $\gamma \sum_{n\in\mathbb{N}} nk^n < 1$. For $x_0 \in \mathrm{Fix}(F)$, choose $x_1 \in Gx_0$ such that

$$\omega_\lambda(x_0, x_1) \leq \Omega_\lambda(Fx_0, Gx_0) + \frac{\gamma\epsilon}{1-k}. \tag{4.13}$$

It is possible to choose $x_2 \in Gx_1$ such that

$$\omega_\lambda(x_1, x_2) \leq \Omega_\lambda(Gx_0, Gx_1) + \frac{\gamma\epsilon}{1-k}k. \tag{4.14}$$

By induction, we can construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\omega_\lambda(x_n, x_{n+1}) \leq \Omega_\lambda(Gx_{n-1}, Gx_n) + \frac{\gamma\epsilon}{1-k}k^n. \tag{4.15}$$

Observe that

$$\begin{aligned}
\omega_\lambda(x_n, x_{n+1}) &\leq \Omega_\lambda(Fx_{n-1}, Fx_n) + \frac{\gamma\epsilon}{1-k}k^n \\
&\leq k[\omega_{2\lambda}(x_{n-1}, Fx_{n-1}) + \omega_{2\lambda}(x_n, Fx_n)] + \frac{\gamma\epsilon}{1-\mathrm{k}}k^n \\
&\leq k[\omega_\lambda(x_{n-1}, Fx_{n-1}) + \omega_\lambda(x_n, Fx_n)] + \frac{\gamma\epsilon}{1-k}k^n \\
&\leq k[\omega_\lambda(x_{n-1}, Fx_{n-1}) + \omega_\lambda(x_n, x_{n+1})] + \frac{\gamma\epsilon}{1-k}k^n.
\end{aligned} \tag{4.16}$$

Further, set $\xi := k/(1-k) < 1$ and $\eta := \gamma\epsilon/(1-k)$, we obtain

$$\begin{aligned}
\omega_\lambda(x_n, x_{n+1}) &\leq \xi\omega_\lambda(x_{n-1}, x_n) + \frac{\gamma\epsilon}{(1-k)^2}k^n \\
&\leq \xi^2\omega_\lambda(x_{n-2}, x_{n-1}) + 2\frac{\gamma\epsilon}{(1-k)^3}k^n \\
&\ \ \vdots \\
&\leq \xi^n\omega_\lambda(x_0, x_1) + n\eta\xi^n.
\end{aligned} \tag{4.17}$$

Similar to the proof of Theorem 4.1, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. The completeness of X_ω implies that $\{x_n\}_{n\in\mathbb{N}}$ converges to some limit $x \in X_\omega$. We can further see that $x \in \mathrm{Fix}(G)$. Now, observe that

$$\begin{aligned}
\omega_\lambda(x_0, x) &\leq \omega_{\lambda/2n}(x_0, x_1) + \omega_{\lambda/2n}(x_1, x_2) + \cdots + \omega_{\lambda/2n}(x_{n-1}, x_n) + \omega_{\lambda/2}(x_n, x) \\
&\leq \sum_{m\in\mathbb{N}} \omega_{\lambda/2n}(x_{m-1}, x_m) + \omega_{\lambda/2}(x_n, x) \\
&\leq \omega_{\lambda/2n}(x_0, x_1)\sum_{m\in\mathbb{N}} k^{m-1} + \eta\sum_{m\in\mathbb{N}}(m-1)k^{m-1} + \omega_{\lambda/2}(x_n, x) \\
&\leq (1-k)^{-1}[\omega_{\lambda/2n}(\mathrm{x}_0, x_1) + \epsilon] + \omega_{\lambda/2}(x_n, x) \\
&\leq (1-k)^{-1}[\Omega_{\lambda/2n}(Fx_0, Gx_0) + 2\epsilon] + \omega_{\lambda/2}(x_n, x).
\end{aligned} \tag{4.18}$$

Since $\omega_\lambda(x_0, x) \geq \omega_\lambda(x_0, \mathrm{Fix}(G))$ and together with (4.18), we have, as $n \to +\infty$, that

$$\begin{aligned}\delta_\lambda(\mathrm{Fix}(F),\ \mathrm{Fix}(G)) &\leq (1-k)^{-1}\left[\sup_{x\in \mathrm{Fix}(F)} \zeta_{(Fx,Gx)} + 2\epsilon\right] \\ &\leq (1-k)^{-1}\left[\sup_{x\in X_\omega} \zeta_{(Fx,Gx)} + 2\epsilon\right].\end{aligned} \tag{4.19}$$

Similarly, we have

$$\delta_\lambda(\mathrm{Fix}(G),\ \mathrm{Fix}(F)) \leq (1-k)^{-1}\left[\sup_{x\in X_\omega} \zeta_{(Fx,Gx)} + 2\epsilon\right]. \tag{4.20}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof. □

Corollary 4.4. *Let X_ω be a complete modular metric space, and let $F_n : X_\omega \to \mathcal{CB}(X_\omega)$, for $n \in \mathbb{N}$, be multivalued mappings such that there exists $0 \leq k < 1$ such that*

$$\Omega_\lambda(F_n x, F_n y) \leq k\left[\omega_{2\lambda}(x, F_n x) + \omega_{2\lambda}(y, F_n y)\right] \tag{4.21}$$

for all $x, y \in X_\omega$ and $\lambda > 0$. Suppose for any $A, B \in \mathcal{CB}(X_\omega)$, $\lim_{\lambda\downarrow 0}\Omega_\lambda(A, B) = \zeta_{(A,B)} < +\infty$. If $\lim_{n\to+\infty}\zeta_{(F_n x, Fx)} = 0$ uniformly for $x \in X_\omega$, then $\lim_{n\to+\infty}\Omega_\lambda(\mathrm{Fix}(F_n), \mathrm{Fix}(F)) = 0$.

Proof. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to+\infty}\zeta_{(F_n x, Fx)} = 0$ uniformly for $x \in X_\omega$ and $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in X_\omega} \zeta_{(F_n x, Fx)} < (1-k)\epsilon \tag{4.22}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$. By Theorem 4.3, we have

$$\Omega_\lambda(\mathrm{Fix}(F_n), \mathrm{Fix}(F)) < \epsilon \tag{4.23}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$ and $\lambda > 0$. □

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