Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2014, Article ID 982013, 6 pages
http://dx.doi.org/10.1155/2014/982013

Research Article

On k-Gamma and k-Beta Distributions and Moment Generating Functions

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Received 10 February 2014; Revised 29 June 2014; Accepted 4 July 2014; Published 15 July 2014

Academic Editor: Chin-Shang Li

The main objective of the present paper is to define k-gamma and k-beta distributions and moments generating function for the said distributions in terms of a new parameter $k > 0$. Also, the authors prove some properties of these newly defined distributions.

1. Basic Definitions

In this section we give some definitions which provide a base for our main results. The definitions (1.1–1.3) are given in [1] while (1.4–1.6) are introduced in [2]. Also, we have taken some statistics related definitions (1.7–1.11) from [3–5].

1.1. Pochhmmer Symbol. The factorial function is denoted and defined by

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1); & \text{for } n \geq 1,\ a \neq 0, \\ 1; & \text{if } n = 0. \end{cases} \tag{1}$$

The function $(a)_n$ defined in relation (1) is also known as Pochhmmer symbol.

1.2. Gamma Function. Let $z \in \mathbb{C}$; the Euler gamma function is defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^{z-1}}{(z)_n} \tag{2}$$

and the integral form of gamma function is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathbb{R}(z) > 0. \tag{3}$$

From the relation (3), using integration by parts, we can easily show that

$$\Gamma(z+1) = z\Gamma(z). \tag{4}$$

The relation between Pochhammer symbol and gamma function is given by

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}. \tag{5}$$

1.3. Beta Function. The beta function of two variables is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \operatorname{Re}(x), \operatorname{Re}(y) > 0 \tag{6}$$

and, in terms of gamma function, it is written as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{7}$$

1.4. Pochhammer k-Symbol. For $k > 0$, the Pochhammer k-symbol is denoted and defined by

$$(a)_{n,k} = \begin{cases} a(a+k)(a+2k)\cdots(a+(n-1)k); & \text{for } n \geq 1,\ a \neq 0, \\ 1; & \text{if } n = 0. \end{cases} \tag{8}$$

1.5. k-Gamma Function. For $k > 0$ and $z \in \mathbb{C}$, the k-gamma function is defined as

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n! k^n (nk)^{z/k-1}}{(z)_{n,k}} \quad (9)$$

and the integral representation of k-gamma function is

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k} dt. \quad (10)$$

1.6. k-Beta Function. For $\text{Re}(x), \text{Re}(y) > 0$, the k-beta function of two variables is defined by

$$B_k(x, y) = \frac{1}{k}\int_0^\infty t^{x/k-1}(1-t)^{y/k-1} dt \quad (11)$$

and, in terms of k-gamma function, k-beta function is defined as

$$B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \quad (12)$$

Also, the researchers [6–10] have worked on the generalized k-gamma and k-beta functions and discussed the following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x), \quad (13)$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \quad (14)$$

$$\Gamma_k(k) = 1, \quad k > 0. \quad (15)$$

Using the above relations, we see that, for $x, y > 0$ and $k > 0$, the following properties of k-beta function are satisfied by authors (see [6, 7, 11]):

$$\beta_k(x+k, y) = \frac{x}{x+y}\beta_k(x, y), \quad (16)$$

$$\beta_k(x, y+k) = \frac{y}{x+y}\beta_k(x, y), \quad (17)$$

$$\beta_k(xk, yk) = \frac{1}{k}\beta(x, y), \quad (18)$$

$$\beta_k(x, k) = \frac{1}{x}, \qquad \beta_k(k, y) = \frac{1}{y}. \quad (19)$$

Note that when $k \to 1$, $\beta_k(x, y) \to \beta(x, y)$.

For more details about the theory of k-special functions like k-gamma function, k-beta function, k-hypergeometric functions, solutions of k-hypergeometric differential equations, contegious functions relations, inequalities with applications and integral representations with applications involving k-gamma and k-beta functions and so forth. (See [12–17].)

