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Research Article

Application of Symbolic Computation in Nonlinear Differential-Difference Equations

Fuding Xie,[1,2] Zhen Wang,[3] and Min Ji[1]

[1] *Department of Computer Science, Liaoning Normal University, Liaoning, Dalian 116081, China*
[2] *Key Laboratory of Mathematics and Mechanization (KLMM), Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 100080, China*
[3] *School of Physics and Electronic Technology, Liaoning Normal University, Liaoning, Dalian 116029, China*

Correspondence should be addressed to Fuding Xie, xiefd@sohu.com

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A method is proposed to construct closed-form solutions of nonlinear differential-difference equations. For the variety of nonlinearities, this method only deals with such equations which are written in polynomials in function and its derivative. Some closed-form solutions of Hybrid lattice, Discrete mKdV lattice, and modified Volterra lattice are obtained by using the proposed method. The travelling wave solutions of nonlinear differential-difference equations in polynomial in function tanh are included in these solutions. This implies that the proposed method is more powerful than the one introduced by Baldwin et al. The results obtained in this paper show the validity of the proposal.

1. Introduction

Wadati [1] introduced the following nonlinear differential-difference equation (NDDE):

$$\frac{du_n(t)}{dt} = \left(\alpha + \beta u_n + \gamma u_n^2\right)(u_{n-1} - u_{n+1}), \tag{1.1}$$

where α, β, and $\gamma \neq 0$ are constants.

However, (1.1) can be thought as a discrete version of a nonlinear partial differential equation:

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx} = 0, \tag{1.2}$$

which can be solved by the inverse scattering method [2].

However, (1.1) obviously includes the following famous NDDEs, namely,

(a) hybrid lattice [3]:

$$\frac{du_n(t)}{dt} = \left(1 + \beta u_n + \gamma u_n^2\right)(u_{n-1} - u_{n+1}), \tag{1.3}$$

(b) discrete mKdV lattice [1, 4]:

$$\frac{du_n(t)}{dt} = \left(1 + u_n^2\right)(u_{n+1} - u_{n-1}), \tag{1.4}$$

(c) modified Volterra lattice [5]:

$$\frac{du_n(t)}{dt} = u_n^2(u_{n+1} - u_{n-1}). \tag{1.5}$$

The travelling wave solutions of (1.3) in polynomial in function tanh are reported in [3]. Ü. Göktas and W. Hereman [4] investigated the conservations laws of (1.4). However, (1.5) is a very well studied integrable model. It is a bi-Hamiltonian, possesses a Lax pair, recursion operator, local master-symmetry, infinite hierarchy of higher symmetries, and conservation laws [5]. In this study, searching for the closed-form solutions, especially solitary wave solutions and periodic solutions of (1.4) and (1.5), is considered.

In the theory of lattice-soliton, there existed several classical methods to seek for solutions of NDDEs, such as the inverse scattering method [6, 7], bilinear form [8, 9], symmetries [10], and numerical methods [11]. As far as we know, little work is being done to find closed-form solutions of NDDEs by using of symbolic computation. Baldwin et al. [3] recently presented an adaptation of the tanh-method to solve NDDEs. Some analytical (closed-form) solutions of several lattices in polynomial in function tanh have been obtained [3]. Their work may be thought as a breakthrough in solving NDDEs symbolically.

In this study, the method proposed in [3] where tanh-solutions are only considered is firstly generalized, and then is applied to solve (1.1). As a result, a variety of closed-form solutions of (1.1) have been found in terms of trigonometric and Jacobi elliptic functions. The proposed approach allows us to exactly solve NDDEs with the aid of symbolic computation. The solutions presented here not only cover the known one presented by Baldwin et al.[3] , but also introduce new solutions for some NDDEs.

The rest of the paper is organized as follows: in the following section, an improved method is proposed and how to obtain the solitary wave solutions and periodic solutions to NDDEs is depicted. Section 3 is devoted to illustrating the application of the proposal in exactly solving (1.1). As a result, some new solitary wave solutions and periodic solutions of Hybrid lattice, discrete mKdV lattice, and modified Volterra lattice have been obtained. The final is conclusions.

2. The Improved Method

To solve NDDEs directly, in this section, one would like to describe the improved method and its algorithm. Suppose the NDDE we study in this work is in the following form

$$P\Big(u_{n+p_1}(t), u_{n+p_2}(t), \ldots, u_{n+p_s}(t), u'_{n+p_1}(t), u'_{n+p_2}(t), \ldots, u'_{n+p_s}(t), \ldots, \\ u^{(k)}_{n+p_1}(t), u^{(k)}_{n+p_2}(t), \ldots, u^{(k)}_{n+p_s}(t)\Big) = 0, \quad (2.1)$$

where P is a polynomial; $u_n(t)$ is a dependent variable; t is a continuous variable; the superscript denotes the order of derivative; n, $p_i \in Z$.

