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Research Article

Hypersurfaces with Null Higher Order Anisotropic Mean Curvature

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Given a positive function F on $\mathbb{S}^n$ which satisfies a convexity condition, for $1 \le r \le n$, we define for hypersurfaces in $\mathbb{R}^{n+1}$ the rth anisotropic mean curvature function $H_{r;F}$, a generalization of the usual rth mean curvature function. We call a hypersurface anisotropic minimal if $H_F = H_{1;F} = 0$, and anisotropic r-minimal if $H_{r+1;F} = 0$. Let W be the set of points which are omitted by the hyperplanes tangent to M. We will prove that if an oriented hypersurface M is anisotropic minimal, and the set W is open and nonempty, then $x(M)$ is a part of a hyperplane of $\mathbb{R}^{n+1}$. We also prove that if an oriented hypersurface M is anisotropic r-minimal and its rth anisotropic mean curvature $H_{r;F}$ is nonzero everywhere, and the set W is open and nonempty, then M has anisotropic relative nullity $n - r$.

1. Introduction

Let $F : \mathbb{S}^n \to \mathbb{R}^+$ be a smooth function which satisfies the following convexity condition:

$$\left(D^2F + FI\right)_x > 0, \quad \forall x \in \mathbb{S}^n, \tag{1}$$

where $\mathbb{S}^n$ is the standard unit sphere in $\mathbb{R}^{n+1}$, D^2F denotes the intrinsic Hessian of F on $\mathbb{S}^n$, I denotes the identity on $T_x\mathbb{S}^n$, and >0 means that the matrix is positive definite. We consider the map

$$\begin{gathered} \phi : \mathbb{S}^n \longrightarrow \mathbb{R}^{n+1}, \\ x \longrightarrow F(x)\,x + \left(\operatorname{grad}_{\mathbb{S}^n} F\right)_x; \end{gathered} \tag{2}$$

its image $W_F = \phi(\mathbb{S}^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of F (see [1–9]). When $F \equiv 1$, the Wulff shape W_F is just $\mathbb{S}^n$.

Now let $x : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an oriented hypersurface. Let $N : M \to \mathbb{S}^n$ denote its Gauss map. The map $\nu = \phi \circ N : M \to W_F$ is called the anisotropic Gauss map of x.

Let $S_F = -d\nu$. S_F is called the F-Weingarten operator, and the eigenvalues of S_F are called anisotropic principal curvatures. Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\kappa_1, \kappa_2, \ldots, \kappa_n$:

$$\sigma_r = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r} \quad (1 \le r \le n). \tag{3}$$

We set $\sigma_0 = 1$. The rth anisotropic mean curvature $H_{r;F}$ is defined by $H_{r;F} = \sigma_r / C_n^r$, also see Reilly [10]. $H_F := H_{1;F}$ is called the anisotropic mean curvature. When $F \equiv 1$, S_F is just the Weingarten operator of hypersurfaces, and $H_{r;F}$ is just the rth mean curvature H_r of hypersurfaces which has been studied by many authors (see [11–14]). Thus, the rth anisotropic mean curvature $H_{r;F}$ generalizes the rth mean curvature H_r of hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.

We say that $x : M \to \mathbb{R}^{n+1}$ is anisotropic r-minimal if $H_{r+1;F} = 0$.

For $p \in M$, we define $\nu(p) = \dim \ker(S_F)$. We call $\nu = \min_{p \in M} \nu(p)$ the anisotropic relative nullity; it generalized the usual relative nullity.

For a smooth immersion $x : M \to \mathbb{Q}_c^{n+1}$ of a hypersurface into an $(n+1)$-dimensional space form with constant sectional curvature c, we denote by

$$W = \mathbb{Q}_c^{n+1} - \bigcup_{p\in M} (\mathbb{Q}_c^n)_p, \tag{4}$$

where for every $p \in M$, $(\mathbb{Q}_c^n)_p$ is the totally geodesic hypersurface of $\mathbb{Q}_c^{n+1}$ tangent to $x(M)$ at $x(p)$. So, in the case of $c = 0$, W is the set of points which are omitted by the hyperplanes tangent to $x(M)$.

We will study immersion with W nonempty. In this direction, Hasanis and Koutroufiotis (see [15]) proved the following.

Theorem 1. *Let $x : M \to \mathbb{Q}_c^3$ be a complete minimal immersion with $c \geq 0$. If W is nonempty, then x is totally geodesic.*

Later, in [16], Alencar and Frensel extended the result above assuming an extra condition. They proved the following.

