

## Research Article

# The Local Stability of Solutions for a Nonlinear Equation

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The approach of Kruzkov's device of doubling the variables is applied to establish the local stability of strong solutions for a nonlinear partial differential equation in the space  $L^1(R)$  by assuming that the initial value only lies in the space  $L^1(R) \cap L^\infty(R)$ .

## 1. Introduction and Main Results

Coclite and Karlsen [1] studied the well-posedness for the nonlinear equation

$$\begin{aligned} v_t - v_{txx} + 4h'(v)v_x \\ = h'''(v)v_x^3 + 3h''(v)v_x v_{xx} + h'(v)v_{xxx}, \end{aligned} \quad (1)$$

where function  $h(v)$  satisfies

$$|h'(v)| \leq c|v|, \quad |h(v)| \leq c|v|^2, \quad (2)$$

or

$$|h'(v)| \leq c, \quad |h(v)| \leq c|v|, \quad (3)$$

where  $c$  is a positive constant. The existence of entropy weak solutions and several other dynamic properties for (1) are investigated in [1].

Consider the following partial differential equation:

$$\begin{aligned} v_t - v_{txx} + mh'(v)v_x \\ = h'''(v)v_x^3 + 3h''(v)v_x v_{xx} + h'(v)v_{xxx}, \end{aligned} \quad (4)$$

where  $m$  is a positive constant and  $h(v) \in C^3$ . If  $m = 4$  and  $h(v) = v^2/2$ , (4) becomes the Degasperis-Procesi equation (see [2–12]). If  $h(v) = v^3$ , we note that  $h(v)$  does not satisfy conditions (2) and (3). Recently, Wu [13] established the local well-posedness of strong solutions for (4) in the Sobolev space  $C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R))$  provided that  $h(v) = v^3$  and the initial value lies in the space  $H^s(R)$  with  $s > 3/2$ . The objective of this work is to study (4) in the case

$h(v) = v^3$ . The local stability of strong solutions for this case is established in the space  $L^1(R)$ . We think that this stability result is a new conclusion for (4).

When  $h(v) = v^3$ , the Cauchy problem of (4) takes the form

$$\begin{aligned} v_t - v_{txx} + 3mv^2v_x = 6v_x^3 + 18vv_x v_{xx} + 3v^2 v_{xxx}, \\ v(0, x) = v_0(x), \end{aligned} \quad (5)$$

which is equivalent to

$$\begin{aligned} v_t + 3v^2v_x + (m-1)(1 - \partial_x^2)^{-1} \partial_x(v^3) = 0, \\ v(0, x) = v_0(x), \end{aligned} \quad (6)$$

where the operator  $\Lambda^{-2}g = (1 - \partial_x^2)^{-1}g = (1/2) \int_{-\infty}^{\infty} e^{-|x-y|} g(y)dy$  for any  $g \in L^\infty(R)$  or  $g \in L^{p_0}(R)$  with a parameter  $p_0$  satisfying  $1 \leq p_0 < \infty$ .

Using the approach of Kruzkov's device of doubling the variables in [14], we obtain the following result.

**Theorem 1.** Let  $v_1(t, x)$  and  $v_2(t, x)$  be two strong solutions of problem (5) or (6) with initial data  $v_1(0, x) = v_{10}(x) \in L^1(R) \cap L^\infty(R)$  and  $v_2(0, x) = v_{20}(x) \in L^1(R) \cap L^\infty(R)$ . Then, for an arbitrary  $T > 0$ ,

$$\int_{-\infty}^{\infty} |v_1(t, x) - v_2(t, x)| dx \leq e^{ct} \int_{-\infty}^{\infty} |v_{10}(x) - v_{20}(x)| dx, \quad t \in [0, T] \quad (7)$$

holds, where  $c$  is a constant depending on  $\|v_{10}\|_{L^1(R)}$ ,  $\|v_{10}\|_{L^\infty(R)}$ ,  $\|v_{20}\|_{L^1(R)}$ ,  $\|v_{20}\|_{L^\infty(R)}$ , and  $T$ .

