Hindawi Publishing Corporation
Journal of Mathematics
Volume 2013, Article ID 630934, 4 pages
http://dx.doi.org/10.1155/2013/630934

Research Article

Coefficient Inequalities of Analytic Functions Related to Robertson Functions

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Received 5 January 2013; Revised 19 March 2013; Accepted 2 April 2013

Academic Editor: S. T. Ali

We introduce and study a subclass of analytic functions related to Robertson functions. Here we discuss the coefficient estimate for function in this class.

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Also let S^* and C denote the well-known classes of starlike and convex functions, respectively.

For any two analytic functions f given by (1) and g with

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \text{for } z \in E, \quad (2)$$

the convolution (Hadamard product) is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad \text{for } z \in E. \quad (3)$$

Using the concept of convolution, Ruscheweyh [1] introduced a differential operator D^δ given by

$$D^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = z + \sum_{n=2}^{\infty} \varphi_n(\delta) a_n z^n, \quad (\delta > -1) \quad (4)$$

with

$$\varphi_n(\delta) = \frac{(\delta+1)_{n-1}}{(n-1)!}, \quad (5)$$

where $(x)_n$ is a Pochhammer symbol given as

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n \in \mathbb{N}. \end{cases} \quad (6)$$

It is obvious that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$, and

$$D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}, \quad \forall \delta = n \in N_0 = \{0, 1, 2, \ldots\}. \quad (7)$$

The following identity can easily be established:

$$(\delta+1) D^{\delta+1} f(z) = \delta D^\delta f(z) + z\left(D^\delta f(z)\right)'. \quad (8)$$

Now with the help of Ruscheweyh derivative, we define a class $VD_\lambda(\alpha, \beta, b, \delta)$ of analytic functions as follows.

Definition 1. Let $f(z) \in A$. Then, $f(z) \in VD_\lambda(\alpha, \beta, b, \delta)$, if and only if

$$\operatorname{Re}\left\{e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1} f(z)}{D^\delta f(z)}\right)\right\} > \alpha\left|\frac{2}{b}\left(\frac{D^{\delta+1} f(z)}{D^\delta f(z)} - 1\right)\right| + \beta\cos\lambda, \quad (9)$$

where $\alpha \geq 0$, $0 \leq \beta < 1$, $\delta > -1$, λ is real with $|\lambda| < \pi/2$, and $b \in \mathbb{C} \setminus \{0\}$.

By giving specific values to α, β, λ, b, and δ in $VD_\lambda(\alpha,\beta,b,\delta)$, we obtain many important subclasses studied by various authors in earlier papers, see for details [2–5], and list some of them as follows:

(i) $VD_\lambda(0,0,2,0) \equiv S_\lambda^*$ and $VD_\lambda(0,0,1,1) \equiv K_\lambda$, studied by Spacek [6] and Robertson [7], respectively; for the advancement work, see [8, 9].

(ii) $VD_0(\alpha,\beta,2,0) \equiv SD(\alpha,\beta)$ and $VD_0(\alpha,\beta,1,1) \equiv KD(\alpha,\beta)$, studied by both Owa et al. and Shams et al. [10, 11].

(iii) $VD_\lambda(1,0,2,0) \equiv USP(\lambda)$, $VD_\lambda(1,0,1,1) \equiv \mathrm{UCSP}(\lambda)$, introduced by Ravichandran et al. [12].

(iv) $VD_0(\alpha,\beta,b,\delta) \equiv VD(\alpha,\beta,b,\delta)$, considered by Latha [13].

(v) $VD_0(0,\beta,2,0) \equiv S^*(\beta)$, $VD_0(0,\beta,1,1) \equiv C(\beta)$, the well-known classes of starlike and convex functions of order β.

From the above special cases, we note that this class provides a continuous passage from the class of starlike functions to the class of convex functions.

We will assume throughout our discussion, unless otherwise stated, that $\alpha \geq 0$, $0 \leq \beta < 1$, $\delta > -1$, λ is real with $|\lambda| < \pi/2$, and $b \in \mathbb{C} \setminus \{0\}$.

