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Research Article

On Prime Near-Rings with Generalized Derivation

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Let N be a 3-prime 2-torsion-free zero-symmetric left near-ring with multiplicative center Z. We prove that if N admits a nonzero generalized derivation f such that $f(N) \subseteq Z$, then N is a commutative ring. We also discuss some related properties.

1. Introduction

Let N be a zero-symmetric left near-ring, not necessarily with a multiplicative identity element; and let Z be its multiplicative center. Define N to be 3-prime if for all $a, b \in N\setminus\{0\}$, $aNb \neq \{0\}$; and call N 2-torsion-free if $(N,+)$ has no elements of order 2. A derivation on N is an additive endomorphism D of N such that $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$. A generalized derivation f with associated derivation D is an additive endomorphism $f : N \to N$ such that $f(xy) = f(x)y + xD(y)$ for all $x, y \in N$. In the case of rings, generalized derivations have received significant attention in recent years.

In [1], we proved the following.

Theorem A. *If N is 3-prime and 2-torsion-free and D is a derivation such that $D^2 = 0$, then $D = 0$.*

Theorem B. *If N is a 3-prime 2-torsion-free near-ring which admits a nonzero derivation D for which $D(N) \subseteq Z$, then N is a commutative ring.*

Theorem C. *If N is a 3-prime 2-torsion-free near-ring admitting a nonzero derivation D such that $D(x)D(y) = D(y)D(x)$ for all $x, y \in N$, then N is a commutative ring.*

In this paper, we investigate possible analogs of these results, where D is replaced by a generalized derivation f.

We will need three easy lemmas.

Lemma 1.1 (see [1, Lemma 3]). *Let N be a 3-prime near-ring.*

(i) *If $z \in Z \setminus \{0\}$, then z is not a zero divisor.*

(ii) *If $Z \setminus \{0\}$ contains an element z such that $z + z \in Z$, then $(N, +)$ is abelian.*

(iii) *If D is a nonzero derivation and $x \in N$ is such that $xD(N) = \{0\}$ or $D(N)x = \{0\}$, then $x = 0$.*

Lemma 1.2 (see [2, Proposition 1]). *If N is an arbitrary near-ring and D is a derivation on N, then $D(xy) = D(x)y + xD(y)$ for all $x, y \in N$.*

Lemma 1.3. *Let N be an arbitrary near-ring and let f be a generalized derivation on N with associated derivation D. Then*

$$(f(a)b + aD(b))c = f(a)bc + aD(b)c \quad \forall a, b, c \in N. \tag{1.1}$$

Proof. Clearly $f((ab)c) = f(ab)c + abD(c) = (f(a)b + aD(b))c + abD(c)$; and by using Lemma 1.2, we obtain $f(a(bc)) = f(a)bc + aD(bc) = f(a)bc + aD(b)c + abD(c)$.

Comparing these two expressions for $f(abc)$ gives the desired conclusion. □

2. The main theorem

Our best result is an extension of Theorem B.

Theorem 2.1. *Let N be a 3-prime 2-torsion-free near-ring. If N admits a nonzero generalized derivation f such that $f(N) \subseteq Z$, then N is a commutative ring.*

In the proof of this theorem, as well as in a later proof, we make use of a further lemma.

Lemma 2.2. *Let R be a 3-prime near-ring, and let f be a generalized derivation with associated derivation $D \neq 0$. If $D(f(N)) = \{0\}$, then $f(D(N)) = \{0\}$.*

Proof. We are assuming that $D(f(x)) = 0$ for all $x \in N$. It follows that $D(f(xy)) = D(f(x)y) + D(xD(y)) = 0$ for all $x, y \in N$, that is,

$$f(x)D(y) + D(x)D(y) + xD^2(y) = 0 \quad \forall x, y \in N. \tag{2.1}$$

Applying D again, we get

$$f(x)D^2(y) + D^2(x)D(y) + D(x)D^2(y) + D(x)D^2(y) + xD^3(y) = 0 \quad \forall x, y \in N. \tag{2.2}$$