1.7. Probability Distribution and Expected Values. In a random experiment with n outcomes, suppose a variable X assumes the values $x_1, x_2, x_3, \ldots, x_n$ with corresponding probabilities $P_1, P_2, P_3, \ldots, P_n$; then this collection is called probability distribution and $\Sigma p_i = 1$ (in case of discrete distributions). Also, if $f(x)$ is a continuous probability distribution function defined on an interval $[a, b]$, then $\int_a^b f(x)dx = 1$.

In statistics, there are three types of moments which are (i) moments about any point $x = a$, (ii) moments about $x = 0$, and (iii) moments about mean position of the given data. Also, expected value of the variate is defined as the first moment of the probability distribution about $x = 0$ and the rth moment about mean of the probability distribution is defined as $E(x_i - \overline{x})^r$ where $\overline{x}$ is the mean of the distribution.

Also, $E(x)$ shows the expected value of the variate x and is defined as the first moment of the probability distribution about $x = 0$; that is,

$$\mu_1' = E(x) = \int_a^b x f(x)\, dx. \quad (20)$$

1.8. Gamma Distribution. A continuous random variable Z is said to have a gamma distribution with parameter $m > 0$, if its probability distribution function is defined by

$$f(z) = \begin{cases} \dfrac{1}{\Gamma(m)} z^{m-1} e^{-z}, & 0 \leqq z < \infty, \\ 0, & \text{elsewhere} \end{cases} \quad (21)$$

and its distribution function $F(z)$ is defined by

$$F(z) = \begin{cases} \displaystyle\int_0^z \frac{1}{\Gamma(m)} z^{m-1} e^{-z} dz, & z \geq 0, \\ 0, & z < 0, \end{cases} \quad (22)$$

which is also called the incomplete gamma function.

1.9. Moment Generating Function of Gamma Distribution. The moment generating function of Z is defined by

$$\begin{aligned} M_0(t) = E\left(e^{tZ}\right) &= \int_0^\infty e^{tZ} f(z)\, dz \\ &= \int_0^\infty \frac{1}{\Gamma(m)} z^{m-1} e^{-z(1-t)} dz. \end{aligned} \quad (23)$$

1.10. Beta Distribution of the First Kind. A continuous random variable Z is said to have a beta distribution with two parameters m and n, if its probability distribution function is defined by

$$f(z) = \begin{cases} \dfrac{1}{B(m, n)} z^{m-1}(1-z)^{n-1}, & 0 \leqq z \leqq 1;\ m, n > 0 \\ 0, & \text{elsewhere.} \end{cases} \quad (24)$$

This distribution is known as a beta distribution of the first kind and a beta variable of the first kind is referred to as $\beta_1(m,n)$. Its distribution function $F(z)$ is given by

$$F(z) = \begin{cases} 0, & z < 0, \\ \int_0^z \frac{1}{B(m,n)} z^{m-1}(1-z)^{n-1} dz, & 0 \leqq z \leqq 1;\ m,n > 0, \\ 0, & z > 1. \end{cases} \tag{25}$$

1.11. Beta Distribution of the Second Kind. A continuous random variable Z is said to have a beta distribution of the second kind with parameters m and n, if its probability distribution function is defined by

$$f(z) = \begin{cases} \frac{1}{\beta(m,n)} \frac{z^{m-1}}{(1+z)^{m+n}}, & 0 \leqq z < \infty;\ m,n > 0, \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

and its probability distribution function is given by

$$F(z) = \int_0^{\infty} \frac{1}{\beta(m,n)} \frac{z^{m-1}}{(1+z)^{m+n}} dz, \quad 0 \leqq z < \infty;\ m,n > 0. \tag{27}$$

2. Main Results: k-Gamma and k-Beta Distributions

In this section, we define gamma and beta distributions in terms of a new parameter $k > 0$ and discuss some properties of these distributions in terms of k.