Theorem 2. *Let $x : M \to \mathbb{Q}_c^{n+1}$ be an oriented, minimally immersed hypersurface. If W is open and nonempty, then x is totally geodesic.*

In [17], Alencar and Batista studied hypersurfaces with null higher order mean curvature; they proved the following.

Theorem 3. *Let M be a complete and orientable Riemannian manifold and let $x : M \to \mathbb{Q}_c^{n+1}$ be an isometric immersion with $H_{r+1} = 0$ and $H_r \neq 0$ everywhere, $r \geq 1$. If W is open and nonempty, then the relative nullity $\nu = n - r$.*

We note that, Alencar in [18] provides examples of nontotally geodesic minimal hypersurfaces in $\mathbb{R}^{2n}$, $n \geq 4$, with nonempty W; in [17], Alencar and Batista provides examples of 1-minimal hypersurfaces with $H_1 \neq 0$ everywhere in $\mathbb{R}^{2n}$, $n \geq 5$, with nonempty W but $\nu \neq n-1$. These examples show that it is necessary to add an extra hypothesis.

In this paper, we prove the anisotropic version of Theorems 2 and 3 for an immersion $x : M \to \mathbb{R}^{n+1}$. Explicitly, we prove the following two theorems.

Theorem 4. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented, anisotropic minimally immersed hypersurface. If W is open and nonempty, then $x(M)$ is a part of a hyperplane of $\mathbb{R}^{n+1}$.*

Theorem 5. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented immersed hypersurface with $H_{r+1;F} = 0$ and $H_{r;F} \neq 0$ everywhere, $r \geq 1$. If W is open and nonempty, then the anisotropic relative nullity $\nu = n - r$.*

2. Preliminaries

In this paper, we use the summation convention of Einstein and the following convention of index ranges unless otherwise stated:

$$1 \leq i, j, \ldots \leq n; \qquad 1 \leq \alpha, \beta, \ldots \leq n+1. \tag{5}$$

We define $F^* : \mathbb{R}^{n+1} \to \mathbb{R}$ to be

$$F^*(y) = \sup\left\{\frac{\langle y, z\rangle}{F(z)} \mid z \in \mathbb{R}^{n+1} \setminus \{0\}\right\}; \tag{6}$$

then F^* is a Minkowski norm on $\mathbb{R}^{n+1}$. In fact, as proved in [19], $F^* : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is smooth and we have the following.

Proposition 6. *(1) $F^*(y) > 0$, for all $y \in \mathbb{R}^{n+1} \setminus \{0\}$;*

(2) $F^*(ty) = tF^*(y)$, for all $y \in \mathbb{R}^{n+1}$, $t > 0$;

(3) $F^*(y+z) \leq F^*(y) + F^*(z)$, for all $y, z \in \mathbb{R}^{n+1}$, and the equality holds if and only if $y = 0$, or $z = 0$ or $y = kz$ for some $k > 0$.

(4) $W_F = \{y \in \mathbb{R}^{n+1} \mid F^*(y) = 1\}$.

We define

$$\begin{aligned}\overline{g}_{\alpha\beta}(y) &= \frac{1}{2}\frac{\partial^2 (F^*)^2}{\partial y^\alpha \partial y^\beta}(y), \\ g_y(X, Y) &= \overline{g}_{\alpha\beta}(y) X^\alpha Y^\beta,\end{aligned} \tag{7}$$

where $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $X = (X^1, X^2, \ldots, X^{n+1})$, $Y = (Y^1, Y^2, \ldots, Y^{n+1}) \in T_y\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.

From the Euler's theorem for homogeneous functions, we have

$$\frac{\partial \overline{g}_{\alpha\beta}}{\partial y^\gamma}(z) z^\beta = \frac{1}{2}\frac{\partial^3 (F^*)^2}{\partial y^\alpha \partial y^\beta \partial y^\gamma}(z) z^\beta = 0, \tag{8}$$

where $z = (z^1, z^2, \ldots, z^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$. Thus,

$$\frac{\partial g_z(X, z)}{\partial y^\gamma} = \overline{g}_{\alpha\beta}(z)\frac{\partial X^\alpha}{\partial y^\gamma} z^\beta + \overline{g}_{\alpha\gamma}(z) X^\alpha \frac{\partial z^\beta}{\partial y^\gamma}, \tag{9}$$

where $z = (z^1, z^2, \ldots, z^{n+1}) \in T\mathbb{R}^{n+1}$ is nonzero everywhere and $X = (X^1, X^2, \ldots, X^{n+1}) \in T\mathbb{R}^{n+1}$.

