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Research Article

On a Generalized Discrete Ratio-Dependent Predator-Prey System

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Verifiable criteria are established for the permanence and existence of positive periodic solutions of a delayed discrete predator-prey model with monotonic functional response. It is shown that the conditions that ensure the permanence of this system are similar to those of its corresponding continuous system. And the investigations generalize some well-known results. In particular, a more acceptant method is given to study the bounded discrete systems rather than the comparison theorem.

1. Introduction

Since the end of the 19th century, many biological models have been established to illustrate the evolutionary of species, among them, predator-prey models attracted more and more attention of biologists and mathematicians. There are many different kinds of predator-prey models in the literature. And since 1990s, the so-called ratio-dependent predator-prey models play an important role in the investigations on predator-prey models, which can be roughly stated as that the per capita predator growth rate should be a function of the ratio of prey to predator abundance. Under some simple assumptions, a general form of a ratio-dependent model is

$$\begin{aligned} x' &= rx\left(1-\frac{x}{K}\right)-\varphi\left(\frac{x}{y}\right)y, \\ y' &= y\left(\mu\varphi\left(\frac{x}{y}\right)-D\right). \end{aligned} \tag{1.1}$$

Here the predator-prey interactions are described by $\varphi(x/y)$; this function replaces the functional response function $\varphi(x)$ in the traditional prey-dependent model. For the study of ratio-dependent predator-prey models, most works have been done on the Michaelis-Menten type model

$$\begin{aligned} x' &= rx\left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{my + x}, \\ y' &= y\left(-d + \frac{fx}{my + x}\right), \end{aligned} \tag{1.2}$$

or its periodic type

$$\begin{aligned} x' &= xa(t) - b(t)x - \frac{\alpha(t)xy}{my(t) + x}, \\ y' &= y\left(-d(t) + \frac{f(t)x}{my + x}\right), \end{aligned} \tag{1.3}$$

see [1–7] and references therein. It is easy to see that here the functional response function is $\varphi(u) = cu/(m + u)$, $u = x/y$, as we know, this functional response function was first used by Holling [8], and later biologists call it Holling type II functional response function, it usually describes the uptake of substrate by the microorganisms in microbial dynamics or chemical kinetics [9]. And in the present paper, we will concentrate on the general form of the ratio-dependent predator-prey model.

For the sake of convenience, we introduce some notations and definitions. Denote $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}^+$ as the sets of all integers, real numbers, and nonnegative real numbers, respectively. Let C denote the set of all bounded sequences $f : \mathbb{Z} \to \mathbb{R}$, C_+ is the set of all $f \in C$ such that $f > 0$, and $C_\omega = \{f \in C_+ \mid f(k + \omega) = f(k), k \in \mathbb{Z}\}$, $I_\omega = \{0, 1, \ldots, \omega - 1\}$. We define

$$f^M = \sup_{k\in\mathbb{Z}} f(k), \qquad f^L = \inf_{k\in\mathbb{Z}} f(k), \tag{1.4}$$

for any $f \in C$. Obviously, if f is an ω-periodic sequence, then

$$f^M = \max_{k\in I_\omega} f(k), \qquad f^L = \min_{k\in I_\omega} f(k). \tag{1.5}$$

We also define

$$\overline{f} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} f(k), \tag{1.6}$$

if f is an ω-periodic sequence. And denote

$$\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)dt, \qquad f^L = \min_{t\in[0,\omega]} |f(t)|, \qquad f^M = \max_{t\in[0,\omega]} |f(t)|, \tag{1.7}$$

when f is a periodic continuous function with period ω.

In view of the periodicity of the actual environment, we begin with the following periodic continuous ratio-dependent predator-prey system:

$$\begin{aligned} x'(t) &= x(t)[b(t) - a(t)x(t-\tau_1)] - c(t)g\left(\frac{x(t)}{y(t)}\right)y(t), \\ y'(t) &= y(t)\left[e(t)g\left(\frac{x(t-\tau_2)}{y(t-\tau_2)}\right) - d(t)\right], \end{aligned} \tag{1.8}$$

where $x(t)$ and $y(t)$ represent the densities of the prey population and predator population at time t respectively; $\tau_1 \geq 0$, $\tau_2 \geq 0$ are real constants; $b, d : \mathbb{R} \to \mathbb{R}$ and $a, c, e : \mathbb{R} \to \mathbb{R}^+$ are continuous periodic functions with period $\omega > 0$ and $\int_0^\omega b(t)dt > 0$, $\int_0^\omega d(t)dt > 0, a, c, e$ is not always zero; and $g(u(t))$ (here $u(t) = x(t)/y(t)$) denotes the ratio-dependent response function, which reflects the capture ability of the predator. Here we assume that $g(u)$ satisfies the following monotonic condition, for short, we call it (M):

(i) $g \in C^1[0, +\infty)$, $g(0) \geq 0$;

(ii) $g'(u) \geq 0$ for $u \in [0, +\infty)$;

(iii) $\lim_{u\to\infty} g(u) = \alpha \neq 0$.