This paper is organized as follows. Several lemmas are given in Section 2 and the proof of Theorem 1 is completed in Section 3.

**2. Several Lemmas**

**Lemma 1** (see [13]). *Assume that initial value  $v_0 = v(0, x) \in L^2(R)$ . Then the solution of problem (5) with  $m > 0$  satisfies*

$$\int_R K_1 K dx = \int_R \frac{1 + \xi^2}{m + \xi^2} |\widehat{v}(\xi)|^2 d\xi = \int_R \frac{1 + \xi^2}{m + \xi^2} |\widehat{v}_0(\xi)|^2 d\xi, \tag{8}$$

where  $K_1 = v - \partial_{xx}^2 v$  and  $K = (m - \partial_{xx}^2)^{-1} v$ . Moreover, there exist two constants  $c_1 > 0$  and  $c_2 > 0$  depending only on  $m$  such that

$$c_1 \|v_0\|_{L^2(R)} \leq c_1 \|v\|_{L^2(R)} \leq c_2 \|v_0\|_{L^2(R)}. \tag{9}$$

In fact, if  $v \in L^1(R) \cap L^\infty(R)$ , we know that  $v \in L^2(R)$ .

**Lemma 2** (see [13]). *Assume that  $v_0 \in L^2(R)$ . Then*

$$\|v(t, x)\|_{L^\infty} \leq \|v_0\|_{L^\infty} e^{ct}, \quad \forall t \in [0, \infty), \tag{10}$$

where  $c > 0$  is a constant independent of  $t$ .

**Lemma 3.** *Let  $J_v = (m - 1)(1 - \partial_x^2)^{-1} \partial_x(v^3)$  and  $v_0 \in L^1(R) \cap L^\infty(R)$ . Consider that*

$$\|J_v\|_{L^\infty(R)} \leq |m - 1| \|v_0\|_{L^\infty(R)}^3 e^{ct}, \quad t \in [0, \infty) \tag{11}$$

holds, where  $c > 0$  is a constant independent of  $t$ .

*Proof.* Since

$$\begin{aligned} |\Lambda^{-2} \partial_x(v^3)| &= \left| \frac{1}{2} \partial_x \int_{-\infty}^{\infty} e^{-|x-y|} v^3(t, y) dy \right| \\ &= \left| -\frac{1}{2} e^{-x} \int_{-\infty}^x e^y v^3(t, y) dy \right| \end{aligned}$$

$$U_k = \frac{1}{k^2} \iiint_{|(t-\tau)/2| \leq k, \delta \leq (t+\tau)/2 \leq T-\delta, |(x-y)/2| \leq k, |(x+y)/2| \leq r-\delta} |u(t, x) - u(\tau, y)| dx dt dy d\tau \tag{16}$$

satisfies  $\lim_{k \rightarrow 0} U_k = 0$ .

**Lemma 5** (see [14]). *If the function  $|\partial F(u)/\partial u|$  is bounded, then the function  $H(u, v) = \text{sign}(u - v)(F(u) - F(v))$  satisfies the Lipschitz condition in  $u$  and  $v$ , respectively.*

$$\begin{aligned} &+ \frac{1}{2} e^x \int_x^{\infty} e^{-y} v^3(t, y) dy \Big| \\ &\leq \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} |v^3(t, y)| dy, \end{aligned} \tag{12}$$

we complete the proof by using Lemma 2 and  $\int_{-\infty}^{\infty} e^{|x-y|} dy = 2$ .  $\square$