As F^* is a Minkowski norm on $\mathbb{R}^{n+1}$, the following lemma holds (see [20, 21]).

Lemma 7. *For any $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $u \in \mathbb{R}^{n+1}$ one has*

$$g_y(y, z) \leq F^*(y) F^*(z), \tag{10}$$

and the equality holds if and only if there exists $t \geq 0$ such that $z = ty$.

Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Let $\nu : M \to W_F$ denote its anisotropic Gauss map. Then for any $p \in M$, $\nu(p)$ is perpendicular to $x_*(T_pM)$ with respect to the inner product $g_{\nu(p)}$ and $F^*(\nu(p)) = 1$. Thus, we call $\nu(p)$ an anisotropic unit normal vector of T_pM.

3. A Connection on Hypersurfaces of Minkowski Space

Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ and denote $\nu : M \to W_F$ its anisotropic Gauss map.

Let $\overline{\nabla}$ be the standard connection on the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For vector fields X, Y on M, we decompose $\overline{\nabla}_X Y$ as the tangent part $\nabla_X Y$ and the anisotropic normal part $\mathrm{II}(X,Y)\nu$ with respect to the inner product g_ν. That is,

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X,Y)\,\nu, \tag{11}$$

where $g_\nu(\nabla_X Y, \nu) = 0$.

We also have the Weingarten formula:

$$\begin{gathered}\overline{\nabla}_X \nu = -S_F X, \\ g_\nu(S_F X, Y) = \mathrm{II}(X,Y),\end{gathered} \tag{12}$$

where we have used (9).

It is easy to verify that ∇ is a torsion free connection on M and II is a symmetric second order covariant tensor field on M. We call II the anisotropic second fundamental form.

Let $\{e_i\}_{i=1}^n$ be a local frame of M and $\{\omega^i\}_{i=1}^n$ its dual frame. Let $g_{ij} = g_\nu(e_i, e_j)$, $\nabla e_i = \omega_i^j \otimes e_j$, $\mathrm{II}(e_i, e_j) = h_{ij}$, $h_i^j = g^{jk}h_{ki}$, where (g^{ij}) is the inverse matrix of (g_{ij}). Then we have

$$dx = \omega^i e_i, \tag{13}$$

$$de_i = \omega_i^j e_j + h_{ij}\omega^j \nu, \tag{14}$$

$$d\nu = -h_i^j \omega^i e_j. \tag{15}$$

Differentiating (13) and using (14), we get

$$\begin{gathered}d\omega^i = \omega^j \wedge \omega_j^i, \\ h_{ij} = h_{ji}.\end{gathered} \tag{16}$$

Differentiating (14) and using (14)-(15), we get

$$\begin{gathered}h_{ijk} = h_{ikj}, \\ d\omega_i^j - \omega_i^k \wedge \omega_k^j = -\frac{1}{2}R_{i\ kl}^{\ j}\omega^k \wedge \omega^l,\end{gathered} \tag{17}$$

where

$$dh_{ij} - h_{ik}\omega_j^k - h_{kj}\omega_i^k = h_{ijk}\omega^k, \tag{18}$$

and $R_{i\ kl}^{\ j} = -R_{i\ lk}^{\ j} = h_{ik}h_l^j - h_{il}h_k^j$.

Differentiating (15) and using (14), we get

$$h_{i\ k}^{\ j} = h_{k\ i}^{\ j}, \tag{19}$$

where

$$dh_i^j + h_i^k\omega_k^j - h_k^j\omega_i^k = h_{ik}^j\omega^k. \tag{20}$$

Note (h_i^j) is the matrix of the F-Weingarten operator $S_F = -d\nu$, its eigenvalues are called the anisotropic principal curvatures, and we denote them by $\kappa_1, \ldots, \kappa_n$.