2. Some Properties of the Class $VD_\lambda(\alpha,\beta,b,\delta)$

Theorem 2. *If $f(z) \in VD_\lambda(\alpha,\beta,b,\delta)$ with $0 \leq \alpha \leq \beta$, then*

$$f(z) \in VD_\lambda\left(0,\left(\frac{\beta-\alpha}{1-\alpha}\right),b,\delta\right). \quad (10)$$

Proof. Since $\operatorname{Re} w \leq |w|$ for any complex number w, $f(z) \in VD_\lambda(\alpha,\beta,b,\delta)$ implies that

$$\begin{aligned}\operatorname{Re}&\left\{e^{i\lambda}\left(1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^\delta f(z)}\right)\right\}\\&>\alpha\left|\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^\delta f(z)}-\frac{2}{b}\right|+\beta\cos\lambda\\&\geq\alpha\operatorname{Re}\left\{e^{i\lambda}\left(1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^\delta f(z)}\right)\right\}\\&\quad+(\beta-\alpha)\cos\lambda\end{aligned} \quad (11)$$

which implies that

$$\operatorname{Re}\left\{e^{i\lambda}\left(1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^\delta f(z)}\right)\right\} > \frac{(\beta-\alpha)\cos\lambda}{(1-\alpha)}, \quad (z\in E). \quad (12)$$

And hence, we obtain the required result.

Put $\lambda = 0$, $b = 2$, and $\delta = 0$ in Theorem 2; we obtain the following result. □

Corollary 3 (see [10]). *If $f(z) \in SD(\alpha,\beta)$ with $0 \leq \alpha \leq \beta$, then*

$$f(z) \in S^*\left(\frac{\beta-\alpha}{1-\alpha}\right). \quad (13)$$

Set $\lambda = 0$, $b = 1$, and $\delta = 1$ in Theorem 2; one has the following result.

Corollary 4 (see [10]). *If $f(z) \in KD(\alpha,\beta)$ with $0 \leq \alpha \leq \beta$, then*

$$f(z) \in K\left(\frac{\beta-\alpha}{1-\alpha}\right). \quad (14)$$

Theorem 5. *If $f(z) \in VD_\lambda(\alpha,\beta,b,\delta)$, then*

$$|a_2| \leq \frac{|b|\,|\eta|}{|1-\alpha|}, \quad (15)$$

$$|a_n| \leq \frac{(\delta+1)\,|b|\,|\eta|}{(n-1)\,|1-\alpha|\,\varphi_n(\delta)}\prod_{j=1}^{n-2}\left(1+\frac{(\delta+1)\,|b|\,|\eta|}{j\,|1-\alpha|}\right), \quad n\geq 3, \quad (16)$$

where

$$|\eta| = \sqrt{(1-\beta)^2\cos^2\lambda+(1-\alpha)^2\sin^2\lambda}. \quad (17)$$

Proof. We note that for $f(z) \in VD_\lambda(\alpha,\beta,b,\delta)$,

$$\operatorname{Re}\left\{e^{i\lambda}\left(1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^\delta f(z)}\right)\right\} > \frac{(\beta-\alpha)}{(1-\alpha)}\cos\lambda, \quad z\in E. \quad (18)$$

Let us define the function $p(z)$ by

$$\begin{aligned}p(z)=&((1-\alpha)\\&\times\left[e^{i\lambda}\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^\delta f(z)\right)\right)\right]\\&-(\beta-\alpha)\cos\lambda)\times((1-\beta)\cos\lambda+i(1-\alpha)\sin\lambda)^{-1}.\end{aligned} \quad (19)$$

Then, $p(z)$ is analytic in E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. Let

$$p(z) = 1+\sum_{n=1}^{\infty}p_n z^n. \quad (20)$$

Then, (19) can be written as

$$\begin{aligned}1&-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^\delta f(z)}\\&=1+\frac{(1-\beta)\cos\lambda+i(1-\alpha)\sin\lambda}{e^{i\lambda}(1-\alpha)}\sum_{n=1}^{\infty}p_n z^n,\end{aligned} \quad (21)$$

and using (8), we have

$$\begin{aligned}2e^{i\lambda}&\frac{(1-\alpha)}{(\delta+1)}\left(z\left(D^\delta f(z)\right)'-\left(D^\delta f(z)\right)\right)\\&=b\eta\left(D^\delta f(z)\right)\left(\sum_{n=1}^{\infty}p_n z^n\right)\end{aligned} \quad (22)$$