To compute the travelling wave solutions of (2.1), we first set $u_n(t) = u(\xi_n)$ and

$$\xi_n = dn + ct + \xi_0, \quad (2.2)$$

where d and c are constants to be determined later and ξ_0 is a constant.

Step 1. We assume that the travelling wave solutions of (2.1) we are looking for are in the following frame:

$$u_n(t) = a_0 + \sum_{i=1}^{m} g^{i-1}(\xi_n)\left[a_i f(\xi_n) + b_j g(\xi_n)\right], \quad (2.3)$$

with

$$f(\xi_n) = \frac{\sinh(\xi_n)}{\cosh\ (\xi_n) + r}, \\ g(\xi_n) = \frac{1}{\cosh(\xi_n) + r}, \quad (2.4)$$

where a_i, b_j are all constants to be determined later and r a constant.

The degree m in (2.3) can be determined by balancing the highest nonlinear term and the highest-order derivative term in (2.1) as in the continuous case.

The functions $f(\xi_n)$ and $g(\xi_n)$ in (2.4) satisfy the following equations:

$$f'(\xi_n) = \left(1 + r^2\right) g^2(\xi_n) - r g(\xi_n), \quad g'(\xi_n) = -f(\xi_n) g(\xi_n), \\ f^2(\xi_n) = (1 - r g(\xi_n))^2 - g^2(\xi_n). \quad (2.5)$$

However, (2.5) reveals that $f^j(\xi_n)$ $(j \geq 2)$ can be expressed by $f^l(\xi_n) g^h(\xi_n)$ $(l = 0, 1)$.

Step 2 (to derive and solve the algebraic system). The following identity is easily obtained in terms of (2.2):

$$\xi_{n+p_i} = d(n + p_i) + ct + \xi_0 = \xi_n + dp_i. \quad (2.6)$$

It says the relationship between ξ_{n+p_i} and ξ_n. To find the solutions of NDDEs, one also utilizes the following formulae:

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y),$$
$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y). \tag{2.7}$$

If the ξ_n in (2.3) is replaced by ξ_{n+p_i}, then expression of u_{n+p_i} in $\sinh(\xi_n)$ and $\cosh(\xi_n)$ will be obtained in terms of (2.7).

With the aid of symbolic computation software Maple 8, we substitute u_{n+p_i} into (2.1) and apply the rule

$$\sinh^2(x) = \cosh^2(x) - 1, \tag{2.8}$$

to simplify the expression. Clearing the denominator and collecting the coefficients of $\cosh^k(\xi_n)\sinh^l(\xi_n)$ ($k = 0, 1, \ldots, h$; $l = 0, 1$) and setting them to zero, we can obtain the nonlinear algebraic equations. The values of unknowns will be found by using the Wu's method [12] to solve algebraic system.

Step 3 (construct and test the exact solutions). Substitute the values obtained in Step 2 along with (2.2) and (2.4) into (2.3), one can find the solutions of (2.1). To assure the correctness of the solutions, it is necessary to substitute them into the original equation.

Remark 2.1. If we set $r = 0$ in (2.4), the travelling wave solutions of (2.1) in polynomial in tanh and sech will be found. If one further requires that $b_j = 0$, then the polynomial travelling wave solutions in tanh will be obtained. In this sense, we can say that this method covers the one given in [3].

Remark 2.2. More important, if the hyperbolic functions in (2.4) are replaced by the trigonometric function, that is,

$$f(\xi_n) = \frac{\sin(\xi_n)}{\cos(\xi_n) + r},$$
$$g(\xi_n) = \frac{1}{\cos(\xi_n) + r}, \tag{2.9}$$

the periodic solutions to (2.1) will be obtained. While doing so, it is necessary to modify the formulae (2.7) and (2.8) properly.

Remark 2.3. We can obviously write (2.4) into the following solitary wave form which have physical relevance:

$$f(\xi_n) = \frac{\tanh(\xi_n)}{1 + r\,\mathrm{sech}(\xi_n)} = \frac{1}{\coth(\xi_n) + r\mathrm{csch}(\xi_n)},$$
$$g(\xi_n) = \frac{\mathrm{sech}(\xi_n)}{1 + r\,\mathrm{sech}(\xi_n)} = \frac{\mathrm{csch}(\xi_n)}{\coth(\xi_n) + r\mathrm{csch}(\xi_n)}. \tag{2.10}$$

In what follows, we will apply this method to solve (1.1). As a result, their abundant exact solutions have been derived.

3. Soliton Wave Solutions and Periodic Solutions of (1.1)

By balancing the highest nonlinear term u_n^2 and the highest-order derivative term $du_n(t)/dt$ in (1.1), we have $m+1=2m$, that is, $m=1$.