We have n invariants, the elementary symmetric function σ_r of the anisotropic principal curvatures:

$$\sigma_r = \sum_{i_1<\cdots i_r} \kappa_{i_1}\cdots\kappa_{i_n} \quad (1 \le r \le n). \tag{21}$$

For convenience, we set $\sigma_0 = 1$. The rth anisotropic mean curvature $H_{r;F}$ is defined by

$$H_{r;F} = \frac{\sigma_r}{C_n^r}, \qquad C_n^r = \frac{n!}{r!\,(n-r)!}. \tag{22}$$

Using the characteristic polynomial of S_F, σ_r is defined by

$$\det(tI - S_F) = \sum_{r=0}^{n}(-1)^r\sigma_r t^{n-r}. \tag{23}$$

So, we have

$$\sigma_r = \frac{1}{r!}\sum_{i_1,\ldots,i_r;j_1,\ldots,j_r}\delta_{i_1\cdots i_r}^{j_1\cdots j_r}h_{j_1}^{i_1}\cdots h_{j_r}^{i_r}, \tag{24}$$

where $\delta_{i_1\cdots i_r}^{j_1\cdots j_r}$ is the usual generalized Kronecker symbol; that is, $\delta_{i_1\cdots i_r}^{j_1\cdots j_r}$ equals +1 (resp., −1) if $i_1\cdots i_r$ are distinct and $(j_1\cdots j_r)$ is an even (resp., odd) permutation of $(i_1\cdots i_r)$ and in other cases it equals zero.

Definition 8. Let $f : M \to \mathbb{R}$ be a smooth function. One defines the gradient (with respect to the induced metric g_ν on M) $\operatorname{grad} f$ of the function f by

$$g_\nu(\operatorname{grad} f, X) = X(f), \tag{25}$$

where X is any smooth vector field on M.

Define f_i by $df = f_i\omega^i$; then

$$\operatorname{grad} f = g^{ij}f_j e_i. \tag{26}$$

We define

$$dV = |e_1, \ldots, e_n, \nu|\,\omega^1 \wedge \cdots \wedge \omega^n, \tag{27}$$

where $|e_1, \ldots, e_n, \nu|$ is the determinant of the matrix $(e_1, \ldots, e_n, \nu)$. Then dV is a volume element on M.

Definition 9. Let X be a smooth vector field on M. One defines the divergence (with respect to the volume element dV) $\operatorname{div} X$ by $d\{i(X)dV\} = (\operatorname{div} X)dV$, where

$$\begin{aligned}(i(X)\,dV)(Y_1, \ldots, Y_{n-1}) &\equiv dV(X, Y_1, \ldots, Y_{n-1}), \\ &\forall Y_1, \ldots, Y_{n-1} \in \mathcal{X}(M).\end{aligned} \tag{28}$$

Lemma 10. *Let $X = X^i e_i$; then $\operatorname{div} X = X_i^i$, where*

$$dX^i + X^j\omega_j^i = X_j^i\omega^j. \tag{29}$$

Proof. By (14), (15), we get

$$d\,|e_1, \ldots, e_n, \nu| = \omega_i^i\,|e_1, \ldots, e_n, \nu|. \tag{30}$$

From the definition of $i(X)$, we have

$$\begin{aligned}i(X)\,dV = \sum_i (-1)^{i+1}X^i\,|e_1, \ldots, e_n, \nu|\,\omega^1 \\ \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^n.\end{aligned} \tag{31}$$

So,

$$
\begin{aligned}
d\{i(X)\,dV\} = {} & \sum_i (-1)^{i+1}\left(dX^i\right)\wedge \left|e_1,\ldots,e_n,\nu\right|\omega^1 \\
& \wedge\cdots\wedge\widehat{\omega}^i\wedge\cdots\wedge\omega^n \\
& + \sum_i (-1)^{i+1}X^i\left(d\left|e_1,\ldots,e_n,\nu\right|\right) \\
& \wedge\omega^1\wedge\cdots\wedge\widehat{\omega}^i\wedge\cdots\wedge\omega^n \\
& + \sum_{j<i}(-1)^{i+j}X^i\left|e_1,\ldots,e_n,\nu\right| d\omega^j\wedge\omega^1 \\
& \wedge\cdots\wedge\widehat{\omega}^j\wedge\cdots\wedge\widehat{\omega}^i\wedge\cdots\wedge\omega^n \\
& + \sum_{j>i}(-1)^{i+j+1}X^i\left|e_1,\ldots,e_n,\nu\right| d\omega^j \\
& \wedge\omega^1\wedge\cdots\wedge\widehat{\omega}^i\wedge\cdots\wedge\widehat{\omega}^j\wedge\cdots\wedge\omega^n \\
= {} & X^i_i dV.
\end{aligned}
\tag{32}
$$