Definition 1. Let Z be a continuous random variable; then it is said to have a k-gamma distribution with parameters $m > 0$ and $k > 0$, if its probability density function is defined by

$$f_k(z) = \begin{cases} \frac{1}{\Gamma_k(m)} z^{m-1} e^{-z^k/k}, & 0 \leqq z < \infty,\ k > 0, \\ 0, & \text{elsewhere} \end{cases} \tag{28}$$

and its distribution function $F_k(z)$ is defined by

$$F_k(z) = \begin{cases} \int_0^z \frac{1}{\Gamma_k(m)} z^{m-1} e^{-z^k/k} dz, & z > 0, \\ 0, & z < 0. \end{cases} \tag{29}$$

Proposition 2. *The newly defined $\Gamma_k(m)$ distribution satisfies the following properties.*

(i) *The k-gamma distribution is the probability distribution that is area under the curve is unity.*

(ii) *The mean of k-gamma distribution is equal to a parameter m.*

(iii) *The variance of k-gamma distribution is equal to the product of two parameters mk.*

Proof of (i). Using the definition of k-gamma distribution along with the relation (10), we have

$$\int_0^{\infty} f_k(z)\,dz = \frac{1}{\Gamma_k(m)} \int_0^{\infty} z^{m-1} e^{-z^k/k} dz = \frac{\Gamma_k(m)}{\Gamma_k(m)} = 1. \tag{30}$$

□

Proof of (ii). As mean of a distribution is the expected value of the variate, so the mean of the k-gamma distribution is given by

$$\overline{z} = E_k(Z) = \frac{1}{\Gamma_k(m)} \int_0^{\infty} z \cdot z^{m-1} e^{-z^k/k} dz. \tag{31}$$

Using the definition of k-gamma function and the relation (13), we have

$$\overline{z} = \frac{1}{\Gamma_k(m)} \int_0^{\infty} z^m e^{-z^k/k} dz = \frac{\Gamma_k(m+k)}{\Gamma_k(m)} = m\frac{\Gamma_k(m)}{\Gamma_k(m)} = m. \tag{32}$$

□

Proof of (iii). As variance of a distribution is equal to $E(x^2) - (E(x))^2$, so the variance of k-gamma distribution is calculated as

$$\mathrm{Var}_k(Z) = E_k\left(Z^2\right) - \left(E_k(Z)\right)^2. \tag{33}$$

Now, we have to find $E_k(Z^2)$, which is given by

$$\begin{aligned} E_k\left(Z^2\right) &= \frac{1}{\Gamma_k(m)} \int_0^{\infty} z^2 \cdot z^{m-1} e^{-z^k/k} dz \\ &= \frac{1}{\Gamma_k(m)} \int_0^{\infty} z^{m+1} e^{-z^k/k} dz \\ &= \frac{\Gamma_k(m+2k)}{\Gamma_k(m)} = \frac{(m+k)\,m\Gamma_k(m)}{\Gamma_k(m)} \\ &= m(m+k). \end{aligned} \tag{34}$$

Thus we obtain the variance of k-gamma distribution as

$$\sigma_k^2 = m(m+k) - m^2 = mk, \tag{35}$$

where σ_k^2 is the notation of variance present in the literature. □

2.1. k-Beta Distribution of First Kind. Let Z be a continuous random variable; then it is said to have a k-beta distribution of the first kind with two parameters m and n, if its probability distribution function is defined by

$$f_k(z) = \begin{cases} \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1}, & 0 \leqq z \leqq 1;\ m,n,k > 0, \\ 0, & \text{elsewhere.} \end{cases} \tag{36}$$

In the above distribution, the beta variable of the first kind is referred to as $\beta_{1,k}(m,n)$ and its distribution function $F_k(z)$ is given by

$$F_k(z) = \begin{cases} 0, & z < 0, \\ \int_0^z \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1} dz, & 0 \leqq z \leqq 1; \\ & m,n > 0, \\ 0, & z > 1. \end{cases} \quad (37)$$

Proposition 3. *The k-beta distribution $\beta_{1,k}(m,n)$ satisfies the following basic properties.*

(i) *k-beta distribution is the probability distribution that is the area of $\beta_{1,k}(m,n)$ under a curve $f_k(z)$ is unity.*