which implies that

$$e^{i\lambda}a_n = \frac{b\eta(\delta+1)}{2(1-\alpha)(n-1)\varphi_n(\delta)} \times \{p_{n-1} + \varphi_2(\delta)a_2p_{n-2} + \cdots + \varphi_{n-1}(\delta)a_{n-1}p_1\}, \quad (23)$$

where we have used (4) and (5). Now applying the coefficient estimates $|p_n| \le 2$ for Caratheodory function [14], we obtain

$$|a_n| \le \frac{(\delta+1)|b||\eta|}{(n-1)|1-\alpha|\varphi_n(\delta)} \times [1 + \varphi_2(\delta)|a_2| + \cdots + \varphi_{n-1}(\delta)|a_{n-1}|]. \quad (24)$$

For $n = 2$,

$$|a_2| \le \frac{|b||\eta|}{|1-\alpha|} \quad (25)$$

which proves (15).

For $n = 3$,

$$|a_3| \le \frac{(\delta+1)|b||\eta|}{2|1-\alpha|\varphi_3(\delta)}\left[1 + \frac{(\delta+1)|b||\eta|}{|1-\alpha|}\right]. \quad (26)$$

Therefore, (16) holds for $n = 3$. Assume that (16) is true for all $n = 3, 4, \ldots k$ and consider

$$\begin{aligned} |a_{k+1}| \le{}& \frac{(\delta+1)|b||\eta|}{(k+1-1)|1-\alpha|\varphi_{k+1}(\delta)} \\ &\times \Bigg\{\left(1 + \frac{(\delta+1)|b||\eta|}{|1-\alpha|}\right) \\ &+ \frac{(\delta+1)|b||\eta|}{2|1-\alpha|}\left(1 + \frac{(\delta+1)|b||\eta|}{|1-\alpha|}\right) \\ &+ \cdots + \frac{(\delta+1)|b||\eta|}{(k-1)\,|\,|1-\alpha|}\prod_{j=1}^{k-2} 1 + \frac{(\delta+1)|b||\eta|}{j|1-\alpha|}\Bigg\} \\ ={}& \frac{(\delta+1)|b||\eta|}{k|1-\alpha|\varphi_{k+1}(\delta)}\prod_{j=1}^{k-1}\left(1 + \frac{(\delta+1)|b||\eta|}{j|1-\alpha|}\right). \end{aligned} \quad (27)$$

Thus, the result is true for $n = k+1$, and hence by induction, (16) holds for all $n \ge 3$.

If we set $\lambda = 0$, $b = 2$, and $\delta = 0$ in Theorem 5, we get the result proved in [10]. □

Corollary 6. *If $f(z) \in SD(\alpha, \beta)$, then*

$$|a_2| \le \frac{2(1-\beta)}{|1-\alpha|},$$
$$|a_n| \le \frac{2(1-\beta)}{(n-1)|1-\alpha|}\prod_{j=1}^{n-2}\left(1 + \frac{2(1-\beta)}{j|1-\alpha|}\right), \quad (n \ge 3). \quad (28)$$

Remark 7. If we take $\alpha = 0$ in Corollary 6, we have

$$|a_n| \le \frac{1}{(n-1)!}\prod_{j=2}^{n}(j - 2\beta), \quad (n \ge 2) \quad (29)$$

which was proved by Robertson [15].

By setting $\lambda = 0$, $b = 1$, and $\delta = 1$ in Theorem 5, we obtain the result in [10].

Corollary 8. *If $f(z) \in KD(\alpha, \beta)$, then*

$$|a_2| \le \frac{(1-\beta)}{|1-\alpha|},$$
$$|a_n| \le \frac{2(1-\beta)}{n(n-1)|1-\alpha|}\prod_{j=1}^{n-2}\left(1 + \frac{2(1-\beta)}{j|1-\alpha|}\right), \quad (n \ge 3). \quad (30)$$

Remark 9. Letting $\alpha = 0$ in Corollary 8, we have

$$|a_n| \le \frac{1}{n!}\prod_{j=2}^{n}(j - 2\beta), \quad (n \ge 2) \quad (31)$$

given by Robertson [15].