□

4. $L_{r;F}$ Operator for Hypersurfaces

We introduce the Newton transformation defined by

$$P_r = \sigma_r I - \sigma_{r-1}S_F + \cdots + (-1)^r S_F^r,\quad r = 0,\ldots,n; \tag{33}$$

then

$$P_0 = I,\qquad P_n = 0,\qquad P_r = \sigma_r I - P_{r-1}S_F. \tag{34}$$

Lemma 11. *The matrix of P_r is given by:*

$$(P_r)_i^j = \frac{1}{r!}\delta^{j_1\cdots j_r j}_{i_1\cdots i_r i}h^{i_1}_{j_1}\cdots h^{i_r}_{j_r}. \tag{35}$$

Proof. We prove Lemma 11 inductively. For $r = 0$, it is easy to check that (35) is true.

We can check directly

$$\delta^{j_1\cdots j_q}_{i_1\cdots i_q} = \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_{q-1}}_{i_1} & \delta^{j_q}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_{q-1}}_{i_2} & \delta^{j_q}_{i_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^{j_1}_{i_{q-1}} & \delta^{j_2}_{i_{q-1}} & \cdots & \delta^{j_{q-1}}_{i_{q-1}} & \delta^{j_q}_{i_{q-1}} \\ \delta^{j_1}_{i_q} & \delta^{j_2}_{i_q} & \cdots & \delta^{j_{q-1}}_{i_q} & \delta^{j_q}_{i_q} \end{vmatrix}. \tag{36}$$

Assume that (35) is true for $r = k$, we only need to show that it is also true for $r = k + 1$. For $r = k + 1$, using (24) and (36), we have

$$
\begin{aligned}
&\text{RHS of (35)} \\
&= \frac{1}{(k+1)!}\sum_{i_1,\ldots,i_{k+1};j_1,\ldots,j_{k+1}}\delta^{j_1\cdots j_{k+1}j}_{i_1\cdots i_{k+1}i}h^{j_1}_{i_1}\cdots h^{j_{k+1}}_{i_{k+1}} \\
&= \frac{1}{(k+1)!}\sum\begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_{k+1}}_{i_1} & \delta^{j}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_{k+1}}_{i_2} & \delta^{j}_{i_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^{j_1}_{i_{k+1}} & \delta^{j_2}_{i_{k+1}} & \cdots & \delta^{j_{k+1}}_{i_{k+1}} & \delta^{j}_{i_{k+1}} \\ \delta^{j_1}_{i} & \delta^{j_2}_{i} & \cdots & \delta^{j_{k+1}}_{i} & \delta^{j}_{i} \end{vmatrix} h^{j_1}_{i_1}\cdots h^{j_{k+1}}_{i_{k+1}} \\
&= \frac{1}{(k+1)!}\sum\left(\delta^j_i\delta^{j_1\cdots j_{k+1}}_{i_1\cdots i_{k+1}} - \delta^{j_{k+1}}_i\delta^{j_1\cdots j_k j}_{i_1\cdots i_k i_{k+1}} + \cdots\right)h^{j_1}_{i_1}\cdots h^{j_{k+1}}_{i_{k+1}} \\
&= \sigma_{k+1}\delta^j_i - \frac{1}{(k+1)!}\sum\delta^{j_{k+1}}_i\delta^{j_1\cdots j_k j}_{i_1\cdots i_k i_{k+1}}h^{j_1}_{i_1}\cdots h^{j_{k+1}}_{i_{k+1}} + \cdots \\
&= \sigma_{k+1}\delta^j_i - \sum\left(P_k\right)^{i_{k+1}}_i h^j_{i_{k+1}} \\
&= \left(P_{k+1}\right)^j_i.
\end{aligned}
\tag{37}
$$

□

Lemma 12. *For each r, one has*

(a) $(P_r)_i{}^j{}_j = 0$;

(b) $\operatorname{Trace}(P_r S_F) = (r+1)\sigma_{r+1}$;

(c) $\operatorname{Trace}(P_r) = (n-r)\sigma_r$;

(d) $\operatorname{Trace}(P_r S_F^2) = \sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2}$.

