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Research Article

Some Basic Properties of Certain New Subclass of Meromorphic Functions

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We introduce and investigate a new subclass $\mathscr{M}(\beta,\eta)$ of meromorphic functions. Some interesting properties such as inclusion relationship, coefficient estimates, neighborhoods, and partial sums are proved. Connections of the results with known results are also considered.

1. Introduction

Let Σ denote the class of functions f of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k, \tag{1}$$

which are analytic in the punctured open unit disk:

$$\mathbb{U}^{*}:=\{z:z\in\mathbb{C},0<|z|<1\}=:\mathbb{U}\setminus\{0\}. \tag{2}$$

Let $\mathscr{P}$ denote the class of functions of the form

$$p(z)=1+\sum_{k=1}^{\infty}p_kz^k, \tag{3}$$

which are analytic $\mathbb{U}$ and satisfy the condition

$$\Re(p(z))>0\quad(z\in\mathbb{U}). \tag{4}$$

A function $f\in\Sigma$ is said to be in the class $\mathscr{MS}^{*}(\alpha)$ of meromorphic starlike functions of order α if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right)<-\alpha\quad(z\in\mathbb{U}^{*};0\le\alpha<1). \tag{5}$$

For $\eta>1$, Wang et al. [1] (see also Nehari and Netanyahu [2]) introduced and studied a new subclass $\mathscr{M}(\eta)$ of Σ consisting of functions f satisfying

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>-\eta\quad(z\in\mathbb{U}^{*}). \tag{6}$$

We note that meromorphic starlike functions and related topics attract many authors' attentions; see (for example) the earlier works [3–8] and the references cited therein.

Let

$$\mathfrak{f}(z)=z+\sum_{k=m+1}^{\infty}a_kz^k\quad(m\in\mathbb{N}:=\{1,2,3\ldots\}) \tag{7}$$

be analytic in $\mathbb{U}$. Assuming that $\alpha\in\mathbb{C}$ and $0\le\beta<1$, we say that a function $f\in\mathscr{H}_m(\alpha,\beta)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}+\alpha\frac{zf''(z)}{f(z)}\right)>\alpha\beta\left(\beta+\frac{m}{2}-1\right)+\beta-\frac{m}{2}\quad(z\in\mathbb{U}). \tag{8}$$

The function class $\mathscr{H}_m(\alpha,\beta)$ was introduced and studied recently by Ravichandran et al. [9], Liu et al. [10], Singh and Gupta [11], and Wang et al. [12].

In [13], Wang et al. introduced a subclass of meromorphic function $\widetilde{\mathscr{H}}(\beta,\lambda)$ which satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}+\beta\frac{z^2f''(z)}{f(z)}\right)<\beta\lambda\left(\lambda+\frac{1}{2}\right)+\frac{\beta}{2}-\lambda \qquad \left(\beta\geq 0;\frac{1}{2}\leq\lambda<1;z\in\mathbb{U}^*\right). \tag{9}$$

It was proved that the class $\widetilde{\mathscr{H}}(\beta,\lambda)$ is a subclass of the $\mathscr{MS}^*(\lambda)$ of meromorphically starlike functions of order λ.

Motivated essentially by the above works, we introduce and investigate a new subclass of Σ of meromorphic functions.

Definition 1. Suppose that $\eta>1$. Let $\mathscr{M}(\beta,\eta)$ denote a subclass of Σ consisting of functions satisfying the condition that

$$\Re\left(\frac{zf'(z)}{f(z)}+\beta\frac{z^2f''(z)}{f(z)}\right)>\frac{1}{2}\beta\left(2\eta^2+\eta+1\right)-\eta \qquad \left(z\in\mathbb{U}^*\right). \tag{10}$$

We note that, for $\beta=0$, the class $\mathscr{M}(0,\eta)$ reduces to $\mathscr{M}(\eta)$.

In the present paper, we aim at proving some interesting properties such as inclusion relationship, coefficient estimates, neighborhoods, and partial sums for functions in the class $\mathscr{M}(\beta,\eta)$.

The following lemmas will be required in our investigation.