In [10], we gave a sufficient condition for the permanence of the continuous model

$$\begin{aligned} x'(t) &= x(t)[a(t) - b(t)x(t)] - c(t)g\left(\frac{x(t)}{y(t)}\right)y(t), \\ y'(t) &= y(t)\left[e(t)g\left(\frac{x(t-\tau)}{y(t-\tau)}\right) - d(t)\right]. \end{aligned} \tag{1.9}$$

In which $a(t)$, $b(t), c(t)$, $d(t)$, and $e(t)$ are all positive periodic continuous functions with period $\omega > 0$; τ is a positive constant. In addition to condition (M), the functional response function g also satisfies the following.

(iv) There exists a positive constant h such that $u^2 g'(u) \leq h$.

Without loss of generality, in this paper, we always assume that $\alpha = 1$ (if $\alpha \neq 1$, let $c^*(t) = \alpha c(t)$ and still denote $c^*(t)$ as $c(t)$).

Some special cases of system (1.8) have been studied, see [11, 12] and so forth. In those papers, the authors mainly concentrated on the existence of periodic solutions and permanence for systems they considered.

Set

$$m = \sup_{u\in[0,+\infty)} \frac{g(u)}{u}, \tag{1.10}$$

from (ii) and (iii), we can easily obtain $0 < m < +\infty$.

Through the above assumptions, we can see that, one of the main results in [10] can be given as follows.

Theorem 1.1. *If*

(H1) $\overline{b} > m\overline{c}$

and

(H2) $\overline{e} > \overline{d}$

hold, then system (1.9) *is permanent.*

Remark 1.2. By similar methods proposed in [10], we can show that under conditions (H1) and (H2), system (1.8) is also permanent.

We also need to mention that conditions (H1) and (H2) are sufficient to assure the existence of positive periodic solutions of (1.8); this problem has been solved in [13].

However, when the size of the population is rarely small or the population has nonoverlapping generations [14, 15], a more realistic model should be considered, that is, the discrete time model. Just as pointed out in [16], even if the coefficients are constants, the asymptotic behavior of the discrete system is rather complex and "chaotic" than the continuous one, see [16] for more details. Similar to the arguments of [17], we can obtain a discrete time analogue of (1.8):

$$N_1(k+1) = N_1(k)\exp\left\{b(k) - a(k)N_1(k-[\tau_1]) - c(k)h\left(\frac{N_1(k)}{N_2(k)}\right)\right\},$$
$$N_2(k+1) = N_2(k)\exp\left\{-d(k) + e(k)g\left(\frac{N_1(k-[\tau_2])}{N_2(k-[\tau_2])}\right)\right\}, \tag{1.11}$$

where $[t]$ denotes the integer part of t $(t > 0)$ and $h(N_1(k)/N_2(k)) = N_2(k)g(N_1(k)/N_2(k))/N_1(k) = g(u(k))/u(k), u(k) = N_1(k)/N_2(k)$. Correspondingly, the basic assumptions of (1.11) is the same as that in (1.8), of cause, here $b, d : \mathbb{Z} \to \mathbb{R}$ and $a, c, e : \mathbb{Z} \to \mathbb{R}^+$ are periodic sequences with period $\omega > 0$ and $\sum_{k=0}^{\omega-1} b(k) > 0$, $\sum_{k=0}^{\omega-1} d(k) > 0$, and g satisfies (M). To the best of our knowledge, a few investigations have been carried out for the permanence on delayed discrete ecological systems, since the dynamics of these systems are usually more complicated than the continuous ones.

The exponential form of system (1.11) assures that, for any initial condition $N(0) > 0$, $N(k)$ remains positive. In the remainder of this paper, for biological reasons, we only consider solutions $N(k)$ with

$$N_i(-k) \geq 0, \quad k = 1, 2, \ldots, \max\{[\tau_1], [\tau_2]\}, \qquad N_i(0) > 0, \quad i = 1, 2. \tag{1.12}$$

System (1.11) includes many biological models as its special cases, which have been studied by many authors; see [17–20] and so forth. Among them, Fan and Wang (see [17]) considered the existence of positive periodic solutions for delayed periodic Michaelis-Menten type ratio-dependent predator-prey system

$$N_1(k+1) = N_1(k)\exp\left\{b_1(k) - a_1(k)N_1(k) - \frac{\alpha_1(k)N_2(k)}{N_1(k)+m(k)N_2(k)}\right\},$$
$$N_2(k+1) = N_2(k)\exp\left\{-b_2(k) + \frac{\alpha_2(k)N_1(k)}{N_1(k)+m(k)N_2(k)}\right\}, \tag{1.13}$$

and obtained the following theorem.

Theorem 1.3. *Assume that the following conditions hold:*

(A1) $\overline{b_1} > \overline{(\alpha_1(k)/m(k))}$,

(A2) $\overline{\alpha_2} > \overline{b_2}$.

Then (1.13) *has at least one positive* ω*-periodic solution.*

Later in [20], we proved that under conditions (A1) and (A2), system (1.13) is also permanent, so by the main result in [21], we can also obtain Theorem 1.3, which gives another method to prove the existence of periodic solutions.

From the works above, it is not difficult to find: that for the continuous time model (1.3) and the discrete time model (1.13), conditions that assure the existence of positive periodic solutions are exactly the same. In addition, when we comparing the work in [11] with that in [20], we found amazedly that conditions that assure the permanence of the discrete models are also the same as those of the continuous models. This motivated us to consider the permanence of system (1.11) only under conditions (H1) and (H2), since we have already obtained the permanence of system (1.9).