Let  $\zeta_T = [0, T] \times R$  for an arbitrary  $T > 0$ .  $C_0^\infty(\zeta_T)$  is the space of all infinitely differentiable functions with compact support in  $\zeta_T$ . We define  $\theta(\omega)$  as a function which is infinitely differentiable on  $(-\infty, +\infty)$  such that  $\theta(\omega) \geq 0$ ,  $\theta(\omega) = 0$  for  $|\omega| \geq 1$ , and  $\int_{-\infty}^{\infty} \theta(\omega) d\omega = 1$ . For any number  $k > 0$ , we let  $\theta_k(\omega) = (\theta(k^{-1}\omega)/k)$ . Then we have that  $\theta_k(\omega)$  is a function in  $C^\infty(-\infty, \infty)$  and

$$\begin{aligned} \theta_k(\omega) &\geq 0, \quad \theta_k(\omega) = 0 \quad \text{if } |\omega| \geq k, \\ |\theta_k(\omega)| &\leq \frac{c}{k}, \quad \int_{-\infty}^{\infty} \theta_k(\omega) d\omega = 1, \end{aligned} \tag{13}$$

where  $c$  is a positive constant. Assume that the function  $u(x)$  is locally integrable in  $(-\infty, \infty)$ . We define the approximation of function  $u(x)$  as

$$u^k(x) = \frac{1}{k} \int_{-\infty}^{\infty} \theta\left(\frac{x-y}{k}\right) u(y) dy, \quad k > 0. \tag{14}$$

We know that a point  $x_0$  is defined as a Lebesgue point of function  $u(x)$  if

$$\lim_{k \rightarrow 0} \frac{1}{k} \int_{|x-x_0| \leq k} |u(x) - u(x_0)| dx = 0. \tag{15}$$

For the Lebesgue point  $x_0$  of the function  $u(x)$ , we get  $\lim_{k \rightarrow 0} u^k(x_0) = u(x_0)$ . Since the measure for the set of points which are not the Lebesgue points of  $u(x)$  is zero, we have  $u^k(x) \rightarrow u(x)$  as  $k \rightarrow 0$  almost everywhere.

For any  $N > 0$ , let  $R_0 > \sup_{t \in [0, \infty)} \|v\|_{L^\infty(R)}^2 < \infty$ . Let  $\square$  represent the cone  $\{(t, x) : |x| \leq N - R_0 t, 0 < t < T_0 = \min(T, NR_0^{-1})\}$ . We let  $S_\tau$  represent the cross section of the cone  $\square$  by the plane  $t = \tau$ ,  $\tau \in [0, T_0]$ . Let  $K_r = \{x : |x| \leq r\}$ , where  $r > 0$ .

**Lemma 4** (see [14]). *Let the function  $u(t, x)$  be bounded and measurable in cylinder  $\Omega = [0, T] \times K_r$ . If  $\delta \in (0, \min[r, T])$  and  $h \in (0, \delta)$ , then the function*

**Lemma 6.** *Let  $v$  be the strong solution of problem (6) and  $\phi(t, x) \in C_0^\infty(\zeta_T)$ . Then*

$$\begin{aligned} &\iint_{\zeta_T} \left\{ |v - \alpha| \phi_t + \text{sign}(v - k) (v^3 - \alpha^3) \phi_x \right. \\ &\quad \left. - \text{sign}(v - \alpha) J_v(t, x) \phi \right\} dx dt = 0, \end{aligned} \tag{17}$$

where  $\alpha$  is an arbitrary constant.

*Proof.* Let  $\Phi(v)$  be an arbitrary twice smooth function on the line  $-\infty < v < \infty$ . We multiply the first equation of problem (6) by the function  $\Phi'(v)\phi(t, x)$ , where  $\phi(t, x) \in C_0^\infty(\zeta_T)$ . Integrating over  $\zeta_T$  and transferring the derivatives with respect to  $t$  and  $x$  to the test function  $\phi$ , for any constant  $\alpha$ , we obtain

$$\iint_{\zeta_T} \left\{ \Phi(v) \phi_t + \left[ \int_\alpha^v \Phi'(z) 3z^2 dz \right] \phi_x - \Phi'(v) J_v(t, x) \phi \right\} dx dt = 0, \tag{18}$$

in which we have used  $\int_{-\infty}^\infty \left[ \int_\alpha^v \Phi'(z) 3z^2 dz \right] \phi_x dx = - \int_{-\infty}^\infty [\phi \Phi'(v) 3v^2 v_x] dx$ .  
Integration by parts yields

$$\int_{-\infty}^\infty \left[ \int_\alpha^v \Phi'(z) 3z^2 dz \right] \phi_x dx = \int_{-\infty}^\infty \left[ (v^3 - \alpha^3) \Phi'(v) - \int_\alpha^v (z^3 - \alpha^3) \Phi''(z) dz \right] \phi_x dx. \tag{19}$$