Lemma 2 (see [14]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that Φ is mapping from $\mathbb{C}^2\times\mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix,y;z)\notin\Omega$ for $z\in\mathbb{U}$ and for all real x, y such that $y\leq-(1+x^2)/2$. If the function $\psi(z)=1+c_1z+c_2z^2+\cdots$ is analytic in $\mathbb{U}$ and $\Phi(\psi(z),z\psi'(z);z)\in\Omega$ for all $z\in\mathbb{U}$, then $\Re(\psi(z))>0$.*

Lemma 3. *Let $\eta>1$, $0\leq\beta<1$, and $-1+2\beta+\gamma>0$. Suppose also that the sequence $\{A_k\}_{k=1}^{\infty}$ is defined by*

$$A_1=\frac{-1+2\beta+\gamma}{1-\beta},\qquad A_k=\frac{2(-1+2\beta+\gamma)}{(k+1)[1+(k-2)\beta]}\left[1+\sum_{m=1}^{k-1}A_m\right]. \tag{11}$$

Then

$$A_k=\frac{-1+2\beta+\gamma}{1-\beta}\cdot\prod_{m=1}^{k-1}\frac{(m+1)[1+(m-2)\beta]-2(1-2\beta-\gamma)}{(m+2)[1+(m-1)\beta]} \qquad (k\geq 2). \tag{12}$$

Proof. From (11), we have

$$(k+1)[1+(k-2)\beta]A_k=2(-1+2\beta+\gamma)\left[1+\sum_{m=1}^{k-1}A_m\right],$$
$$(k+2)[1+(k-1)\beta]A_{k+1}=2(-1+2\beta+\gamma)\left[1+\sum_{m=1}^{k}A_m\right]. \tag{13}$$

Combining (13), we find that

$$\frac{A_{k+1}}{A_k}=\frac{(k+1)[1+(k-2)\beta]+2(-1+2\beta+\gamma)}{(k+2)[1+(k-1)\beta]} \qquad (k\in\mathbb{N}). \tag{14}$$

Thus, for $k\geq 2$, we deduce from (14) that

$$\begin{aligned}A_k&=\frac{A_k}{A_{k-1}}\times\frac{A_{k-1}}{A_{k-2}}\times\cdots\times\frac{A_2}{A_1}\times A_1\\&=\frac{k[1+(k-3)\beta]+2(-1+2\beta+\gamma)}{(k+1)[1+(k-2)\beta]}\\&\quad\times\frac{(k-1)[1+(k-4)\beta]+2(-1+2\beta+\gamma)}{k[1+(k-3)\beta]}\\&\quad\times\cdots\times\frac{2(1-\beta)+2(-1+2\beta+\gamma)}{3[1+0\beta]}\times\frac{-1+2\beta+\gamma}{1-\beta}\\&=\frac{-1+2\beta+\gamma}{1-\beta}\\&\quad\cdot\prod_{m=1}^{k-1}\frac{(m+1)[1+(m-2)\beta]-2(1-2\beta-\gamma)}{(m+2)[1+(m-1)\beta]}.\end{aligned} \tag{15}$$

This completes the proof of Lemma 3. □

Lemma 4. *Let*

$$\eta>1,\quad 0\leq\beta<\frac{2\eta-2}{2\eta^2+\eta-1}. \tag{16}$$

Suppose also that $f\in\Sigma$ is given by (1) and

$$\sum_{k=1}^{\infty}[k+\beta k(k-1)+\gamma]|a_k|\leq\gamma-1, \tag{17}$$

where (and throughout this paper unless otherwise mentioned) the parameter γ is defined as

$$\gamma:=\eta-\frac{1}{2}\beta\left(2\eta^2+\eta+1\right). \tag{18}$$

Then $f\in\mathscr{M}(\beta,\eta)$.

The proof of Lemma 4 is similar to that of Theorem 1 in Wang et al. [1] and so is omitted.

2. Main Results

We begin by proving the following result which shows that $\mathcal{M}(\beta,\eta)$ is a subclass of $\mathcal{M}(\eta)$.