(ii) *The mean of this distribution is $m/(m+n)$.*

(iii) *The variance of $\beta_{1,k}(m,n)$ is $mnk/((m+n)^2(m+n+k))$.*

Proof of (i). By using the above definition of k-beta distribution, we have

$$\int_0^z F_k(z)\,dz = \int_0^z \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1} dz, \quad 0 \leqq z \leqq 1; \quad m,n > 0. \quad (38)$$

By the relation (11), we get

$$\int_0^z F_k(z)\,dz = \int_0^1 \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1} dz = \frac{B_k(m,n)}{B_k(m,n)} = 1. \quad (39)$$

□

Proof of (ii). The mean of the distribution, $\mu'_{1,k}$, is given by

$$\mu'_{1,k} = E_k(Z) = \int_0^z zF_k(z)\,dz = \int_0^z \frac{1}{kB_k(m,n)} z \cdot z^{m/k-1}(1-z)^{n/k-1} dz, \quad 0 \leqq z \leqq 1; \quad m,n > 0. \quad (40)$$

Using the relations (12), (13), and (16), we have

$$\mu'_{1,k} = \int_0^1 \frac{1}{kB_k(m,n)} z^{m/k}(1-z)^{n/k-1} dz = \frac{B_k(m+k,n)}{B_k(m,n)} = \frac{\Gamma_k(m+k)\,\Gamma_k(n)\,\Gamma_k(m+n)}{\Gamma_k(m)\,\Gamma_k(n)\,\Gamma_k(m+n+k)} = \frac{m}{m+n}. \quad (41)$$

□

Proof of (iii). The variance of $\beta_{1,k}(m,n)$ is given by

$$\sigma_k^2 = (\mathrm{Var})_k = E_k\left(Z^2\right) - \left(E_k(Z)\right)^2, \quad (42)$$

$$E_k\left(Z^2\right) = \int_0^1 \frac{1}{kB_k(m,n)} z^{m/k+1}(1-z)^{n/k-1} dz = \frac{B_k(m+2k,n)}{B_k(m,n)} = \frac{\Gamma_k(m+2k)\,\Gamma_k(n)\,\Gamma_k(m+n)}{\Gamma_k(m)\,\Gamma_k(n)\,\Gamma_k(m+n+2k)} = \frac{m(m+k)}{(m+n)(m+n+k)}. \quad (43)$$

Thus substituting the values of $E_k(Z^2)$ and $E_k(Z)$ in (42) along with some algebraic calculations we have the desired result. □

2.2. k-Beta Distribution of the Second Kind. A continuous random variable Z is said to have a k-beta distribution of the second kind with parameters m and n, if its probability distribution function is defined by

$$f_k(z) = \begin{cases} \frac{1}{k\beta_k(m,n)} \frac{z^{m/k-1}}{(1+z)^{(m+n)/k}}, & 0 \leqq z < \infty;\ m,n,k > 0, \\ 0, & \text{otherwise}. \end{cases} \quad (44)$$

Note. The k-beta distribution of the second kind is denoted by $\beta_{2,k}(m,n)$.

Theorem 4. *The k-beta function of the second kind represents a probability distribution function that is*

$$\int_0^\infty f_k(z)\,dz = 1. \quad (45)$$

Proof. We observe that

$$\int_0^\infty f_k(z)\,dz = \int_0^\infty \frac{1}{k\beta_k(m,n)} \frac{z^{m/k-1}}{(1+z)^{(m+n)/k}} dz. \quad (46)$$

Let $1+z = 1/y$, so that $dz = -dy/y^2$; thus by using the relation (11), the above equation gives

$$= \frac{1}{\beta_k(m,n)} \frac{1}{k} \int_0^1 y^{n/k-1}(1-y)^{m/k-1} dy = \frac{\beta_k(m,n)}{\beta_k(m,n)} = 1. \quad (47)$$