Let  $\Phi^k(v)$  be an approximation of the function  $|v - \alpha|$  and set  $\Phi(v) = \Phi^k(v)$ . Using the properties of the  $\text{sign}(v - \alpha)$ , (18), and (19) and sending  $k \rightarrow 0$ , we have

$$\iint_{\zeta_T} \left\{ |v - \alpha| \phi_t + \text{sign}(v - \alpha) [v^3 - \alpha^3] \phi_x - \text{sign}(v - \alpha) J_v(t, x) \phi \right\} dx dt = 0, \tag{20}$$

which completes the proof.  $\square$

We note that the proof of (17) can also be found in [14].

**Lemma 7.** Assume that  $v_1(t, x)$  and  $v_2(t, x)$  are two strong solutions of problem (6) associated with the initial data  $v_{10} = v_1(0, x) \in L^1(R) \cap L^\infty(R)$  and  $v_{20} = v_2(0, x) \in L^1(R) \cap L^\infty(R)$ . For any  $\phi \in C_0^\infty(\zeta_T)$ ,

$$\left| \int_{-\infty}^\infty \text{sign}(v_1 - v_2) [J_{v_1}(t, x) - J_{v_2}(t, x)] \phi dx \right| \leq c \int_{-\infty}^\infty |v_1 - v_2| dx \tag{21}$$

holds, where  $c$  depends on  $\|v_{10}\|_{L^1(R)}$ ,  $\|v_{10}\|_{L^\infty(R)}$ ,  $\|v_{20}\|_{L^1(R)}$ ,  $\|v_{20}\|_{L^\infty(R)}$ ,  $T$ , and  $\phi$ .

*Proof.* We obtain

$$\Lambda^{-2} [v_1^3(t, x) - v_2^3(t, x)] = \frac{1}{2} \int_{-\infty}^\infty e^{-|x-y|} [v_1^3(t, y) - v_2^3(t, y)] dy, \tag{22}$$

$$\begin{aligned} & \left| \Lambda^{-2} \partial_x [v_1^3(t, x) - v_2^3(t, x)] \right| \\ &= \left| -\frac{1}{2} e^{-x} \int_{-\infty}^x e^y [v_1^3(t, y) - v_2^3(t, y)] dy + \frac{1}{2} e^x \int_x^\infty e^{-y} [v_1^3(t, y) - v_2^3(t, y)] dy \right| \\ &\leq c \int_{-\infty}^\infty e^{-|x-y|} |v_1^3(t, y) - v_2^3(t, y)| dy \\ &\leq c \int_{-\infty}^\infty e^{-|x-y|} |v_1(t, y) - v_2(t, y)| dy, \end{aligned} \tag{23}$$

in which we have used Lemma 2. Using (23) and the Fubini theorem completes the proof.  $\square$

### 3. Proof of Theorem 1

Here we state that the techniques used in this paper to establish the local stability of solutions for problem (6) come from the methods of Kruzkov's device of doubling the variables presented in Kruzkov's paper [14].

*Proof of Theorem 1.* For an arbitrary  $T > 0$ , set  $\zeta_T = [0, T] \times R$ . Let  $f(t, x) \in C_0^\infty(\zeta_T)$ . We assume that  $f(t, x) = 0$  outside the cylinder

$$\Theta = \{(t, x)\} = [\delta, T - 2\delta] \times H_{r-2\delta}, \quad 0 < 2\delta \leq \min(T, r). \tag{24}$$