Theorem 5. *Suppose that $\eta > 1$ and $\beta \geq 0$. Then*

$$\mathcal{M}(\beta,\eta) \subset \mathcal{M}(\eta). \tag{19}$$

Proof. Define

$$\rho(z) := \frac{zf'(z)/f(z)+\eta}{\eta-1} \quad (\eta > 1; z \in \mathbb{U}). \tag{20}$$

Then ρ is analytic in $\mathbb{U}$. It follows from (20) that

$$-\frac{zf''(z)}{f'(z)} = \eta+1-(\eta-1)\rho(z)+\frac{(\eta-1)z\rho'(z)}{\eta-(\eta-1)\rho(z)}. \tag{21}$$

Combining (20) and (21), we obtain that

$$\begin{aligned}\frac{zf'(z)}{f(z)}&\left(\beta\frac{zf''(z)}{f'(z)}+1\right)\\ &= \beta(\eta-1)z\rho'(z)+\beta(\eta-1)^2\rho^2(z)\\ &\quad-(\eta-1)(2\beta\eta+\beta-1)\rho(z)+\eta(\beta\eta+\beta-1)\\ &= \Phi\left(\rho(z),z\rho'(z);z\right),\end{aligned} \tag{22}$$

where

$$\begin{aligned}\Phi(r,s;t) &= \beta(\eta-1)s+\beta(\eta-1)^2r^2\\ &\quad-(\eta-1)(2\beta\eta+\beta-1)r+\eta(\beta\eta+\beta-1).\end{aligned} \tag{23}$$

For all real x and y satisfying $y \leq -(1+x^2)/2$, we have

$$\begin{aligned}\mathfrak{R}&\{\phi(ix,y;z)\}\\ &= \beta(\eta-1)y-\beta(\eta-1)^2x^2+\eta(\beta\eta+\beta-1)\\ &\leq -\beta(\eta-1)\frac{1+x^2}{2}-\beta(\eta-1)^2x^2\\ &\quad+\eta(\beta\eta+\beta-1) \leq \frac{1}{2}\beta\left(2\eta^2+\eta+1\right)-\eta.\end{aligned} \tag{24}$$

If we set

$$\Omega = \left\{\xi : \mathfrak{R}(\xi) > \frac{1}{2}\beta\left(2\eta^2+\eta+1\right)-\eta\right\}, \tag{25}$$

then $\Phi(ix,y;z) \notin \Omega$ for all real x, y such that $y \leq -(1+x^2)/2$. Moreover, from definition (10), we know that $\Phi(\rho(z),z\rho'(z);z) \in \Omega$. Using Lemma 2, we conclude that $\mathfrak{R}(\rho(z)) > 0$ for all $z \in \mathbb{U}$, which implies that $f \in \mathcal{M}(\eta)$. This completes the proof of Theorem 5. □

Now we consider the coefficient estimates for functions belonging to the class $\mathcal{M}(\beta,\eta)$.

Theorem 6. *Suppose that*

$$\eta > 1, \quad 0 \leq \beta < \frac{2\eta-2}{2\eta^2+\eta-1}. \tag{26}$$

If $f \in \mathcal{M}(\beta,\eta)$, then

$$\begin{aligned}|a_1| &\leq \frac{-1+2\beta+\gamma}{1-\beta},\\ |a_k| &\leq \frac{-1+2\beta+\gamma}{1-\beta}\\ &\quad\cdot\prod_{m=1}^{k-1}\frac{(m+1)[1+(m-2)\beta]-2(1-2\beta-\gamma)}{(m+2)[1+(m-1)\beta]}\\ &\qquad (k \in \mathbb{N}\setminus\{1\}).\end{aligned} \tag{27}$$

Proof. Suppose that $f \in \mathcal{M}(\beta,\eta)$. Then there exists $\tau \in \mathcal{P}$ such that

$$\frac{zf'(z)}{f(z)}+\beta\frac{z^2f''(z)}{f(z)}+\gamma = (-1+2\beta+\gamma)\tau(z) \quad (z \in \mathbb{U}^*). \tag{28}$$

It follows from (28) that

$$zf'(z)+\beta z^2f''(z) = [(-1+2\beta+\gamma)\tau(z)-\gamma]f(z). \tag{29}$$

Combining (1) and (29), we have

$$\begin{aligned}&\left(-\frac{1}{z}+\sum_{k=1}^{\infty}ka_kz^k\right)+\beta\left(\frac{2}{z}+\sum_{k=1}^{\infty}k(k-1)a_kz^k\right)\\ &= \left[(-1+2\beta+\gamma)\left(1+\sum_{k=1}^{\infty}\tau_kz^k\right)-\gamma\right]\cdot\left(\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k\right).\end{aligned} \tag{30}$$

