

## Research Article

# Renormalisation Group Analysis of Turbulent Hydrodynamics

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Turbulent hydrodynamics is characterised by universal scaling properties of its structure functions. The basic framework for investigations of these functions has been set by Kolmogorov in 1941. His predictions for the scaling exponents, however, deviate from the numbers found in experiments and numerical simulations. It is a challenge for theoretical physics to derive these deviations on the basis of the Navier-Stokes equations. The renormalization group is believed to be a very promising tool for the analysis of turbulent systems, but a derivation of the scaling properties of the structure functions has so far not been achieved. In this work, we recall the problems involved, present an approach in the framework of the exact renormalisation group to overcome them, and present first numerical results.

## 1. Introduction

The theoretical understanding of hydrodynamical turbulence still represents one of the big challenges of theoretical physics. In his fundamental work on this subject, Kolmogorov [1] introduced structure functions, describing the moments of velocity differences in a fluid. Assuming scale independence in a certain range of distances, he predicted scaling behaviour of the structure functions, associated with certain classical scaling exponents. The numbers for the exponents found experimentally and later in numerical simulations deviate, however, significantly from their classical values. It is still one of the unsolved problems of classical physics to derive the scaling behaviour from first principles.

It is generally accepted that the behaviour of an incompressible fluid is on a fundamental level described by the Navier-Stokes equations, expressing the conservation of momentum of fluid elements. It should therefore in principle be possible to deduce the scaling exponents on the basis of the Navier-Stokes equations. This has, however, not been achieved so far.

A promising approach seems to be the Renormalization Group (RG), which aims to describe the dependence of the correlation functions of a given field theory on the scale on which the system is observed. Beginning with the work of Forster et al. [2], numerous attempts have been made

to apply the various formulations of the RG to turbulent hydrodynamics, but until today the observables proposed by Kolmogorov could not be deduced in accordance with experiment. See, for example, [3, 4] for some work on the RG approach to turbulence.

There are basically two different approaches to the RG, the “field theoretic” and the “exact” RG. They are related to each other, so that the distinction may appear artificial, but, nevertheless, in practice their differences show up in applications. The field theoretic RG is based on the Callan-Symanzik equations [5, 6] or closely related approaches; see, for example, [7]. Under suitable conditions the perturbative calculation of the RG functions allows to derive the scaling behaviour of correlation functions. The work of [2] is based on this approach.

A problem with the application of the field theoretic RG to the study of turbulence is the fact that it treats only part of the space of all Hamiltonians and essentially amounts to an expansion around the case of laminar flow. Therefore, it is not sufficient to capture the essential properties of the structure functions. The formulation of the RG most suitable for the study of turbulence appears to be the Exact Renormalisation Group (ERG) due to Wilson [8, 9]; see, for example, [10]. It explicitly involves the generating functional of the correlation functions to be studied and is not restricted to a small number of couplings. The scaling behaviour to be studied does not

have to be located in the vicinity of the free or laminar theory. The generating functional can be simplified and reformulated to suit the analytic methods involved. The ERG has been applied to the problem of turbulence by Collina and Tomassini [11]. Based on the Martin-Siggia-Rose functional [12], they derive an RG flow equation, different from ours, which is studied in an approximation scheme.

The aim of this work is the following. First, for the generating functional a functional integral is formulated, which explicitly incorporates all constraints, and which resolves the constraints and nonlocalities by means of Lagrange multiplier and auxiliary fields. The incompressibility condition is implemented in the functional integral, too. Here we differ from previous work, which in one way or another omitted the incompressibility condition and/or the pressure term of the Navier-Stokes equations.

Then, starting from the action contained in the functional integral, we formulate a renormalisation group transformation. The approach presented in this paper is based on the ERG. It is especially helpful, as we shall see, for the analysis of a theory with constraints, like in our case the incompressibility condition. The RG-flow, as we will discuss in detail, can be understood as the continuous way of calculating all Feynman graphs of the theory. Keeping this in mind, we establish a numerical algorithm that calculates the RG-flow by integrating out the corresponding graphs. We take advantage of the freedom of choice of a cutoff function for the propagator. Our approach leads to rate equations for the RG flow, which are quite lengthy, but straightforward. They can be iterated quickly and up to a high number of involved field operators.

In this context we show that the predictions of Kolmogorov can be identified as the trivial scaling solutions of this theory. We also show that non trivial structures in coupling space exist. In order to demonstrate the utility of the method, we have tested the algorithm on well-known theories. The identification of intermittent exponents in turbulence, however, has not yet been accomplished due to the numerical complexity of the problem and is left for future work.