We define

$$\begin{aligned} h &= f\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \theta_k\left(\frac{t-\tau}{2}\right) \theta_k\left(\frac{x-y}{2}\right) \\ &= f(\dots) \lambda_k(*), \end{aligned} \tag{25}$$

where  $(\dots) = ((t + \tau)/2, (x + y)/2)$  and  $(*) = ((t - \tau)/2, (x - y)/2)$ . The function  $\theta_k(\omega)$  is defined in (13). Note that

$$h_t + h_\tau = f_t(\dots) \lambda_k(*), \quad h_x + h_y = f_x(\dots) \lambda_k(*). \tag{26}$$

Taking  $v = v_1(t, x)$  and  $\alpha = v_2(\tau, y)$  and assuming that  $f(t, x) = 0$  outside the cylinder  $\Theta$ , from Lemma 6, we have

$$\begin{aligned} & \iiint_{\zeta_T \times \zeta_T} \left\{ |v_1(t, x) - v_2(\tau, y)| h_t + \text{sign}(v_1(t, x) - v_2(\tau, y)) \right. \\ & \quad \times (v_1^3(t, x) - v_2^3(\tau, y)) h_x \\ & \quad - \text{sign}(v_1(t, x) - v_2(\tau, y)) \\ & \quad \left. \times J_{v_1}(t, x) h \right\} dx dt dy d\tau = 0. \end{aligned} \tag{27}$$

Similarly, it has

$$\begin{aligned} & \iiint_{\zeta_T \times \zeta_T} \left\{ |v_2(\tau, y) - v_1(t, x)| g_\tau \right. \\ & \quad + \text{sign}(v_2(\tau, y) - v_1(t, x)) \\ & \quad \times (v_2^3(\tau, y) - v_1^3(t, x)) h_y \\ & \quad - \text{sign}(v_2(\tau, y) - v_1(t, x)) \\ & \quad \left. \times J_{v_2}(\tau, y) h \right\} dx dt dy d\tau = 0, \end{aligned} \tag{28}$$

from which we obtain

$$\begin{aligned} 0 & \leq \iiint_{\zeta_T \times \zeta_T} |v_1(t, x) - v_2(\tau, y)| (h_t + h_\tau) dx dt dy d\tau \\ & \quad + \iiint_{\zeta_T \times \zeta_T} \text{sign}(v_1(t, x) - v_2(\tau, y)) \\ & \quad \times (v_1^3(t, x) - v_2^3(\tau, y)) \\ & \quad \times [h_x + h_y] dx dt dy d\tau \\ & \quad + \left| \iiint_{\zeta_T \times \zeta_T} \text{sign}(v_1(t, x) - v_2(\tau, y)) \right. \\ & \quad \quad \left. \times (J_{v_1}(t, x) - J_{v_2}(\tau, y)) h dx dt dy d\tau \right| \\ & = \iiint_{\zeta_T \times \zeta_T} (I_1 + I_2) dx dt dy d\tau \\ & \quad + \left| \iiint_{\zeta_T \times \zeta_T} I_3 dx dt dy d\tau \right|. \end{aligned} \tag{29}$$

We claim that

$$\begin{aligned} 0 & \leq \iint_{\zeta_T} |v_1(t, x) - v_2(t, x)| f_t + \text{sign}(v_1(t, x) - v_2(t, x)) \\ & \quad \times (v_1^3(t, x) - v_2^3(t, x)) f_x dx dt \end{aligned}$$

$$|B_1(k)| \leq c \left[ k + \frac{1}{k^2} \iiint_{|(t-\tau)/2| \leq k, \delta \leq (t+\tau)/2 \leq T-\delta, |(x-y)/2| \leq k, |(x+y)/2| \leq r-\delta} |v_2(t, x) - v_2(\tau, y)| dx dt dy d\tau \right], \tag{33}$$

where the constant  $c$  does not depend on  $k$ . Using Lemma 4, we obtain  $B_1(k) \rightarrow 0$  as  $k \rightarrow 0$ . The integral  $B_2$  does not depend on  $k$ . In fact, substituting  $t = \alpha_1, (t - \tau)/2 = \beta, x = \eta, (x - y)/2 = \xi$  and noting that

$$\int_{-k}^k \int_{-\infty}^{\infty} \lambda_k(\beta, \xi) d\xi d\beta = 1, \tag{34}$$

$$\begin{aligned} & + \left| \iint_{\zeta_T} \text{sign}(v_1(t, x) - v_2(t, x)) \right. \\ & \quad \left. \times [J_{v_1}(t, x) - J_{v_2}(t, x)] f dx dt \right|. \end{aligned} \tag{30}$$