Taking $D(y)$ instead of y in (2.1) gives $f(x)D^2(y) + D(x)D^2(y) + xD^3(y) = 0$, hence (2.2) yields

$$D^2(x)D(y) + D(x)D^2(y) = 0 \quad \forall\ x, y \in N. \tag{2.3}$$

Now, substitute $D(x)$ for x in (2.1), obtaining $f(D(x))D(y) + D^2(x)D(y) + D(x)D^2(y) = 0$; and use (2.3) to conclude that $f(D(x))D(y) = 0$ for all $x, y \in N$. Thus, by Lemma 1.1(iii), $f(D(x)) = 0$ for all $x \in N$. □

Proof of Theorem 2.1. Since $f \neq 0$, there exists $x \in N$ such that $0 \neq f(x) \in Z$. Since $f(x)+f(x) = f(x+x) \in Z$, $(N,+)$ is abelian by Lemma 1.1(ii). To complete the proof, we show that N is multiplicatively commutative.

First, consider the case $D = 0$, so that $f(xy) = f(x)y \in Z$ for all $x, y \in N$. Then $f(x)yw = wf(x)y$, hence $f(x)(yw - wy) = 0$ for all $x, y, w \in N$. Choosing x such that $f(x) \neq 0$ and invoking Lemma 1.1(i), we get $yw - wy = 0$ for all $y, w \in N$.

Now assume that $D \neq 0$, and let $c \in Z \setminus \{0\}$. Then $f(xc) = f(x)c + xD(c) \in Z$; therefore, $(f(x)c + xD(c))y = y(f(x)c + xD(c))$ for all $x, y \in N$, and by Lemma 1.3, we see that $f(x)cy + xD(c)y = yf(x)c + yxD(c)$. Since both $f(x)$ and $D(c)$ are in Z, we have $D(c)(xy - yx) = 0$ for all $x, y \in N$, and provided that $D(Z) \neq \{0\}$, we can conclude that N is commutative.

Assume now that $D \neq 0$ and $D(Z) = \{0\}$. In particular, $D(f(x)) = 0$ for all $x \in N$. Note that for $c \in N$ such that $f(c) = 0$, $f(cx) = cD(x) \in Z$; hence by Lemma 2.2, $D(x)D(y) \in Z$ and $D(y)D(x) \in Z$ for each $x, y \in N$. If one of these is 0, the other is a central element squaring to 0, hence is also 0. The remaining possibility is that $D(x)D(y)$ and $D(y)D(x)$ are nonzero central elements, in which case $D(x)$ is not a zero divisor. Thus $D(x)D(x)D(y) = D(x)D(y)D(x)$ yields $D(x)(D(x)D(y) - D(y)D(x)) = 0 = D(x)D(y) - D(y)D(x)$. Consequently, N is commutative by Theorem C. □

3. On Theorems A and C

Theorem C does not extend to generalized derivations, even if N is a ring. As in [3], consider the ring H of real quaternions, and define $f : H \to H$ by $f(x) = ix + xi$. It is easy to check that f is a generalized derivation with associated derivation given by $D(x) = xi - ix$, and that $f(x)f(y) = f(y)f(x)$ for all $x, y \in H$.

Theorem A also does not extend to generalized derivations, as we see by letting N be the ring $M_2(F)$ of 2×2 matrices over a field F and letting f be defined by $f(x) = e_{12}x$. However, we do have the following results.