## 2. Basics of Turbulence

In this section, we introduce the basic notions needed in this work and recall some of Kolmogorov's predictions from 1941 (K41). Reviews can be found, for example, in [10, 13–15].

*2.1. Navier-Stokes Equations.* The starting point of our considerations is the full Navier-Stokes equations (NSEs) given by

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha - \nu \nabla^2 v_\alpha + \frac{1}{\rho} \partial_\alpha p = 0, \quad (1)$$

where  $\nu$  denotes the  $D$ -dimensional velocity field (in Navier-Stokes turbulence,  $D$  is either 2 or 3),  $\nu$  the kinematic viscosity,  $p$  the scalar pressure field, and  $\rho$  the density of the fluid. In incompressible turbulence, the velocity field is required to be divergence-free;

$$\mathbf{M}(\nu) := \partial_\alpha v_\alpha = 0. \quad (2)$$

For fully developed turbulence, statistically homogeneous in space and time and statistically isotropic in space, a mechanism is needed to insert energy into the system, so that an equilibrium flow can develop. The standard way of providing this is to add a stochastic force (stirring force)  $f_\alpha$  to (1) that is long-range correlated as follows:

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha - \nu \nabla^2 v_\alpha + \frac{1}{\rho} \partial_\alpha p = f_\alpha. \quad (3)$$

The idea is to bring energy into the flow on large scales. Let large structures decay freely into smaller ones until the energy is finally dissipated into heat (Richardson-cascade). We model the stochastic force to be Gaussian distributed, with  $\delta$ -correlation in time and a long-range correlation function in space as follows:

$$\begin{aligned} \langle f(x_1, t) f(x_2, t') \rangle &\equiv F^{-1}(x_1, t, x_2, t') \\ &= -\epsilon \delta(t - t') \nabla^{-2}(x_1, x_2), \end{aligned} \quad (4)$$

where  $\epsilon$  is the local energy dissipation rate.  $\nabla^{-2}$  denotes the fundamental solution of the Laplacian, for example, in 3 dimensions as follows:

$$\nabla^{-2}(x_1, x_2) = \frac{1}{4\pi |x_1 - x_2|}. \quad (5)$$

Different forms have been tried for  $F$ , though in the context of the NSE it is widely believed that the form of the stochastic force does not influence the statistical characteristics of turbulence. It should be mentioned, however, that in the case of Burgers turbulence the intermittent exponents (to be defined later) clearly depend on the choice of the stirring.

Equation (2) is sufficient to eliminate the pressure term, as can be seen in the derivation of the solenoidal NSE as follows. Operating onto (3) with a divergence operator, the first and the third terms drop out, as the field is divergence-free as follows:

$$\frac{1}{\rho} \nabla^2 p = \partial_\beta f_\beta - \partial_\beta (v_\gamma \partial_\gamma v_\beta). \quad (6)$$

Inverting the Laplacian then yields

$$\frac{1}{\rho} p = \frac{\partial_\beta}{\nabla^2} f_\beta - \frac{\partial_\beta}{\nabla^2} (v_\gamma \partial_\gamma v_\beta), \quad (7)$$

which is the afore mentioned condition for the pressure field.

One might ask whether the inversion of the Laplacian leads to a unique solution for  $p$ . Two different solutions might at best differ by a harmonic function, which is either constant or growing without limits. The second option is not physical, the first one not relevant as we are only working with pressure differences.

To obtain the solenoidal NSE, replace the “solved” pressure field into (3) as

$$\partial_t v_\alpha - \nu \nabla^2 v_\alpha + \left( \delta_{\alpha\beta} - \frac{\partial_\alpha \partial_\beta}{\nabla^2} \right) (v_\gamma \partial_\gamma v_\beta) \quad (8)$$

$$= \left( \delta_{\alpha\beta} - \frac{\partial_\alpha \partial_\beta}{\nabla^2} \right) f_\beta$$

$$\iff \partial_t v_\alpha - \nu \nabla^2 v_\alpha + P_{\alpha\beta} (v_\gamma \partial_\gamma v_\beta) = P_{\alpha\beta} f_\beta. \quad (9)$$

From now on we shall investigate the solenoidal NSE. It is important to keep in mind that these are only equivalent to the full NSE as long as incompressibility is ensured.

Also observe that (9) is nonlocal, as it involves the inverse of the Laplacian operator, the integral kernel of which is of the form (5).