Until very recently, Yang [22] studied the permanence of system (1.11) when $\tau_1 = 0$ and obtained the following conclusion.

Theorem 1.4. *Assume that*

(B1)

$$e^M > d^L, \tag{1.14}$$

(B2)

$$b^L > mc^M, \tag{1.15}$$

(B3)

$$e^L g\left(\frac{m_1}{M_2}\right) = d^M, \tag{1.16}$$

hold, where

$$m_1 = \frac{b^L - mc^M}{a^M} \exp\left\{b^L - mc^M - a^M M_1\right\},$$
$$M_1 = \frac{1}{a^L} \exp\left\{b^M - 1\right\}, \qquad M_2 = \exp\left\{2\left(e^M - d^L\right)\right\}. \tag{1.17}$$

Then (1.11) *is permanent.*

Remark 1.5. In Theorem 1.4, condition (B3) implies condition (B1); and (B3) is an equality, it is too strong to satisfy.

As pointed out in [23], if we use the method of comparison theorem, then the additional condition (like (B3)), to some extent, is necessary. But for system itself, this condition may be not necessary. In this paper, our aim is to improve the above results. One of the main results in this paper is given below, furthermore, we can conclude Corollary 3.5 similarly, from which we could show that condition (B3) can be deleted. Now we list the main result in the following.

Theorem 1.6. *Assume that (H1) and (H2) hold. Then system* (1.11) *is permanent.*

Remark 1.7. Condition (H2) is a necessary condition.

Corollary 1.8. *Assume that (H1) and (H2) hold, then system* (1.11) *has at least one positive ω-periodic solution.*

Clearly, Theorem 1.6 extends and improves [19, Theorem 3.1], [20, Theorem 1.4]; Theorem 1.6 also extends and improves Theorem 1.4 by weaker conditions (H1) and (H2) instead of (B1–B3) when the coefficients are all periodic. In particular, our investigation gives a more acceptant method to study the bounded discrete systems, which is better than the comparison theorem.

For the permanence of biology systems, one can refer to [24–33] and the references cited therein.

The tree of this paper is arranged as follows. In the next section, we give some useful lemmas which are essential to prove our conclusions. And in the third section, we give a proof to the main result.

2. Preliminary

In this section, we list the definition of permanence and establish some useful lemmas.

Definition 2.1. System (1.11) is said to be *permanent* if there exist two positive constants λ_1, λ_2 such that

$$\lambda_1 \le \liminf_{k\to\infty} N_i(k) \le \limsup_{k\to\infty} N_i(k) \le \lambda_2, \quad i = 1, 2, \tag{2.1}$$

for any solution $(N_1(k), N_2(k))$ of (1.11).

Lemma 2.2 ([20]). *The problem*

$$x(k+1) = x(k)\exp\{a(k) - b(k)x(k)\},$$
$$x(0) = x_0 > 0, \tag{2.2}$$

has at least one periodic solution U if $b \in C_\omega$, $a \in C$ and a is an ω-periodic sequence with $\overline{a} > 0$, moreover, the following properties hold.

(a) *U is positive ω-periodic.*

(b) *U has the following estimations for it's boundary:*

$$\frac{\overline{a}}{\overline{b}}\exp\left\{-\left(\overline{|a|} + \overline{a}\right)\omega\right\} \le U(k) \le \frac{\overline{a}}{\overline{b}}\exp\left\{\left(\overline{|a|} + \overline{a}\right)\omega\right\}, \tag{2.3}$$

especially,

$$\frac{\overline{a}}{\overline{b}}\exp\{-\overline{a}\omega\} \le U(k) \le \frac{\overline{a}}{\overline{b}}\exp\{\overline{a}\omega\}, \tag{2.4}$$

if $a \in C_\omega$.

Lemma 2.3. *For any positive constant K, the problem*

$$x(k+1) = x(k)\exp\left\{-d(k) + e(k)g\left(\frac{K}{x(k)}\right)\right\}, \tag{2.5}$$

has at least one periodic solution U if $d \in C$ and d is an ω-periodic sequence provided that (H2) holds. Moreover, the following properties hold.

(a) *U is positive ω-periodic.*

(b) *U has the following estimations for its boundary:*

$$\frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}\exp\left\{-\left(\overline{|d|} + \overline{d}\right)\omega\right\} \le U(k) \le \frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}\exp\left\{\left(\overline{|d|} + \overline{d}\right)\omega\right\}, \tag{2.6}$$

especially,

$$\frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}\exp\left\{-\overline{d}\omega\right\} \le U(k) \le \frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}\exp\left\{\overline{d}\omega\right\}, \tag{2.7}$$

if $d \in C_\omega$, where g^{-1} represents the inverse of g.