□

3. Moment Generating Function of k-Gamma Distribution

In this section, we derive the moment generating function of continuous random variable Z of newly defined k-gamma distribution in terms of a new parameter $k > 0$, which is illustrated as

$$M_{0,k}(t) = E_k\left(e^{tZ^k}\right) = \int_0^\infty \frac{1}{\Gamma_k(m)} e^{tz^k} z^{m-1} e^{-z^k/k} dz$$
$$= \frac{1}{\Gamma_k(m)} \int_0^\infty z^{m-1} e^{(-z^k/k)(1-kt)} dz. \quad (48)$$

Let $u = z(1-kt)^{1/k}$, so that $z = u/(1-kt)^{1/k}$ and $dz = du/(1-kt)^{1/k}$. Then substituting these values in (48), we obtain

$$M_{0,k}(t) = \frac{1}{(1-kt)^{(m-1)/k}\Gamma_k(m)} \int_0^\infty u^{m-1} e^{-u^k/k} \frac{du}{(1-kt)^{1/k}}$$
$$= \frac{1}{(1-kt)^{m/k}\Gamma_k(m)} \int_0^\infty u^{m-1} e^{-u^k/k} du$$
$$= \frac{\Gamma_k(m)}{(1-kt)^{m/k}\Gamma_k(m)} = (1-kt)^{-m/k}, \quad |kt| < 1. \quad (49)$$

Now differentiating r times $M_{0,k}(t)$ with respect to t and putting $t = 0$, we get

$$\mu'_{r,k} = m(m+k)(m+2k)\cdots(m+(r-1)k). \quad (50)$$

Thus when $r = 1$, we obtain $\mu'_{1,k} = m$, when $r = 2$, $\mu'_{2,k} = m(m+k)$, and hence $\mu_{2,k} = \mu'^2_{1,k} - \mu'_{2,k} = mk$ = variance of the k-gamma distribution proved in Proposition 2.

3.1. Higher Moment in terms of k. The rth moment in terms of k is given by

$$\mu'_{r,k}$$
$$= E(Z^r) = \frac{1}{kB_k(m,n)} \int_0^1 z^r \cdot z^{m/k-1}(1-z)^{n/k-1} dz$$
$$= \frac{1}{kB_k(m,n)} \int_0^1 z^{m/k+r-1}(1-z)^{n/k-1} dz$$
$$= \frac{B_k(m+rk,n)}{B_k(m,n)} = \frac{\Gamma_k(m+rk)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(m+rk+n)}$$
$$= \frac{m(m+k)(m+2k)\cdots(m+(r-1)k)}{(m+n)(m+n+k)(m+n+2k)\cdots(m+n+(r-1)k)}. \quad (51)$$

Theorem 5. *The moments of the higher order of k-beta distribution of the second kind are given as*

$$\mu'_{r,k} = \frac{m(m+k)(m+2k)\cdots(m+(r-1)k)}{(n-k)(n-2k)\cdots(n-rk)}. \quad (52)$$

Proof. Consider

$$\mu'_{r,k} = E(Z^r) = \int_0^\infty \frac{1}{k\beta_k(m,n)} \frac{z^{m/k-1+r}}{(1+z)^{(m+n)/k}} dz. \quad (53)$$

Changing the variables as $z = (1-y)/y \Rightarrow dz = (-1/y^2)dy$, above equation becomes

$$= \frac{1}{k\beta_k(m,n)} \int_0^1 y^{n/k-r-1}(1-y)^{m/k+r-1} dy. \quad (54)$$

Replacing $(1-y)$ by t, we have

$$\mu'_{r,k} = \frac{1}{\beta_k(m,n)} \frac{1}{k} \int_0^1 t^{m/k+r-1}(1-t)^{n/k-r-1} dt$$
$$= \frac{\beta_k(m+rk, n-rk)}{\beta_k(m,n)}$$
$$= \frac{\Gamma_k(m+rk)\Gamma_k(n-rk)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)\Gamma_k(m+n)}$$
$$= \frac{\Gamma_k(m+rk)\Gamma_k(n-rk)}{\Gamma_k(m)\Gamma_k(n)}. \quad (55)$$

Now using $\Gamma_k(n-rk) = \Gamma_k(n)/(n-k)(n-2k)\cdots(n-rk)$ in the above equation we get the desired result. □

4. Conclusion

In this paper the authors conclude that we have the following.