The operator  $P_{\alpha\beta} = \delta_{\alpha\beta} - (\partial_\alpha \partial_\beta / \nabla^2)$  is identical to the transverse projector known from electrodynamics. Due to its appearance the formulae can be rewritten in a gauge invariant way. This will facilitate to properly formulate the functional integral discussed later. As the transverse operator  $P$  projects a field onto its incompressible parts and observing that the fields we are interested in are transverse a priori, it is easy to see that

$$P_{\alpha\beta} v_\beta = \left( \delta_{\alpha\beta} - \frac{\partial_\alpha \partial_\beta}{\nabla^2} \right) v_\beta = \delta_{\alpha\beta} v_\beta = v_\alpha, \quad (10)$$

so that we are free to replace  $v$  by  $Pv$  in (3) as

$$\begin{aligned} \mathbf{N}(v) := & \partial_t P_{\alpha\beta} v_\beta - \nu \nabla^2 P_{\alpha\beta} v_\beta \\ & + P_{\alpha\beta} (P_{\gamma\delta} v_\delta \partial_\gamma P_{\beta\epsilon} v_\epsilon) = P_{\alpha\beta} f_\beta. \end{aligned} \quad (11)$$

The resulting equation looks more complicated, but it is invariant under the same local gauge transformations

$$v_\alpha \longrightarrow v_\alpha + \partial_\alpha \Lambda(x) =: U(v), \quad (12)$$

as the vector potential in electrodynamics. Constraint (2) is still required, but it now acts as a gauge fixing term.

**2.2. Structure Functions and Intermittent Exponents.** In 1941 Kolmogorov introduced a statistical framework for turbulent hydrodynamics [1]. As the theory is Galilean invariant, he proposed that the observables should be functions of velocity differences, and more specific, he considered the so-called velocity increment

$$v_{\text{inc}}(r; x) = (v(r+x) - v(r)) \cdot e_x, \quad (13)$$

the difference of the velocities at two points separated by a vector  $x$ , projected onto the unit vector in  $x$ -direction. Suitable observables are the structure functions of order  $p$ . They are defined as the  $p$ th moment of the distribution of the absolute value of the velocity increment as follows:

$$S_p(x) := \langle |v_{\text{inc}}(r; x)|^p \rangle_r. \quad (14)$$

The average is taken over all spatial points  $r$  of a realisation of the turbulent flow. In the case of homogeneous turbulence, this is supposed to be equivalent to an average over all histories  $v(x, t)$ . This average can be defined in terms of a functional integral.

Kolmogorov proposed the existence of a smallest length scale  $\lambda$ , the “dissipation scale,” below which physics is no longer dominated by turbulence, but by dissipation. Dimensional analysis leads to

$$\lambda = \left( \frac{\nu^3}{\epsilon} \right)^{1/4}, \quad (15)$$

where  $\epsilon$  is the (constant) dissipation rate. Assuming that the turbulent cascade of decaying vortices happens on scales much larger than  $\lambda$ , it is argued that observables do not depend on it and are thus self-similar, which means power-law functions of the scale

$$S_p(x) \propto (\epsilon x)^{\xi_p}. \quad (16)$$

By dimensional analysis Kolmogorov deduced

$$\xi_p = \frac{p}{3}. \quad (17)$$

It has long been pointed out [16] that the fundamental assumption, namely, the independence of the smallest scale  $\lambda$ , is by no means natural and is in general not fulfilled in critical systems. This could lead to a scale-dependent dissipation rate (or viscosity) and the breakdown of scaling law (16). Even though general agreement on this point seems to be common, the scale dependence could not yet be deduced.

In case that a typical (macroscopic) length scale  $L$  can be identified in the system, the Reynolds number is defined as

$$\text{Re} = \left( \frac{L}{\lambda} \right)^{10/3}. \quad (18)$$

$L$  might be the radius of an obstacle of the flow, or, in the context considered here, the correlation length of a choice of the stochastic force. Equation (18) coincides with the more common definition

$$\text{Re} = \frac{LU}{\nu} \quad (19)$$

if the typical velocity  $U$  is defined to be

$$U = (\epsilon L)^{1/3}. \quad (20)$$

### 3. Generating Functional

The basic object of the Exact Renormalisation Group (ERG), as well as many other field theoretical methods, is the generating functional of correlation functions. For the case of turbulence, several approaches to define the generating functional can be found in the literature; see, for example, [10, 12, 13, 17]. In the work of Martin et al. [12] the functional is characterised by means of an infinite hierarchy of equations,

analogous to the field theoretic Dyson-Schwinger equations. The results of [18] are obtained in a similar framework.