Proof. We only prove that (2.6) holds, for the rest of the proof, one can refer to [17]. Let $x(n)$ be any possible ω-periodic positive solution of (2.5), then

$$\ln x(n+1) - \ln x(n) = -d(n) + e(n) g\left(\frac{K}{x(n)}\right), \tag{2.8}$$

therefore

$$\sum_{n=0}^{\omega-1} [\ln x(n+1) - \ln x(n)] = \sum_{n=0}^{\omega-1} \left[-d(n) + e(n) g\left(\frac{K}{x(n)}\right)\right] = 0, \tag{2.9}$$

this leads to

$$\sum_{n=0}^{\omega-1} e(n) g\left(\frac{K}{x(n)}\right) = \overline{d}\omega. \tag{2.10}$$

We claim that there exist some n_1 and n_2 such that

$$n_1, n_2 \in I_\omega, \quad x(n_1) \leq \frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}, \quad x(n_2) \geq \frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}. \tag{2.11}$$

If this is not true, then either

$$x(n) < \frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}, \tag{2.12}$$

or

$$x(n_2) > \frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}, \tag{2.13}$$

for any $n \in I_\omega$, in any case, we can obtain $\sum_{n=0}^{\omega-1} e(n) g(K/x(n)) \neq \overline{d}\omega$, this contradiction shows that our claim is true.

Note that for any $n \in I_\omega$,

$$\begin{aligned} \ln x(n) - \ln x(n_1) &\leq \sum_{n=0}^{\omega-1} \left|-d(n) + e(n) g\left(\frac{K}{x(n)}\right)\right|, \\ \ln x(n) - \ln x(n_2) &\geq -\sum_{n=0}^{\omega-1} \left|-d(n) + e(n) g\left(\frac{K}{x(n)}\right)\right|, \end{aligned} \tag{2.14}$$

then by virtue of equality (2.10), we complete the proof. □

Lemma 2.4. *Consider the inequality problem*

$$x(k+1) \leq x(k)\exp\{a(k) - b(k)x(k)\}. \tag{2.15}$$

If $b \in C_\omega$, $a \in C$, and a is an ω-periodic sequence with $\overline{a} > 0$, then any positive solutions $x(k)$ of (2.15) *satisfy*

$$\limsup_{k\to\infty} x(k) \leq H_1 \exp\left\{b^M H_1\right\}, \tag{2.16}$$

where

$$H_1 = \frac{\overline{a}}{\overline{b}} \exp\left\{\left(\overline{a} + \overline{|a|}\right)\omega\right\}. \tag{2.17}$$

Moreover, if $a \in C_\omega$, then

$$\limsup_{k\to\infty} x(k) \leq H_2 \exp\left\{b^M H_2\right\}, \tag{2.18}$$

where

$$H_2 = \frac{\overline{a}}{\overline{b}} \exp\{\overline{a}\omega\}. \tag{2.19}$$

Proof. Consider the following auxiliary equation:

$$z(k+1) = z(k)\exp\{a(k) - b(k)z(k)\}, \tag{2.20}$$

by Lemma 2.2, (2.20) has at least one positive ω-periodic solution, denote it as $z^*(k)$, then

$$z^*(k) \leq H_1. \tag{2.21}$$

Let

$$x(k) = \exp\{u_1(k)\}, \qquad z^*(k) = \exp\{u_2(k)\}, \tag{2.22}$$

then

$$\begin{aligned} u_1(k+1) - u_1(k) &\leq a(k) - b(k)\exp\{u_1(k)\}, \\ u_2(k+1) - u_2(k) &= a(k) - b(k)\exp\{u_2(k)\}. \end{aligned} \tag{2.23}$$

Make the transformation $u(k) = u_1(k) - u_2(k)$, we can obtain

$$u(k+1) - u(k) \leq -b(k)z^*(k)\left[\exp\{u(k)\} - 1\right]. \tag{2.24}$$

Now we divide the proof into two cases according to the oscillating property of $u(k)$. First we assume that $u(k)$ does not oscillate about zero, then $u(k)$ will be either eventually positive or eventually negative. If the latter holds, that is, $u_1(k) < u_2(k)$, we have

$$x(k) < z^*(k) \le H_1 \quad (\text{if } a \in C_\omega, \text{ then } x(k) \le H_2). \tag{2.25}$$

Either if the former holds, then by (2.24), we know $u(k+1) < u(k)$, which means that $u(k)$ is eventually decreasing, also in terms of its positivity, we know that $\lim_{k\to\infty} u(k)$ exists. Then (2.24) implies $\lim_{k\to\infty} u(k) = 0$, which leads to

$$\limsup_{k\to\infty} x(k) \le H_1 \quad \left(\text{if } a \in C_\omega, \text{ then } \limsup_{k\to\infty} x(k) \le H_2\right). \tag{2.26}$$

Now we assume that $u(k)$ oscillates about zero, by (2.24), we know that $u(k) > 0$ implies $u(k+1) \le u(k)$. Thus, if we let $\{u(k_l)\}$ be a subsequence of $\{u(k)\}$, where $u(k_l)$ is the first element of the lth positive semicycle of $\{u(k)\}$, then $\limsup_{k\to\infty} u(k) = \limsup_{l\to\infty} u(k_l)$. For the definition of semicycle and other related concepts, we refer to [34]. Notice that

$$u(k_l) \le u(k_l - 1) - b(k_l - 1) z^*(k_l - 1)\left[\exp\{u(k_l - 1)\} - 1\right] \tag{2.27}$$

and $u(k_l - 1) < 0$, then we know

$$\begin{aligned} u(k_l) &\le b(k_l - 1) z^*(k_l - 1)\left[1 - \exp\{u(k_l - 1)\}\right] \\ &\le b(k_l - 1) z^*(k_l - 1) \le (b(k_l - 1) z^*(k_l - 1))^M. \end{aligned} \tag{2.28}$$