Theorem 3.1. *Let N be a 3-prime near-ring, and let f be a generalized derivation on N with associated derivation D. If $f^2 = 0$, then $D^3 = 0$. Moreover, if N is 2-torsion-free, then $D(Z) = \{0\}$.*

Proof. We have

$$f^2(xy) = f(f(x)y + xD(y)) = f(x)D(y) + f(x)D(y) + xD^2(y) = 0 \quad \forall x, y \in N. \tag{3.1}$$

Applying f to (3.1) gives

$$f(x)D^2(y) + f(x)D^2(y) + f(x)D^2(y) + xD^3(y) = 0 \quad \forall x, y \in N. \tag{3.2}$$

Substituting $D(y)$ for y in (3.1) gives

$$f(x)D^2(y) + f(x)D^2(y) + xD^3(y) = 0; \tag{3.3}$$

Therefore, by (3.2) and (3.3),

$$f(x)D^2(y) = 0 \quad \forall x, y \in N. \tag{3.4}$$

It now follows from (3.3) that $xD^3(y) = 0$ for all $x, y \in N$; and since N is 3-prime, $D^3 = 0$.

Suppose now that N is 2-torsion-free and that $D(Z) \neq \{0\}$, and let $z \in Z$ be such that $D(z) \neq 0$. Then if $x, y \in N$ and $f(N)x = \{0\}$, then $f(yz)x = f(y)zx + yD(z)x = 0 = yD(z)x$; and since N is 3-prime and $D(z)$ is not a zero divisor, $x = 0$. It now follows from (3.4) that $D^2 = 0$ and hence by Theorem A that $D = 0$. But this contradicts our assumption that $D(Z) \neq \{0\}$, hence $D(Z) = \{0\}$ as claimed. □

Theorem 3.2. *Let N be a 3-prime and 2-torsion-free near-ring with* 1*. If f is a generalized derivation on N such that* $f^2 = 0$ *and* $f(1) \in Z$*, then* $f = 0$*.*

Proof. Note that $f(x) = f(1x) = f(1)x + 1D(x)$, so

$$f(x) = cx + D(x), \quad c \in Z. \tag{3.5}$$

If $c = 0$, then $f = D$ and $D^2 = 0$, so $D = 0$ by Theorem A and therefore $f = 0$.

If $c \neq 0$, then c is not a zero divisor, hence by (3.4) $D^2 = 0$ and $D = 0$. But then $f(x) = cx$ and $f^2(x) = c^2x = 0$ for all $x \in N$. Since c^2 is not a zero divisor, we get $N = \{0\}$—a contradiction. Thus, $c = 0$ and we are finished. □

4. More on Theorem C

In [4], the author studied generalized derivations f with associated derivation D which have the additional property that

$$f(xy) = D(x)y + xf(y) \quad \forall x, y \in N. \tag{$*$}$$

Our final theorem, a weak generalization of Theorem C, was stated in [4]; but the proof given was not correct. (At one point, both left and right distributivity were assumed.) We now have all the results required for a proof.

Theorem 4.1. *Let N be a 3-prime 2-torsion-free near-ring which admits a generalized derivation f with nonzero associated derivation D such that f satisfies* $(*)$*. If* $f(x)f(y) = f(y)f(x)$ *for all* $x, y \in N$*, then N is a commutative ring.*

Proof. It is correctly shown in [4] that $(N, +)$ is abelian and either $f(N) \subseteq Z$ or $D(f(N)) = \{0\}$. Hence, in view of Theorem 2.1, we may assume that $D(f(N)) = 0$ and therefore, by Lemma 2.2, that $f(D(N)) = \{0\}$. We calculate $f(D(x)D(y))$ in two ways. Using the defining property of f, we obtain $f(D(x)D(y)) = f(D(x))D(y) + D(x)D^2(y) = D(x)D^2(y)$; and using $(*)$, we obtain $f(D(x)D(y)) = D^2(x)D(y) + D(x)f(D(y)) = D^2(x)D(y)$. Thus, $D^2(x)D(y) = D(x)D^2(y)$ for all $x, y \in N$. But since $D(f(N)) = \{0\}$, (2.3) holds in this case as well; therefore $D^2(x)D(y) = 0$ for all $x, y \in N$, hence by Lemma 1.1(iii) $D^2 = 0$. Thus, $D = 0$, contrary to our original hypothesis, so that the case $D(f(N)) = \{0\}$ does not in fact occur. □

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