Evaluating the coefficient of z^n in both sides of (30) yields

$$2(1-\beta)a_1 = (-1+2\beta+\gamma)\tau_2, \tag{31}$$

$$\begin{aligned}&(k+1)[1+(k-2)\beta]a_k\\ &= (-1+2\beta+\gamma)\left(\tau_{k+1}+\sum_{l=1}^{k-1}\tau_{k-l}a_l\right).\end{aligned} \tag{32}$$

By observing the fact that $|\tau_k| \leq 2$ for $k \in \mathbb{N}$, we find from (31) and (32) that

$$\begin{aligned}|a_1| &\leq \frac{-1+2\beta+\gamma}{1-\beta},\\ |a_k| &\leq \frac{2(-1+2\beta+\gamma)}{(k+1)[1+(k-2)\beta]}\left[1+\sum_{m=1}^{k-1}|a_m|\right]\\ &\qquad (k \geq 2).\end{aligned} \tag{33}$$

Now we define the sequence $\{A_k\}_{k=1}^{\infty}$ as follows:

$$A_1 = \frac{-1+2\beta+\gamma}{1-\beta},$$
$$A_k = \frac{2(-1+2\beta+\gamma)}{(k+1)[1+(k-2)\beta]}\left[1+\sum_{m=1}^{k-1}A_m\right]. \tag{34}$$

In order to prove that

$$|a_k| \le A_k \quad (k \in \mathbb{N}), \tag{35}$$

we use the principle of mathematical induction by noting that

$$|a_1| \le A_1 = \frac{-1+2\beta+\gamma}{1-\beta}. \tag{36}$$

Therefore, we assume that

$$|a_m| \le A_m \quad (m = 1,2,\ldots,k; k \in \mathbb{N}). \tag{37}$$

Combining (32) and (33), we get

$$\begin{aligned}|a_{k+1}| &\le \frac{2(-1+2\beta+\gamma)}{(k+1)[1+(k-2)\beta]}\left[1+\sum_{m=1}^{k}|a_m|\right] \\ &\le \frac{2(-1+2\beta+\gamma)}{(k+1)[1+(k-2)\beta]}\left[1+\sum_{m=1}^{k}A_m\right] = A_{k+1}.\end{aligned} \tag{38}$$

Hence, by the principle of mathematical induction, we have

$$|a_k| \le A_k \quad (k \in \mathbb{N}) \tag{39}$$

as desired. By means of Lemma 2 and (33), we know that (12) holds. Combining (39) and (12), we readily get the coefficient estimates asserted by Theorem 6. □

Using Lemma 4, we introduce the δ-neighborhood of a function $f \in \Sigma$ of the form (1) by means of the following definition:

$$\mathcal{N}_\delta(f) := \left\{ g \in \Sigma : g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \right.$$
$$\left. \sum_{k=1}^{\infty} \frac{k+\beta k(k-1)+\gamma}{\gamma-1}|a_k - b_k| \le \delta \ (\delta \ge 0) \right\}. \tag{40}$$

By making use of definition (40), we obtain the following result.

Theorem 7. *If $f \in \Sigma$ satisfies the condition*

$$\frac{f(z)+\varepsilon z^{-1}}{1+\varepsilon} \in \mathcal{M}(\beta,\eta) \quad (\varepsilon \in \mathbb{C}; |\varepsilon| < \delta; \delta > 0), \tag{41}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{M}(\beta,\eta). \tag{42}$$

Proof. It is easily seen from (10) that a function $g \in \mathcal{M}(\beta,\eta)$ if and only if

$$\frac{zg'(z)+\beta z^2 g''(z)+g(z)}{zg'(z)+\beta z^2 g''(z)+(2\gamma-1)g(z)} \ne \sigma$$
$$(z \in \mathbb{U}; \sigma \in \mathbb{C}; |\sigma| = 1), \tag{43}$$

which is equivalent to

$$\frac{(g*\varrho)(z)}{z^{-1}} \ne 0 \quad (z \in \mathbb{U}), \tag{44}$$

where

$$\varrho(z) = \frac{1}{z} + \sum_{k=1}^{\infty} c_k z^k$$
$$\left(c_k := \frac{k+\beta k(k-1)+1-[k+\beta k(k-1)+(2\gamma-1)]\sigma}{2\beta+(-2+2\beta+2\gamma)\sigma}\right). \tag{45}$$