Therefore

$$\limsup_{l\to\infty} u(k_l) \le (b(k_l - 1) z^*(k_l - 1))^M. \tag{2.29}$$

By the medium of (2.22), (2.25), and (2.26), we have

$$\limsup_{k\to\infty} x(k) \le H_1 \exp\left\{b^M H_1\right\} \quad \left(\text{if } a \in C_\omega, \text{ then } \limsup_{k\to\infty} x(k) \le H_2 \exp\left\{b^M H_2\right\}\right). \tag{2.30}$$

□

Corollary 2.5. *Any positive solution of the inequality problem* (2.15) *satisfies*

$$\limsup_{k\to\infty} x(k) \le \min\left\{\frac{1}{b^L} \exp\left\{a^M - 1\right\}, \frac{a^M}{b^L} \exp\left\{a^M\right\}\right\}, \tag{2.31}$$

where $a \in C$, $a^M > 0$ *and* $b \in C_+$.

Proof. Define the function

$$f(x) = x\exp\{a - bx\}, \quad x > 0,\ a \in \mathbb{R},\ b > 0, \tag{2.32}$$

it is easy to see

$$f(x) \le \frac{1}{b}\exp\{a - 1\}, \tag{2.33}$$

which immediately leads to

$$\limsup_{k\to\infty} x(k) \le \limsup_{k\to\infty} \frac{1}{b(k)}\exp\{a(k) - 1\} \le \frac{1}{b^L}\exp\left\{a^M - 1\right\}. \tag{2.34}$$

From (2.15), we have

$$x(k+1) \le x(k)\exp\{a(k) - b(k)x(k)\} \le x(k)\exp\left\{a^M - b^L x(k)\right\}. \tag{2.35}$$

By Lemma 2.4, for any $\omega > 0$,

$$\limsup_{k\to\infty} x(k) \le \frac{a^M}{b^L}\exp\left\{a^M\omega\right\}\exp\left\{a^M\exp\left\{a^M\omega\right\}\right\}, \tag{2.36}$$

let $\omega \to 0$, we can obtain

$$\limsup_{k\to\infty} x(k) \le \frac{a^M}{b^L}\exp\left\{a^M\right\}, \tag{2.37}$$

by (2.34) and (2.37), we complete the proof. □

Remark 2.6. Note that when $a^M \le 1/e$, $(a^M/b^L)\exp\{a^M\} \le (1/b^L)\exp\{a^M - 1\}$.

Similarly, we can obtain the following result.

Lemma 2.7. *If any positive solution $x(k)$ of the inequality problem*

$$x(k+1) \ge x(k)\exp\{a(k) - b(k)x(k)\} \tag{2.38}$$

satisfies

$$\limsup_{k\to\infty} x(k) \le H, \tag{2.39}$$

here H is a positive constant. Then if $b \in C_\omega$, $a \in C$, and a is an ω-periodic sequence with $\overline{a} > 0$, one has

$$\liminf_{k\to\infty} x(k) \geq H_3 \exp\{-Hb^M\},$$
$$H_3 = \frac{\overline{a}}{\overline{b}} \exp\left\{-\left(\overline{|a|} + \overline{a}\right)\omega\right\}. \tag{2.40}$$

Moreover, if $a \in C_\omega$, then

$$\liminf_{k\to\infty} x(k) \geq H_4 \exp\left\{-Hb^M\right\}, \tag{2.41}$$

where

$$H_4 = \frac{\overline{a}}{\overline{b}} \exp\{-\overline{a}\omega\}. \tag{2.42}$$

The proof is similar to that of Lemma 2.4.

Corollary 2.8. *If any positive solution $x(k)$ of the inequality problem*

$$x(k+1) \leq x(k) \exp\{a(k) - b(k)x(k)\} \tag{2.43}$$

satisfies

$$\limsup_{k\to\infty} x(k) \leq H, \tag{2.44}$$

here H is a positive constant, then

$$\liminf_{k\to\infty} x(k) \geq \frac{a^L}{b^M} \exp\left\{a^L - b^M H\right\}, \tag{2.45}$$

where $a \in C$ and $b \in C_+$.

3. Proof of the Main Result

For the rest of this paper, we only consider the solution of (1.11) with initial conditions (1.12). To prove Theorem 1.6, we need the following several propositions.

Proposition 3.1. *There exists a positive constant K_1 such that* $\limsup_{k\to+\infty} N_1(k) \leq K_1$.