Proof. (a) Noting (j, j_r) is skew symmetric in $\delta^{j_1\cdots j_r j}_{i_1\cdots i_r i}$ and (j, j_r) is symmetric in $h^{i_1}_{j_1}\cdots h_{j_r}{}^{i_r}{}_j$ (from (19), we have

$$\sum_j (P_r)_i{}^j{}_j = \frac{1}{(r-1)!}\sum_{i_1,\ldots,i_r;j_1,\ldots,j_r;j}\delta^{j_1\cdots j_r j}_{i_1\cdots i_r i}h^{i_1}_{j_1}\cdots h_{j_r}{}^{i_r}{}_j = 0. \tag{38}$$

(b) Using (35) and (24), we have

$$
\begin{aligned}
\operatorname{Trace}\left(P_r S_F\right) &= \sum_{ij}\left(P_r\right)^j_i h^i_j \\
&= \frac{1}{r!}\sum_{i_1,\ldots,i_r;j_1,\ldots,j_r;i,j}\delta^{j_1\cdots j_r j}_{i_1\cdots i_r i}h^{i_1}_{j_1}\cdots h^{i_r}_{j_r}h^i_j \\
&= (r+1)\,\sigma_{r+1}.
\end{aligned}
\tag{39}
$$

(c) Using (b) and the definition of P_r, we have

$$\operatorname{Trace}\left(P_r\right) = \operatorname{tr}\left(\sigma_r I\right) - \operatorname{tr}\left(P_{r-1}S_F\right) = n\sigma_r - r\sigma_r = (n-r)\,\sigma_r. \tag{40}$$

(d) Using (b) and the definition of P_{r+1}, we have

$$
\begin{aligned}
\operatorname{Trace}\left(P_r S_F^2\right) &= \operatorname{Trace}\left(\sigma_{r+1}S_F\right) - \operatorname{Trace}\left(P_{r+1}S_F\right) \\
&= \sigma_1\sigma_{r+1} - (r+2)\,\sigma_{r+2}.
\end{aligned}
\tag{41}
$$

□

Remark 13. When $F = 1$, Lemma 12 was a well-known result (e.g., see Barbosa and Colares [22], or Reilly [23]).

Lemma 14. *One has*

$$(\sigma_r)_k = \sum_{i,j} (P_{r-1})_i^j h_{j\,k}^{\,i}. \tag{42}$$

Proof. From the definition of σ_r, we have the following calculation:

$$\begin{aligned}(\sigma_r)_k &= \frac{1}{r!} \sum_{i_1,\dots,i_r;j_1,\dots,j_r} \delta_{i_1\cdots i_r}^{j_1\cdots j_r} \left(h_{i_1}^{j_1} \cdots h_{i_r}^{j_r}\right)_k \\ &= \frac{1}{(r-1)!} \sum_{i_1,\dots,i_r;j_1,\dots,j_r} \delta_{i_1\cdots i_r}^{j_1\cdots j_r} h_{i_1}^{j_1} \cdots h_{i_r\,k}^{\,j_r} \\ &= \sum_{i_r,j_r} (P_{r-1})_{i_r}^{j_r} h_{i_r\,k}^{\,j_r} = \sum_{i,j} (P_{r-1})_i^j h_{i\,k}^{\,j}.\end{aligned} \tag{43}$$

□

We define an operator $L_{r;F} : C^\infty(M) \to C^\infty(M)$ by

$$L_{r;F}(f) = \operatorname{div}(P_r \nabla f). \tag{44}$$

In the sequel, we will need the following lemma. Item (a) is essentially the content of Lemma 1.1 and Equation (1.3) in [24], while item (b) is quoted as Proposition 1.5 in [25].

Lemma 15. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface, and $0 \le r \le n-1$, $p \in M$.*

(a) *If $\sigma_{r+1}(p) = 0$, then P_r is semidefinite at p;*

(b) *if $\sigma_{r+1}(p) = 0$ and $\sigma_{r+2}(p) \neq 0$, then P_r is definite at p.*

Another important result is as follows (see [26]).

Lemma 16. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface, and $p \in M$.*

(a) *For $1 \le r \le n$, one has $H_{r;F}^2 \ge H_{r-1;F} H_{r+1;F}$. Moreover, if equality happens for $r = 1$ or for some $1 < r < n$, with $H_{r+1;F} \neq 0$ in this case, then p is an anisotropic umbilical point (i.e. $\kappa_1(p) = \kappa_2(p) = \cdots = \kappa_n(p)$);*

(b) *if, for some $1 \le r < n$, one has $H_{r;F} = H_{r+1;F} = 0$, then $H_{j;F} = 0$ for all $r \le j \le n$. In particular, at most $r-1$ of the anisotropic principal curvatures are different from zero.*

The result below is standard, so we omit the proof.