It follows from (45) that

$$\begin{aligned}|c_k| &= \left|\frac{k+\beta k(k-1)+1-[k+\beta k(k-1)+(2\gamma-1)]\sigma}{2\beta+(-2+2\beta+2\gamma)\sigma}\right| \\ &\le \frac{k+\beta k(k-1)+1+[k+\beta k(k-1)+(2\gamma-1)]|\sigma|}{(-2+2\beta+2\gamma)|\sigma|-2\beta} \\ &= \frac{k+\beta k(k-1)+\gamma}{\gamma-1} \quad (|\sigma| = 1).\end{aligned} \tag{46}$$

Furthermore, under the hypotheses of Theorem 7, (44) yields the following inequality:

$$\left|\frac{(f*\varrho)(z)}{z^{-1}}\right| \ge \delta \quad (z \in \mathbb{U}; \delta > 0). \tag{47}$$

Suppose that

$$\chi(z) = \frac{1}{z} + \sum_{k=1}^{\infty} d_k z^k \in \mathcal{N}_\delta(f). \tag{48}$$

It follows from (40) that

$$\begin{aligned}\left|\frac{((f-\chi)*\varrho)(z)}{z^{-1}}\right| &= \left|\sum_{k=1}^{\infty}(a_k-d_k)c_k z^{k+1}\right| \\ &\le |z|\sum_{k=1}^{\infty}\frac{k+\beta k(k-1)+\gamma}{\gamma-1}|a_k-d_k| < \delta.\end{aligned} \tag{49}$$

Combining (47) and (49), we have

$$\begin{aligned}\left|\frac{(\chi*\varrho)(z)}{z^{-1}}\right| &= \left|\frac{([f+(\chi-f)]*\varrho)(z)}{z^{-1}}\right| \\ &\ge \left|\frac{(f*\varrho)(z)}{z^{-1}}\right| - \left|\frac{((\chi-f)*\varrho)(z)}{z^{-1}}\right| > 0,\end{aligned} \tag{50}$$

which implies that

$$\frac{(\chi * \varrho)(z)}{z^{-1}} \neq 0 \quad (z \in \mathbb{U}). \tag{51}$$

Thus, we have

$$\chi(z) \in \mathcal{N}_{\delta}(f) \subset \mathcal{M}(\beta, \eta). \tag{52}$$

This completes the proof of Theorem 7. □

Finally, we derive the partial sums of functions in the class $\mathcal{M}(\beta, \eta)$.

Theorem 8. *Let $f \in \Sigma$ be given by (1) and define the partial sums $f_n(z)$ of f by*

$$f_n(z) = \frac{1}{z} + \sum_{k=1}^{n} a_k z^k \quad (n \in \mathbb{N}). \tag{53}$$

Suppose also that

$$\sum_{k=1}^{\infty} \frac{k + \beta k(k-1) + \gamma}{\gamma - 1} |a_k| \leq 1. \tag{54}$$

Then

(1) $f \in \mathcal{M}(\beta, \eta)$;

(2)

$$\Re\left(\frac{f(z)}{f_n(z)}\right) \geq \frac{n + \beta n(n+1) + 2}{n + \beta n(n+1) + 1 + \gamma} \quad (n \in \mathbb{N}; z \in \mathbb{U}), \tag{55}$$

$$\Re\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{n + \beta n(n+1) + 1 + \gamma}{n + \beta n(n+1) + 2\gamma} \quad (n \in \mathbb{N}; z \in \mathbb{U}). \tag{56}$$

Each of the bounds in (55) and (56) is the best possible for each $n \in \mathbb{N}$.

Proof. (1) It is easy to see that the result follows directly from Lemma 4.