Proof. Given any positive solution $(N_1(k), N_2(k))$ of (1.11), from the first equation of (1.11), we have

$$N_1(k+1) \leq N_1(k) \exp\{b(k) - a(k)N_1(k - [\tau_1])\}. \tag{3.1}$$

Set

$$N_1(k) = \exp\{u_1(k)\}, \tag{3.2}$$

then

$$u_1(k+1) - u_1(k) \leq b(k) - a(k)\exp\{u_1(k - [\tau_1])\}, \tag{3.3}$$

thus

$$\sum_{i=\mathrm{k}-[\tau_1]}^{k-1} (u_1(i+1) - u_1(i)) \leq \sum_{i=k-[\tau_1]}^{k-1} b(i), \tag{3.4}$$

which is equivalent to

$$u_1(k) - \sum_{i=k-[\tau_1]}^{k-1} b(i) \leq u_1(k - [\tau_1]), \tag{3.5}$$

hence

$$\begin{aligned} N_1(k - [\tau_1]) = \exp\{u_1(k - [\tau_1])\} &\geq \exp\left\{ u_1(k) - \sum_{i=k-[\tau_1]}^{k-1} b(i) \right\} \\ &= N_1(k)\exp\left\{ -\sum_{i=k-[\tau_1]}^{k-1} b(i) \right\}. \end{aligned} \tag{3.6}$$

Therefore

$$N_1(k+1) \leq N_1(k)\exp\left\{ b(k) - a(k)\exp\left\{ -\sum_{i=k-[\tau_1]}^{k-1} b(i) \right\} N_1(k) \right\}. \tag{3.7}$$

By Lemma 2.4, we have

$$\limsup_{k\to+\infty} N_1(k) \leq K_1, \tag{3.8}$$

where

$$K_1 = G_1 \exp\left\{\left(a(k)\exp\left\{-\sum_{i=k-[\tau_1]}^{k-1} b(i)\right\}\right)^M G_1\right\},$$
$$G_1 = \frac{\overline{b}}{a(k)\exp\left\{-\sum_{i=k-[\tau_1]}^{k-1} b(i)\right\}}\exp\left\{\left(\overline{b}+\overline{|b|}\right)\omega\right\}. \tag{3.9}$$

□

Proposition 3.2. *Under condition (H1), there exists a positive constant* k_1 *such that* $\liminf_{k\to\infty} N_1(k) \geq k_1$.

Proof. Given any positive solution $(N_1(k), N_2(k))$ of (1.11), from the first equation of (1.11), we have

$$N_1(k+1) \geq N_1(k)\exp\{b(k) - mc(k) - a(k)N_1(k-[\tau_1])\}. \tag{3.10}$$

Set $N_1(k) = \exp\{u_1(k)\}$, then

$$u_1(k+1) - u_1(k) \geq b(k) - mc(k) - a(k)\exp\{u_1(k-[\tau_1])\}, \tag{3.11}$$

which yields

$$\sum_{i=k-[\tau_1]}^{k-1}(u_1(i+1) - u_1(i)) \geq \sum_{i=k-[\tau_1]}^{k-1}(b(i) - mc(i) - a(i)K_1), \tag{3.12}$$

that is,

$$u_1(k-[\tau_1]) \leq u_1(k) - \sum_{i=k-[\tau_1]}^{k-1}(b(i) - mc(i) - a(i)K_1), \tag{3.13}$$

thus

$$\begin{aligned} N_1(k-[\tau_1]) &= \exp\{u_1(k-[\tau_1])\} \\ &\leq \exp\left\{u_1(k) - \sum_{i=k-[\tau_1]}^{k-1}(b(i) - mc(i) - a(i)K_1)\right\} \\ &= N_1(k)\exp\left\{-\sum_{i=k-[\tau_1]}^{k-1}(b(i) - mc(i) - a(i)K_1)\right\}. \end{aligned} \tag{3.14}$$

Therefore

$$N_1(k+1) \geq N_1(k) \times \exp\left\{ b(k) - mc(k) - a(k)\exp\left\{ -\sum_{i=k-[\tau_1]}^{k-1} (b(i) - mc(i) - a(i)K_1) \right\} N_1(k) \right\}. \tag{3.15}$$

Since (H1) holds, then by Lemma 2.7 and Proposition 3.1, we have

$$\liminf_{k\to\infty} x(k) \geq G_2 \exp\left\{ -K_1 \left(a(k)\exp\left\{ -\sum_{i=k-[\tau_1]}^{k-1} (b(i) - mc(i) - a(i)K_1) \right\} \right)^M \right\},$$
$$G_2 = \frac{\overline{b} - m\overline{c}}{a(k)\exp\left\{ -\sum_{i=k-[\tau_1]}^{k-1} (b(i) - m(i) - a(i)K_1) \right\}} \exp\left\{ -\left(\overline{|b - mc|} + \overline{b} - m\overline{c} \right)\omega \right\}. \tag{3.16}$$