Acknowledgment

The principle author would like to thank Prof. Dr. Ihsan Ali, Vice Chancellor Abdul Wali Khan University Mardan for providing excellent research facilities and financial support.

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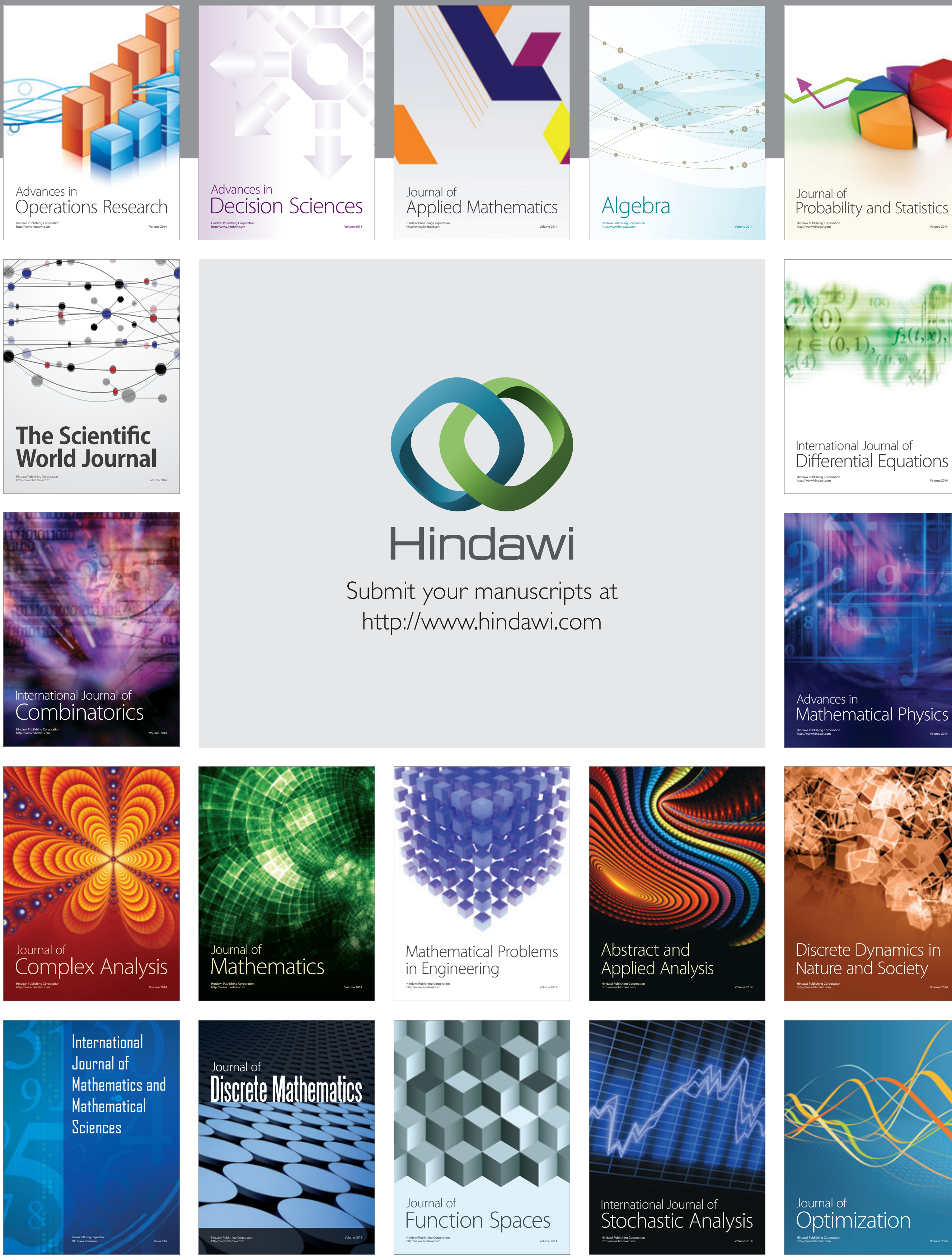
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