(2) Note that

$$\frac{n + 1 + \beta n(n+1) + \gamma}{\gamma - 1} > \frac{n + \beta n(n-1) + \gamma}{\gamma - 1} > 1 \quad (n \in \mathbb{N}). \tag{57}$$

Thus, we have

$$\begin{aligned} &\sum_{k=1}^{n} |a_k| + \frac{n + \beta n(n+1) + 1 + \gamma}{\gamma - 1} \sum_{k=n+1}^{\infty} |a_k| \\ &\quad \leq \sum_{k=1}^{\infty} \frac{k + \beta k(k-1) + \gamma}{\gamma - 1} |a_k| \leq 1. \end{aligned} \tag{58}$$

By setting

$$\begin{aligned} h_1(z) &= \frac{n + \beta n(n+1) + 1 + \gamma}{\gamma - 1} \\ &\quad \cdot \left(\frac{f(z)}{f_n(z)} - \frac{n + \beta n(n+1) + 2}{n + \beta n(n+1) + 1 + \gamma}\right) \\ &= 1 + \frac{((n + \beta n(n+1) + 1 + \gamma)/(\gamma - 1)) \sum_{k=n+1}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{n} a_k z^{k+1}}, \end{aligned} \tag{59}$$

we find from (58) and (59) that

$$\begin{aligned} &\left|\frac{h_1(z) - 1}{h_1(z) + 1}\right| \\ &\leq \frac{((n + \beta n(n+1) + 1 + \gamma)/(\gamma - 1)) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2\sum_{k=1}^{n} |a_k| - ((n + \beta n(n+1) + 1 + \gamma)/(\gamma - 1)) \sum_{k=n+1}^{\infty} |a_k|} \\ &\leq 1 \quad (z \in \mathbb{U}), \end{aligned} \tag{60}$$

which implies inequality (55).

If we put

$$f(z) = \frac{1}{z} - \frac{\gamma - 1}{n + \beta n(n+1) + 1 + \gamma} z^{n+1}, \tag{61}$$

then

$$\begin{aligned} \frac{f(z)}{f_n(z)} &= 1 - \frac{\gamma - 1}{n + \beta n(n+1) + 1 + \gamma} z^{n+2} \\ &\longrightarrow \frac{n + \beta n(n+1) + 2}{n + \beta n(n+1) + 1 + \gamma} \\ &\qquad (z \longrightarrow 1^-), \end{aligned} \tag{62}$$

which shows that the bound in (55) is the best possible for each $n \in \mathbb{N}$.

Now, we set

$$\begin{aligned} h_2(z) &= \frac{n + \beta n(n+1) + 2\gamma}{\gamma - 1} \\ &\quad \cdot \left(\frac{f_n(z)}{f(z)} - \frac{n + \beta n(n+1) + 1 + \gamma}{n + \beta n(n+1) + 2\gamma}\right) \\ &= 1 - \frac{((n + \beta n(n+1) + 2\gamma)/(\gamma - 1)) \sum_{k=n+1}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{\infty} a_k z^{k+1}}. \end{aligned} \tag{63}$$

In view of (58) and (63), we conclude that

$$\begin{aligned} &\left|\frac{h_2(z) - 1}{h_2(z) + 1}\right| \\ &\leq \frac{((n + \beta n(n+1) + 2\gamma)/(\gamma - 1)) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2\sum_{k=1}^{n} |a_k| - ((n + \beta n(n+1) + 2)/(\gamma - 1)) \sum_{k=n+1}^{\infty} |a_k|} \\ &\leq 1 \quad (z \in \mathbb{U}), \end{aligned} \tag{64}$$

which leads to inequality (56) asserted in Theorem 8. The bound in (56) is sharp with the extremal function f given by (61). We thus complete the proof of Theorem 8. □

In what follows, we turn to quotients involving derivatives. The proof of Theorem 9 is similar to that of Theorem 8 and so the details may be omitted.

Theorem 9. *Let $f \in \Sigma$ be given by (1) and define the partial sums $f_n(z)$ of f by (53). If the condition (54) holds, then*

$$\Re\left(\frac{f'(z)}{f_n'(z)}\right) \geq \frac{\beta n(n+1) - n\gamma}{n + \beta n(n+1) + 1 + \gamma} \quad (n \in \mathbb{N}; z \in \mathbb{U}),$$
$$\Re\left(\frac{f_n'(z)}{f'(z)}\right) \geq \frac{n + \beta n(n+1) + 1 + \gamma}{\beta n(n+1) + (n+2)\gamma} \quad (n \in \mathbb{N}; z \in \mathbb{U}). \tag{65}$$

The bounds in (65) are sharp with the extremal function given by (61).

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

The authors jointly worked on the results and they read and approved the final paper.

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