□

Proposition 3.3. *If (H2) holds, then there exists a positive constant K_2 such that*

$$\limsup_{k\to\infty} N_2(k) \leq K_2. \tag{3.17}$$

Proof. Given any positive solution $(N_1(k), N_2(k))$ of (1.11). Set $N_2(k) = \exp\{u_1(k)\}$, from the second equation of (1.11) and notice that conditions (ii) and (iii) on g imply that $g(k, u(k)) \leq 1$, then

$$u_1(k+1) - u_1(k) \leq e(k) - d(k), \tag{3.18}$$

thus

$$\sum_{i=k-[\tau_2]}^{k-1} (u_1(i+1) - u_1(i)) \leq \sum_{i=k-[\tau_2]}^{k-1} (e(i) - d(i)), \tag{3.19}$$

which is equivalent to

$$u_1(k) - \sum_{i=k-[\tau_2]}^{k-1} (e(i) - d(i)) \leq u_1(k - [\tau_2]), \tag{3.20}$$

hence

$$\begin{aligned} N_2(k-[\tau_2]) &= \exp\{u_1(k-[\tau_2])\} \geq \exp\left\{u_1(k) - \sum_{i=k-[\tau_2]}^{k-1}(e(i)-d(i))\right\} \\ &= N_2(k)\left(\exp\left\{-\sum_{i=k-[\tau_2]}^{k-1}(e(i)-d(i))\right\}\right)^L. \end{aligned} \tag{3.21}$$

Therefore for any given $\varepsilon > 0$, we have

$$\begin{aligned} N_2(k+1) &= N_2(k)\exp\left\{-d(k)+e(k)g\left(\frac{N_1(k-[\tau_2])}{N_2(k-[\tau_2])}\right)\right\} \\ &\leq N_2(k)\exp\left\{-d(k)+e(k)g\left(\frac{K_1+\varepsilon}{N_2(k)\left(\exp\left\{-\sum_{i=k-[\tau_2]}^{k-1}(e(i)-d(i))\right\}\right)^L}\right)\right\}, \end{aligned} \tag{3.22}$$

for sufficiently large k. Here we use the monotonicity of the function $g(u)$.

Consider the following auxiliary equation:

$$z(k+1) = z(k)\exp\left\{-d(k)+e(k)g\left(\frac{K_1+\varepsilon}{z(k)\left(\exp\left\{-\sum_{i=k-[\tau_2]}^{k-1}(e(i)-d(i))\right\}\right)^L}\right)\right\}. \tag{3.23}$$

By Lemma 2.3 and condition (H2), we can obtain that (3.23) has at least one positive ω-periodic solution, denote it as $z_1^*(k)$ and

$$z_1^*(k) \leq \frac{K}{g^{-1}\left(\overline{d}/\overline{e}\right)}\exp\left\{\left(\overline{|d|}+\overline{d}\right)\omega\right\} := G_3, \tag{3.24}$$

where

$$K = \frac{K_1}{\left(\exp\left\{-\sum_{i=k-[\tau_2]}^{k-1}(e(i)-d(i))\right\}\right)^L}. \tag{3.25}$$

Let

$$z_1^*(k) = \exp\{u_2(k)\}, \tag{3.26}$$

then

$$u_1(k+1) - u_1(k) \le -d(k) + e(k)g\left(\frac{K}{\exp\{u_1(k)\}}\right),$$
$$u_2(k+1) - u_2(k) = -d(k) + e(k)g\left(\frac{K}{\exp\{u_2(k)\}}\right). \tag{3.27}$$

Denote $u(k) = u_1(k) - u_2(k)$, we have

$$u(k+1) - u(k) \le e(k)\left[g\left(\frac{K}{\exp\{u_1(k)\}}\right) - g\left(\frac{K}{\exp\{u_2(k)\}}\right)\right]. \tag{3.28}$$

First we assume that $u(k)$ does not oscillate about zero, then $u(k)$ will be either eventually positive or eventually negative. If the latter holds, that is, $u_1(k) < u_2(k)$, we have

$$N_2(k) < z_1^*(k) \le G_3. \tag{3.29}$$

Either if the former holds, then by (3.28), we have $u(k+1) < u(k)$, which means that $u(k)$ is eventually decreasing, also in terms of its positivity, we obtain that $\lim_{k\to\infty} u(k)$ exists. Then (3.28) leads to $\lim_{k\to\infty} u(k) = 0$, this implies

$$\limsup_{k\to\infty} N_2(k) \le G_3. \tag{3.30}$$

Now we assume that $u(k)$ oscillates about zero, in view of (3.28), we know that $u(k) > 0$ implies $u(k+1) \le u(k)$. Thus, if we let $\{u(k_l)\}$ be a subsequence of $\{u(k)\}$ where $u(k_l)$ is the first element of the lth positive semicycle of $\{u(k)\}$, then $\limsup_{k\to\infty} u(k) = \limsup_{l\to\infty} u(k_l)$. Also, from

$$u(k_l) \le u(k_l - 1) + e(k_l - 1)\left[g\left(\frac{K}{\exp\{u_1(k_l-1)\}}\right) - g\left(\frac{K}{\exp\{u_2(k_l-1)\}}\right)\right] \tag{3.31}$$

and $u(k_l - 1) < 0$, we know

$$u(k_l) \le e(k_l - 1). \tag{3.32}$$

Therefore

$$\limsup_{l\to\infty} u(k_l) \le (e(k_l - 1))^M. \tag{3.33}$$

Thus we have

$$\limsup_{k\to\infty} N_2(k) \le K_2, \tag{3.34}$$

where

$$K_2 = G_3 \exp\left\{e^M\right\}. \tag{3.35}$$

□

Proposition 3.4. *Under conditions (H1) and (H2), there exists a positive constant* k_2 *such that* $\liminf_{k\to+\infty} N_2(k) \geq k_2$.