We note that the first two terms in the integrand of (29) can be represented in the form

$$H_k = H(t, x, \tau, y, v_1(t, x), v_2(\tau, y)) \lambda_k(*). \tag{31}$$

From Lemma 5, we know that  $H_k$  satisfies the Lipschitz condition in  $v_1$  and  $v_2$ , respectively. By the choice of  $h$ , we have  $H_k = 0$  outside the region as follows:

$$\{(t, x, \tau, y)\} = \left\{ \delta \leq \frac{t+\tau}{2} \leq T-2\delta, \frac{|t-\tau|}{2} \leq k, \frac{|x+y|}{2} \leq r-2\delta, \frac{|x-y|}{2} \leq k \right\},$$

$$\begin{aligned} & \iiint_{\zeta_T \times \zeta_T} H_k dx dt dy d\tau \\ & = \iiint_{\zeta_T \times \zeta_T} [H(t, x, \tau, y, v_1(t, x), v_2(\tau, y)) \\ & \quad - H(t, x, t, x, v_1(t, x), v_2(t, x))] \\ & \quad \times \lambda_k(*) dx dt dy d\tau \\ & \quad + \iiint_{\zeta_T \times \zeta_T} H(t, x, t, x, v_1(t, x), v_2(t, x)) \\ & \quad \times \lambda_k(*) dx dt dy d\tau \\ & = B_1(k) + B_2. \end{aligned} \tag{32}$$

Considering the estimate  $|\lambda(*)| \leq c/k^2$  and the expression of function  $H_k$ , we have

we have

$$\begin{aligned} B_2 & = 2^2 \iint_{\zeta_T} H_k(\alpha_1, \eta, \alpha_1, \eta, v_1(\alpha_1, \eta), v_2(\alpha_1, \eta)) \\ & \quad \times \left\{ \int_{-k}^k \int_{-\infty}^{\infty} \lambda_k(\beta, \xi) d\xi d\beta \right\} d\eta d\alpha_1 \\ & = 4 \iint_{\zeta_T} H(t, x, t, x, v_1(t, x), v_2(t, x)) dx dt. \end{aligned} \tag{35}$$

Hence

$$\begin{aligned} & \lim_{k \rightarrow 0} \iiint_{\zeta_T \times \zeta_T} H_k dx dt dy d\tau \\ &= 4 \iint_{\zeta_T} H(t, x, t, x, v_1(t, x), v_2(t, x)) dx dt. \end{aligned} \tag{36}$$

Since

$$\begin{aligned} I_3 &= \text{sign}(v_1(t, x) - v_2(\tau, y)) \\ &\times (J_{v_1}(t, x) - J_{v_2}(\tau, y)) f \lambda_k(*), \end{aligned} \tag{37}$$

$$\begin{aligned} & \iiint_{\zeta_T \times \zeta_T} I_3 dx dt dy d\tau \\ &= \iiint_{\zeta_T \times \zeta_T} [I_3(t, x, \tau, y) - I_3(t, x, t, x)] dx dt dy d\tau \\ &+ \iiint_{\zeta_T \times \zeta_T} I_3(t, x, t, x) dx dt dy d\tau = C_1(h) + C_2, \end{aligned} \tag{38}$$

we obtain

$$|C_1(k)| \leq c \left( k + \frac{1}{k^2} \iiint_{|(t-\tau)/2| \leq k, \delta \leq (t+\tau)/2 \leq T-\delta, |(x-y)/2| \leq k, |(x+y)/2| \leq r-\delta} |J_{v_2}(t, x) - J_{v_2}(\tau, y)| dx dt dy d\tau \right). \tag{39}$$