(i) If k tends to 1, then k-gamma distribution and k-beta distribution tend to classical gamma and beta distribution.

(ii) The authors also conclude that the area of k-gamma distribution and k-beta distribution for each positive value of k is one and their mean is equal to a parameter m and $m/(m+n)$, respectively. The variance of k-gamma distribution for each positive value of k is equal to k times of the parameter m. In this case if $k = 1$, then it will be equal to variance of gamma distribution. The variance of k-beta distribution for each positive value of k is also defined.

(iii) In this paper the authors introduced moments generating function and higher moments in terms of a new parameter $k > 0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to express profound gratitude to referees for deeper review of this paper and the referee's useful suggestions that led to an improved presentation of the paper.

References

[1] E. D. Rainville, *Special Functions*, The Macmillan, New York, NY, USA, 1960.

[2] R. Diaz, and E. Pariguan, "On hypergeometric functions and Pochhammer k-symbol," *Divulgaciones Matematicas*, vol. 15, no. 2, pp. 179–192, 2007.

[3] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, vol. 2, Charles Griffin and Company, London, UK, 1961.

[4] R. J. Larsen and M. L. Marx, *An Introduction to Mathematical Statistics and Its Applications*, Prentice-Hall International, 5th edition, 2011.

[5] C. Walac, *A Hand Book on Statictical Distributations for Experimentalist*, 2007.

[6] C. G. Kokologiannaki, "Properties and inequalities of generalized k-gamma, beta and zeta functions," *International Journal of Contemporary Mathematical Sciences*, vol. 5, no. 13–16, pp. 653–660, 2010.

[7] C. G. Kokologiannaki and V. Krasniqi, "Some properties of the k-gamma function," *Le Matematiche*, vol. 68, no. 1, pp. 13–22, 2013.

[8] V. Krasniqi, "A limit for the k-gamma and k-beta function," *International Mathematical Forum*, vol. 5, no. 33, pp. 1613–1617, 2010.

[9] M. Mansoor, "Determining the k-generalized gamma function $\Gamma_K \mathscr{X}$ by functional equations," *Journal of Contemporary Mathematical Sciences*, vol. 4, no. 2, pp. 1037–1042, 2009.

[10] S. Mubeen and G. M. Habibullah, "An integral representation of some k-hypergeometric functions," *International Mathematical Forum*, vol. 7, no. 1–4, pp. 203–207, 2012.

[11] S. Mubeen and G. M. Habibullah, "k-fractional integrals and application," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 1–4, pp. 89–94, 2012.

[12] S. Mubeen, M. Naz, and G. Rahman, "A note on k-hypergemetric differential equations," *Journal of Inequalities and Special Functions*, vol. 4, no. 3, pp. 38–43, 2013.

[13] S. Mubeen, G. Rahman, A. Rehman, and M. Naz, "Contiguous function relations for k-hypergeometric functions," *ISRN Mathematical Analysis*, vol. 2014, Article ID 410801, 6 pages, 2014.

[14] S. Mubeen, M. Naz, A. Rehman, and G. Rahman, "Solutions of k-hypergeometric differential equations," *Journal of Applied Mathematics*, vol. 2014, Article ID 128787, 13 pages, 2014.

[15] F. Merovci, "Power product inequalities for the Γ_k function," *International Journal of Mathematical Analysis*, vol. 4, no. 21, pp. 1007–1012, 2010.

[16] S. Mubeen, A. Rehman, and F. Shaheen, "Properties of k-gamma, k-beta and k-psi functions," *Bothalia Journal*, vol. 4, pp. 371–379, 2014.

[17] A. Rehman, S. Mubeen, N. Sadiq, and F. Shaheen, "Some inequalities involving k-gamma and k-beta functions with applications," *Journal of Inequalities and Applications*, vol. 2014, article 224, 16 pages, 2014.