Proof. Given any positive solution $(N_1(k), N_2(k))$ of (1.11), from the second equation of (1.11), we have

$$N_2(k+1) \geq N_2(k)\exp\{-d(k)\}, \tag{3.36}$$

then

$$\sum_{i=k-[\tau_2]}^{k-1} [\ln N_2(i+1) - \ln N_2(i)] \geq -\sum_{i=k-[\tau_2]}^{k-1} d(i), \tag{3.37}$$

hence

$$N_2(k-[\tau_2]) \leq N_2(k)\exp\left\{\sum_{i=k-[\tau_2]}^{k-1} d(i)\right\}. \tag{3.38}$$

Therefore from the second equation of (1.11), we have

$$N_2(k+1) \geq N_2(k)\exp\left\{-d(k) + e(k)g\left(\frac{k_1}{N_2(k)\left(\exp\left\{\sum_{i=k-[\tau_2]}^{k-1} d(i)\right\}\right)^M}\right)\right\}. \tag{3.39}$$

Consider the auxiliary equation

$$z(k+1) = z(k)\exp\left\{-d(k) + e(k)g\left(\frac{k_1}{z(k)\left(\exp\left\{\sum_{i=k-[\tau_2]}^{k-1} d(i)\right\}\right)^M}\right)\right\}, \tag{3.40}$$

by Lemma 2.3 and (H2), (3.40) has at least one positive ω-periodic solution, denoted it as $z_2^*(k)$, then

$$z_2^*(k) \geq \frac{G}{g^{-1}\left(\overline{d}/\overline{e}\right)}\exp\left\{-\left(\overline{|d|} + \overline{d}\right)\omega\right\} := G_4. \tag{3.41}$$

Where

$$G = \frac{k_1}{\left(\exp\left\{\sum_{i=k-[\tau_2]}^{k-1} d(i)\right\}\right)^M}. \tag{3.42}$$

If we set

$$N_2(k) = \exp\{u_1(k)\}, \qquad z_2^*(k) = \exp\{u_2(k)\}, \tag{3.43}$$

then

$$\begin{aligned} u_1(k+1) - u_1(k) &\geq -d(k) + e(k)g\left(\frac{G}{\exp\{u_1(k)\}}\right), \\ u_2(k+1) - u_2(k) &= -d(k) + e(k)g\left(\frac{G}{\exp\{u_2(k)\}}\right). \end{aligned} \tag{3.44}$$

And let $u(k) = u_1(k) - u_2(k)$, we have

$$u(k+1) - u(k) \geq e(k)\left[g\left(\frac{G}{\exp\{u_1(k)\}}\right) - g\left(\frac{G}{\exp\{u_2(k)\}}\right)\right]. \tag{3.45}$$

If $u(k)$ does not oscillate about zero, then by a similar analysis as that in Proposition 3.1, we have

$$\lim_{k\to\infty} \inf N_2(k) \geq \left(z_2^*(k)\right)^L \geq G_4. \tag{3.46}$$

Otherwise, if $u(k)$ oscillates about zero, by (3.45), we know that $u(k) < 0$ implies $u(k+1) \geq u(k)$. Thus, if we denote $\{u(k_l)\}$ as a subsequence of $\{u(k)\}$ where $u(k_l)$ is the first element of the lth negative semicycle of $\{u(k)\}$, then $\liminf_{k\to\infty} u(k) = \liminf_{l\to\infty} u(k_l)$. On the other hand, from

$$u(k_l) \geq u(k_l - 1) + e(k_l - 1)\left[g\left(\frac{G}{\exp\{u_1(k_l - 1)\}}\right) - g\left(\frac{G}{\exp\{u_2(k_l - 1)\}}\right)\right] \tag{3.47}$$

and $u(k_l - 1) > 0$ we can obtain

$$\begin{aligned} u(k_l) &\geq -e(k_l - 1)g\left(\frac{G}{\exp\{u_2(k_l - 1)\}}\right) \\ &\geq -e(k_l - 1). \end{aligned} \tag{3.48}$$

Therefore

$$\liminf_{l\to\infty} u(k_l) \geq \left(-e(k_l - 1)\right)^L. \tag{3.49}$$

By the medium of (3.43), we have

$$\liminf_{k\to\infty} N_1(k) \geq \left(z_2^*(k)\right)^L \exp\left\{\left(-e(k_l-1)\right)^L\right\}. \tag{3.50}$$

Hence $\liminf_{k\to\infty} N_1(k) \geq k_2$, where

$$k_2 = G_4 \exp\left\{-e^M\right\}. \tag{3.51}$$

□

Proof of Theorem 1.6. From the Propositions 3.1–3.4, we can easily know that system (1.11) is permanent. The proof is complete. □

By a similar process as above, we can obtain the following result.

Corollary 3.5. *Assume that* $b, d \in C$ *and* $a, c, e \in C_+$. *If*

(C1)

$$(b - mc)^L > 0, \tag{3.52}$$

(C2)

$$(e - d)^L > 0, \tag{3.53}$$

then system (1.11) *is permanent.*

Obviously, (B2) includes (C2), (B1), and (B3) include (C1). Thus, Corollary 3.5 generalizes and improves Theorem 1.4.

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