Therefore, the following formal solutions for (1.1) can be assumed:

$$u_n(t) = u(\xi_n) = a_0 + \frac{a_1 \sinh(\xi_n)}{\cosh(\xi_n)+r} + \frac{b_1}{\cosh(\xi_n)+r}, \tag{3.1}$$

with ξ_n as (2.2).

Starting from (2.6), we have

$$u_{n\pm1}(t) = u(\xi_{n\pm1}) = u(\xi_n \pm d). \tag{3.2}$$

The expressions of $u_{n+1}(t)$ and $u_{n-1}(t)$ in $\sinh(\xi_n)$ and $\cosh(\xi_n)$ are obtained by (2.7), (3.1), and (3.2). Substituting them and (3.1) into (1.1), clearing the denominator, and setting coefficients of the terms $\sinh^l(\xi_n)\cosh^k(\xi_n)$ $(k=0,1,\dots,3;\ l=0,1)$ to zero give

$$a_1\left[2\sinh(d)\left(\gamma r{a_1}^2+\beta a_0 r+\alpha r-\beta b_1-2\gamma a_0 b_1+\gamma {a_0}^2 r\right)+cr\right]=0,$$

$$2\sinh(d)\left(2\gamma a_0{a_1}^2 r-\gamma {a_1}^2 b_1-\alpha b_1-\beta a_0 b_1-\gamma {a_0}^2 b_1+\beta {a_1}^2 r\right)-b_1 c=0,$$

$$a_1\left[c+2\sinh(d)\cosh(d)\left(\gamma {a_1}^2+\gamma {a_0}^2+\beta a_0+\alpha\right)+4\sinh(d)\left(\gamma {a_0}^2 r^2-\gamma {b_1}^2+\beta a_0 r^2+\alpha r^2\right)\right.$$
$$\left.+2cr^2\cosh(d)\right]=0,$$

$$\sinh(d)\left(\beta {a_1}^2 r^2-2\gamma a_0 {b_1}^2-\beta {b_1}^2-2\gamma {a_0}^2 r b_1-2\alpha r b_1+2\gamma a_0 r^2 {a_1}^2-2\beta a_0 r b_1+\gamma {a_1}^2 b_1 r\right)$$
$$+2{a_1}^2\sinh(d)\cosh(d)\left(2\gamma\ a_0+\beta\right)-rb_1 c\cosh(d)=0,$$

$$a_1\left[2\sinh(d)\cosh(d)\left(2\alpha r+\beta b_1+2\beta\ a_0 r+\gamma a_0 b_1+\gamma {a_0}^2 r\right)\right.$$
$$+2\sinh(d)\left(-\gamma {a_1}^2 r+2\gamma a_0 b_1+\gamma {b_1}^2 r+\gamma {a_0}^2 r^3+\beta b_1 r^2+\alpha r^3+\beta a_0 r^3+\beta b_1+2\gamma a_0 r^2 b_1\right)$$
$$\left.+2cr\cosh(d)+cr\ \cosh^2(d)+cr^3-cr\right]=0,$$

$$2b_1\sinh(d)\left(-\gamma {b_1}^2-\gamma {a_0}^2 r^2+\gamma {a_1}^2-\alpha r^2-\beta a_0 r^2-2\gamma a_0 r b_1-\beta b_1 r\right)$$
$$+2{a_1}^2\sinh(d)\cosh(d)\left(2\gamma a_0 r+2\gamma b_1+\beta r\right)-b_1 c\ \cosh^2(d)+b_1 c-b_1 cr^2=0,$$

$$a_1\left[2\sinh(d)\cosh(d)\left(\beta a_0 r^2-\gamma {a_1}^2+\beta b_1 r+\gamma {a_0}^2 r^2+2\gamma a_0 r b_1+\alpha r^2+\gamma {b_1}^2\right)\right.$$
$$\left.+2b_1\sinh(d)(2\gamma b_1+\beta r)+4\gamma a_0 r a_1 b_1 a_1 c\cosh^2(d)+cr^2-c\right]=0. \tag{3.3}$$

It is difficult to solve this system by hand. Thus, we fall back on symbolic computation software Maple 8 and Wu's method which is a powerful tool to deal with nonlinear algebraic equations. With the aid of them, we can find the solutions to the above system as follows:

Case 1.

$$a_1=0,\qquad a_0=-\frac{\beta}{2\gamma}\mp\frac{r\sqrt{(\beta^2-4\alpha\gamma)(r^2-1)}\tanh(d/2)}{2\gamma(r^2-1)}$$
$$b_1=\pm\frac{\left[2r^2-1-\cosh(d)\right]\sqrt{(\beta^2-4\alpha\gamma)(r^2-1)}\tanh(d/2)}{2\gamma(r^2-1)}, \tag{3.4}$$
$$c=\frac{\left[2r^2-1-\cosh(d)\right](\beta^2-4\alpha\gamma)\tanh(d/2)}{2\gamma(r^2-1)};$$