By Lemmas 3 and 4, we have  $C_1(h) \rightarrow 0$  as  $k \rightarrow 0$ . Using (34), we have

$$\begin{aligned} C_2 &= 2^2 \iint_{\zeta_T} I_3(\alpha_1, \eta, \alpha_1, \eta, v_1(\alpha_1, \eta), v_2(\alpha_1, \eta)) \\ &\times \left\{ \int_{-h}^h \int_{-\infty}^{\infty} \lambda_h(\beta, \xi) d\xi d\beta \right\} d\eta d\alpha_1 \\ &= 4 \iint_{\zeta_T} I_3(t, x, t, x, v_1(t, x), v_2(t, x)) dx dt \\ &= 4 \iint_{\zeta_T} \text{sign}(v_1(t, x) - v_2(t, x)) \\ &\times (J_{v_1}(t, x) - J_{v_2}(t, x)) f(t, x) dx dt. \end{aligned} \tag{40}$$

From (36) and (38)–(40), we prove that inequality (30) holds. Let

$$B(t) = \int_{-\infty}^{\infty} |v_1(t, x) - v_2(t, x)| dx. \tag{41}$$

We define

$$\rho_k(\sigma) = \int_{-\infty}^{\sigma} \theta_k(\omega) d\omega, \quad (\rho'_k(\sigma) = \theta_k(\sigma) \geq 0) \tag{42}$$

and choose the two numbers  $\tau_1$  and  $\tau_2 \in (0, T_0)$ ,  $\tau_1 < \tau_2$ . In (30), we choose

$$\begin{aligned} f &= [\rho_k(t - \tau_1) - \rho_k(t - \tau_2)] \chi(t, x), \\ k &< \min(\tau_1, T_0 - \tau_2), \end{aligned} \tag{43}$$

where

$$\begin{aligned} \chi(t, x) &= \chi_\varepsilon(t, x) = 1 - \rho_\varepsilon(|x| + R_0 t - N + \varepsilon), \\ \varepsilon &> 0. \end{aligned} \tag{44}$$

When  $\varepsilon$  is sufficiently small, we note that function  $\chi(t, x) = 0$  outside the cone  $\sqsupset$  and  $f(t, x) = 0$  outside the set  $\Theta$ . For  $(t, x) \in \Theta$ , we have

$$0 = \chi_t + R_0 |\chi_x| \geq \chi_t + R_0 \chi_x. \tag{45}$$

Applying (30) and (42)–(45) and suitably choosing large  $R_0$ , we have the inequality

$$\begin{aligned} 0 &\leq \iint_{\xi_{T_0}} \{[\theta_k(t - \tau_1) - \theta_k(t - \tau_2)] \\ &\times \chi_\varepsilon |v_1(t, x) - v_2(t, x)|\} dx dt \\ &+ \left| \iint_{\xi_{T_0}} [\rho_k(t - \tau_1) - \rho_k(t - \tau_2)] [J_{v_1}(t, x) - J_{v_2}(t, x)] \right. \\ &\left. \times E(t, x) \chi(t, x) dx dt \right|, \end{aligned} \tag{46}$$

where  $E(t, x) = \text{sign}[v_1(t, x) - v_2(t, x)]$ .

From (46), we obtain

$$\begin{aligned} 0 &\leq \iint_{\xi_{T_0}} \{[\theta_k(t - \tau_1) - \theta_k(t - \tau_2)] \\ &\times \chi_\varepsilon |v_1(t, x) - v_2(t, x)|\} dx dt \\ &+ \int_0^{T_0} (\rho_k(t - \tau_1) - \rho_k(t - \tau_2)) \\ &\times \left| \int_{-\infty}^{\infty} [J_{v_1}(t, x) - J_{v_2}(t, x)] E(t, x) \chi(t, x) dx \right| dt. \end{aligned} \tag{47}$$

Using Lemma 7, we have

$$\begin{aligned} 0 &\leq \iint_{\xi_{T_0}} \{[\theta_k(t - \tau_1) - \theta_k(t - \tau_2)] \\ &\times \chi_\varepsilon |v_1(t, x) - v_2(t, x)|\} dx dt \\ &+ c \int_0^{T_0} (\rho_k(t - \tau_1) - \rho_k(t - \tau_2)) \\ &\times \int_{-\infty}^{\infty} |v_1 - v_2| dx dt, \end{aligned} \tag{48}$$

where  $c > 0$  is a constant as described in (21).