Lemma 17. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface. The operator $L_{r;F}$ associated to the immersion x is elliptic if and only if P_r is positive definite.*

Definition 18. Let $f : M \to \mathbb{R}$ be a smooth function. The Laplacian Δf is defined by $\Delta f := L_{0;F} f = \operatorname{div}(\operatorname{grad} f)$.

It is easy to see that Δ is an elliptic differential operator.

Definition 19. Let $x : M \to \mathbb{R}^{n+1}$ be an immersed hypersurface and ν its anisotropic unit normal vector field. The function $u := g_\nu(x, \nu)$ is called the support function of the immersion x.

Next, we compute $L_{r;F} u$ for the support function $u = g_\nu(x, \nu)$.

Differentiating the decomposition

$$x = g^{ij} g_\nu(x, e_i) e_j + u\nu, \tag{45}$$

we obtain

$$\begin{aligned} dx = {} & \left\{ d\left(g^{ij} g_\nu(x, e_j)\right)\right\} e_i + g^{ij} g_\nu(x, e_j) de_i \\ & + (du)\nu + u\, d\nu. \end{aligned} \tag{46}$$

So, from (13), (14), and (15) we have

$$\begin{aligned} \omega^i e_i = {} & \left\{ d\left(g^{ij} g_\nu(x, e_j)\right) + g^{kj} g_\nu(x, e_j)\,\omega_k^i - u h_j^i \omega^j \right\} e_i \\ & + \left(du + g^{jk} g_\nu(x, e_j) h_{ik} \omega^i\right)\nu. \end{aligned} \tag{47}$$

Thus, we get

$$\begin{aligned} du &= -g^{jk} g_\nu(x, e_j) h_{ik} \omega^i, \\ d\left(g^{ij} g_\nu(x, e_j)\right) &+ g^{kj} g_\nu(x, e_j)\,\omega_k^i - u h_j^i \omega^j = \omega^i. \end{aligned} \tag{48}$$

Denote u^i, $(g^{ij} g_\nu(x, e_j))_k$, u_j^i by

$$\begin{aligned} \operatorname{grad} u &= u^i e_i, \\ \left(g^{ij} g_\nu(x, e_j)\right)_k \omega^k &= d\left(g^{ij} g_\nu(x, e_j)\right) + \left(g^{kj} g_\nu(x, e_j)\right)\omega_k^i, \\ u_j^i \omega^j &= du^i + u^j \omega_j^i, \end{aligned} \tag{49}$$

respectively. Then we have (using (19) the following

$$\begin{aligned} u^i &= -g^{il} h_{kl} g^{jk} g_\nu(x, e_j) = -h_k^i g^{kl} g_\nu(x, e_l), \\ \left(g^{ik} g_\nu(x, e_k)\right)_j &= \delta_j^i + h_j^i g_\nu(x, \nu), \\ u_j^i &= -h_{k\,j}^{\,i} g^{kl} g_\nu(x, e_l) - h_k^i \left(g^{kl} g_\nu(x, e_l)\right)_j \\ &= -h_{jk}^{\,i} g^{kl} g_\nu(x, e_l) - h_j^i - h_k^i h_j^k u. \end{aligned} \tag{50}$$

By using Lemmas 12 and 14, we get

$$\begin{aligned} L_{r;F} u = {} & (P_r)_i^j u_j^i \\ = {} & -(P_r)_i^j h_{jk}^{\,i} g^{kl} g_\nu(x, e_l) \\ & - (P_r)_i^j h_j^i - (P_r)_i^j h_k^i h_j^k u \\ = {} & -(\sigma_{r+1})_k g^{kl} g_\nu(x, e_l) - (P_r)_i^j h_j^i - (P_r)_i^j h_k^i h_j^k u \\ = {} & -g_\nu(\nabla \sigma_{r+1}, x) - (r+1)\sigma_{r+1} \\ & - \left(\sigma_1 \sigma_{r+1} - (r+2)\sigma_{r+2}\right) u. \end{aligned} \tag{51}$$

Thus, we proved the following lemma.