Case 2.

$$a_0=-\frac{\beta}{2\gamma},\qquad a_1=\pm\frac{\sqrt{\beta^2-4\alpha\gamma}\tanh(d/2)}{2\gamma},$$
$$b_1=\pm\frac{\sqrt{(\beta^2-4\alpha\gamma)(r^2-1)}\ \tanh(d/2)}{2\gamma},\qquad c=\frac{(\beta^2-4\alpha\gamma)\tanh(d/2)}{\gamma}; \tag{3.5}$$

Case 3.

$$b_1=0,\qquad r=0,\qquad a_0=-\frac{\beta}{2\gamma},$$
$$a_1=\pm\frac{\sqrt{\beta^2-4\alpha\gamma}\tanh(d)}{2\gamma},\qquad c=\frac{(\beta^2-4\alpha\gamma)\tanh(d)}{2\gamma}. \tag{3.6}$$

Thus the solitary wave solutions of (1.1) are

$$u_n(t)=\frac{\beta}{2\gamma}\mp\frac{\sqrt{(\beta^2-4\alpha\gamma)(r^2-1)}\ \tanh(d/2)}{2\gamma(r^2-1)}\left(r-\frac{2r^2-1-\cosh(d)}{\cosh(\xi_n)+r}\right), \tag{3.7}$$

where $\xi_n = nd + ([2r^2 - 1 - \cosh(d)](\beta^2 - 4\alpha\gamma)\tanh(d/2)/2\gamma(r^2 - 1))t + \xi_0$;

$$u_n(t) = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}\ \tanh(d/2)}{2\gamma} \frac{\sinh(\xi_n) + \sqrt{r^2 - 1}}{\cosh(\xi_n) + r}, \tag{3.8}$$

where $\xi_n = nd + ((\beta^2 - 4\alpha\gamma)\tanh(d/2)/\gamma)t + \xi_0$;

$$u_n(t) = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}\ \tanh(d)}{2\gamma} \tanh\left(nd + \frac{(\beta^2 - 4\alpha\gamma)\tanh(d)}{2\gamma}t + \xi_0\right). \tag{3.9}$$

Similarly, we can also assume that (1.1) possesses the following form solution:

$$u_n(t) = s_0 + \frac{s_1 \sin(\xi_n)}{\cos(\xi_n) + r} + \frac{s_2}{\cos(\xi_n) + r}, \tag{3.10}$$

where s_i $(i = 0, 1, 2)$ is a constant to be determined later.

Repeating the above process and properly modifying the formulae (2.7) and (2.8), we can derive the periodic solutions for (1.1) as follows. For brevity, the procedure of seeking for periodic solutions to (1.1) is omitted:

$$u_n(t) = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{(\beta^2 - 4\alpha\gamma)(r^2 - 1)}\tan(d/2)}{2\gamma(r^2 - 1)} \left(r + \frac{1 + \cosh(d) - 2r^2}{\cos(\xi_n) + r}\right), \tag{3.11}$$

where $\xi_n = nd + ([2r^2 - 1 - \cos(d)](\beta^2 - 4\alpha\gamma)\tan(d/2)/2\gamma(r^2 - 1))t + \xi_0$;

$$u_n(t) = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tan(d/2)}{2\gamma} \frac{\sin(\xi_n) + \sqrt{r^2 - 1}}{\cos(\xi_n) + r}, \tag{3.12}$$

where $\xi_n = nd + ((\beta^2 - 4\alpha\gamma)\tan(d/2)/\gamma)t + \xi_0$;

$$u_n(t) = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tan(d)}{2\gamma} \tan\left(nd + \frac{(\beta^2 - 4\alpha\gamma)\tan(d)}{2\gamma}t + \xi_0\right). \tag{3.13}$$

If we properly set the parameters α, β, and γ, then it is easily to obtain closed-form solutions of Hybrid lattice, discrete mKdV lattice, and modified Volterra lattice, respectively.

In fact, the solution given in [3] for (1.3) is nothing but the solution (3.9). As far as we know, no works have been reported on the closed form solutions of (1.4) and (1.5). Thus, in this sense, these results are novel.

4. Conclusions

A method is proposed to find the solitary wave solutions and periodic solutions for NDDEs. The new closed-form solutions of Hybrid lattice, discrete mKdV lattice, and modified Volterra lattice have been found by using the proposal and symbolic computation. This study reveals that computer algebra plays an important role in exactly solving NDDEs. Meanwhile, it is worthwhile to point out that the proposed method may be applicable to other NDDEs to seek for their travelling wave solutions.

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