Letting  $\varepsilon \rightarrow 0$  in (48) and sending  $N \rightarrow \infty$ , we have

$$\begin{aligned}
 0 \leq & \iint_{\xi_{\tau_0}} \{[\theta_k(t - \tau_1) - \theta_k(t - \tau_2)] \\
 & \times |v_1(t, x) - v_2(t, x)|\} dx dt \\
 & + c \int_0^{T_0} (\rho_k(t - \tau_1) - \rho_k(t - \tau_2)) \\
 & \times \int_{-\infty}^{\infty} |v_1 - v_2| dx dt.
 \end{aligned} \tag{49}$$

By the properties of the function  $\theta_k(\omega)$  for  $k \leq \min(\tau_1, T_0 - \tau_1)$ , we have

$$\begin{aligned}
 & \left| \int_0^{T_0} \theta_k(t - \tau_1) B(t) dt - B(\tau_1) \right| \\
 & = \left| \int_0^{T_0} \theta_k(t - \tau_1) [B(t) - B(\tau_1)] dt \right| \\
 & \leq c \frac{1}{k} \int_{\tau_1 - k}^{\tau_1 + k} |B(t) - B(\tau_1)| dt \rightarrow 0 \quad \text{as } k \rightarrow 0,
 \end{aligned} \tag{50}$$

where  $c$  is independent of  $h$ .

Set

$$P(\tau_1) = \int_0^{T_0} \rho_k(t - \tau_1) B(t) dt = \int_0^{T_0} \int_{-\infty}^{t - \tau_1} \theta_k(\sigma) d\sigma B(t) dt. \tag{51}$$

Using the similar proof of (50), we get

$$\begin{aligned}
 P'(\tau_1) = - \int_0^{T_0} \theta_k(t - \tau_1) B(t) dt \rightarrow -B(\tau_1) \\
 \text{as } k \rightarrow 0,
 \end{aligned} \tag{52}$$

from which we obtain

$$P(\tau_1) \rightarrow P(0) - \int_0^{\tau_1} B(\sigma) d\sigma \quad \text{as } k \rightarrow 0. \tag{53}$$

Similarly, we have

$$P(\tau_2) \rightarrow P(0) - \int_0^{\tau_2} B(\sigma) d\sigma \quad \text{as } k \rightarrow 0. \tag{54}$$

Then, we get

$$P(\tau_1) - P(\tau_2) \rightarrow \int_{\tau_1}^{\tau_2} B(\sigma) d\sigma \quad \text{as } k \rightarrow 0. \tag{55}$$

Letting  $\tau_1 \rightarrow 0$  and  $\tau_2 \rightarrow t$ , from (49), (50), and (55), for any  $t \in [0, T_0]$ , we have

$$\begin{aligned}
 \int_{-\infty}^{\infty} |v_1(t, x) - v_2(t, x)| dx \leq & \int_{-\infty}^{\infty} |v_1(0, x) - v_2(0, x)| dx \\
 & + c \int_0^t \int_{-\infty}^{\infty} |v_1 - v_2| dx,
 \end{aligned} \tag{56}$$

where  $c$  depends on  $\|v_{10}\|_{L^1(R)}$ ,  $\|v_{10}\|_{L^\infty(R)}$ ,  $\|v_{20}\|_{L^1(R)}$ ,  $\|v_{20}\|_{L^\infty(R)}$ , and  $T$ . Using the Gronwall inequality and (56) completes the proof of Theorem 1.  $\square$

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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