Lemma 20. *For $0 \le r \le n-1$, one has the following*

$$L_{r;F}u = -g_\nu(\nabla\sigma_{r+1}, x) - (r+1)\sigma_{r+1} - (\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2})u. \quad (52)$$

Remark 21. Recall $\sigma_1 = nH_F$ and $|\mathrm{II}|^2 = \sigma_1^2 - 2\sigma_2$; let $r = 0$ in (52); we get

$$\Delta u = -n(H_F + g_\nu(\operatorname{grad} H_F, x)) - |\mathrm{II}|^2 u. \quad (53)$$

5. Proof of Theorems 4 and 5

We fix a point $o \in W$ as the origin of $\mathbb{R}^{n+1}$. Without loss of generality, we assume, for each $p \in M$, $\nu(p)$ is the anisotropic unit normal vector of $x(M)$ at $x(p)$ such that $\langle x(p), \nu(p)\rangle_{\nu(p)} > 0$ (otherwise we consider the function $-u$ instead). This gives an orientation to M; indeed, the component of the position vector x perpendicular (with respect to the inner product g_ν) to M defines a never zero, anisotropic normal, vector field on M, such that the support function $u = \langle x(p), \nu(p)\rangle_{\nu(p)}$ is positive on M.

5.1. Proof of Theorem 4. Since x is anisotropic minimal, from (53) we get

$$\Delta u = -|\mathrm{II}|^2 u \le 0, \quad \text{on } M. \quad (54)$$

Let $u_* = \inf_M u$. We claim that u_* is attained at some point $x_0 \in M$. Consider a sequence $\{x_k\} \subset M$ such that $u(x_k) \to u_*$ as $k \to +\infty$. To each x_k we associate $y_k = u(x_k)\nu(x_k)$; then $y_k \in T_{x_k}M$. Since $\|y_k\|_{\mathbb{R}^{n+1}} = u(x_k)\|\nu(x_k)\|_{\mathbb{R}^{n+1}}$ is bounded, there exists a subsequence, which again we call $\{y_k\}$, such that $y_k \to y_0$ for some $y_0 \in \mathbb{R}^{n+1}$. Since $\bigcup_{p\in M} T_pM$ is closed and $\{y_k\} \subset_{p\in M} T_pM$ we deduce that $y_0 \in T_{x_0}M$ for some $x_0 \in M$. Thus, by the continuity of F^* and Lemma 7,

$$u_* = \lim_{k\to+\infty} u(x_k) = \lim_{k\to+\infty} F^*(y_k) = F^*(y_0) \ge g_{\nu(x_0)}(y_0, \nu(x_0)) = u(x_0), \quad (55)$$

so $u^* = u(x_0)$ as needed. Now, from the usual maximum principle u is constant, $u = u_* = u(x_0) > 0$. From (54) we then have $\mathrm{II} \equiv 0$ and x is totally geodesic.

5.2. Proof of Theorem 5. Since $H_{r+1;F} = 0$, from Lemma 20 we get

$$L_{r;F}u = (r+2)\sigma_{r+2}u. \quad (56)$$

Using Lemma 15(a) we have that P_r is semidefinite. Since $H_{r;F}$ does not vanish, we have that $H_{r;F}$ is positive or negative, because $c(r)H_{r;F} = \mathrm{Trace}(P_r)$, where $c(r) = (n-r)C_n^r$. Now we use Lemma 16 and obtain the following:

$$0 = H_{r+1;F}^2 \ge H_{r;F}H_{r+2;F}. \quad (57)$$

Using the information above, we claim that $H_{r+2;F} \equiv 0$.

Case (i) ($H_{r;F} > 0$). In this case, P_r is positive definite, and $L_{r;F}$ is elliptic by Lemma 17. Using (57) we conclude that $H_{r+2;F} \le 0$. Whereas from (56) we have

$$L_{r;F}u \le 0. \quad (58)$$

Following exactly the proof as in Theorem 4, we conclude that u is constant, $u = u_* = u(x_0) > 0$. From (56) we then have $H_{r+2;F} \equiv 0$.

Case (ii) ($H_{r;F} < 0$). In this case, P_r is negative definite, and $-L_{r;F}$ is elliptic by Lemma 17. Using (57) we conclude that $H_{r+2;F} \ge 0$. Whereas from (56) we have

$$-L_{r;F}u \le 0. \quad (59)$$

Now, following exactly the proof as in Theorem 4, we conclude that u is constant, $u = u_* = u(x_0) > 0$. From (56) we then have $H_{r+2;F} \equiv 0$.

Thus we conclude that $H_{r+2;F} \equiv 0$. Now, we use Lemma 16(b) to conclude that $H_{j;F} = 0$ for $j \ge r+1$ and so that $\nu \ge n-r$. Since $H_{r;F}$ does not change sign we have that $\nu = n-r$.

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