In order to set up the ERG, it is necessary to formulate the generating functional in terms of a functional integral. This approach is being followed, for example, in [11, 17]. An apparent problem is that the incompressibility condition has been disregarded in one way or another. This condition, however, leads to nonlocalities which are important for the correlations in the fluid. In this section we sketch the derivation of the Martin-Siggia-Rose functional for the solenoidal NSE and then show how to respect the incompressibility condition (2). As our derivation differs from others in the literature by aspects concerning the functional determinants and constraints, we prefer to show the line of arguments in some detail.

**3.1. Fine-Grained Distribution.** The starting point is the so-called fine-grained probability distribution for the velocity field  $v$ , obtained by counting all possible solutions to the NSE.

Consider

$$Z[J] = \int \mathcal{D}v \langle \delta(v - \mathbf{N}^{-1}(Pf)) e^{(v,J)} \rangle_f, \quad (21)$$

where  $\mathbf{N}$  is defined in (11) (here we adopt the notation of L'vov and Procaccia [17]), and we defined the abbreviation

$$(\mu, Av) = \int d^D x_1 d^D x_2 dt_1 dt_2 \times \mu(x_1, t_1) A(x_1, t_1; x_2, t_2) v(x_2, t_2). \quad (22)$$

It is important to notice that the functional integral is an integral over configurations  $v(x, t)$  of the velocity field, representing histories in space and time, covering the whole range  $-\infty < t < \infty$ . The generating functional is thus not a function of physical time  $t$ . A few remarks are in order.

- (i) In the functional integral aforementioned,  $\mathbf{N}^{-1}$  is not to be understood as the inverse of an operator  $\mathbf{N}$  (which might not exist), but as a multivalued operator counting any solution  $v$  for a given realisation of the random force  $f$ . Observe that the integrand involves a *functional*  $\delta$ -function, meaning that we are searching for histories  $v(x, t)$  that solve the NSE for all  $x$  and  $t$ , rather than a realisation  $v(x, t_1)$  at a given time  $t_1$ , depending on some initial condition.
- (ii) The average  $\langle \cdot \rangle$  is an average over all realisations of  $f$ , replacing the spatial average in (14). Here we adopt the common assumption that for homogeneous and isotropic turbulence, these averages are interchangeable. This assumption is supported by our results for Burgulence published in [19].

Making the average over all realisations of the stochastic force explicit yields

$$Z \propto \int \mathcal{D}v \int \mathcal{D}f \delta(v - \mathbf{N}^{-1}(Pf)) e^{(-1/2)(f, Ff) + (v, J)}. \quad (23)$$

Multiplying the argument of the  $\delta$ -function by  $\mathbf{N}$  leads to a functional determinant that is discussed in detail in the next paragraph:

$$\begin{aligned} \implies Z &\propto \int \mathcal{D}v \int \mathcal{D}f \delta(\mathbf{N}(v) - Pf) e^{(-1/2)(f, Ff)} \\ &\times \det \left\{ \frac{\delta \mathbf{N}_\alpha(v)(x)}{\delta v_\beta(x')} \right\} e^{(v, J)}. \end{aligned} \quad (24)$$

The  $\delta$ -function can be written in terms of a functional Fourier-transformation, introducing an auxiliary field  $u$  as

$$\delta(\mathbf{N}(v) - Pf) \propto \int \mathcal{D}u e^{i(u, \mathbf{N}(v) - Pf)}. \quad (25)$$

We define an action  $S_1$  formally by

$$Z = \int \mathcal{D}v \mathcal{D}u \mathcal{D}f e^{-S_1[v, u, f]} \det \left\{ \frac{\delta \mathbf{N}_\alpha(v)(x)}{\delta v_\beta(x')} \right\} e^{(v, J)}, \quad (26)$$

where the determinant still has to be evaluated. From this the elements of the Feynman-rules of the theory can be identified. Let us focus the attention on two parts, which together lead to the famous  $\theta(0)$ -problem as follows:

- (i)  $uv$ -(diffusion)-propagator:

$$\overset{u}{\longrightarrow} \text{---} \bullet \text{---} \overset{v}{\dashrightarrow} \quad (27)$$

The corresponding bare two-point function, also called response function, is proportional to Green's function of the diffusion equation, applied to transverse fields as follows:

$$\langle u(x_2, t_2) v(x_1, t_1) \rangle \propto \frac{1}{\partial_t P_{\alpha\beta} - \nu \nabla^2 P_{\alpha\beta}}. \quad (28)$$

In order to ensure causality of the theory, the retarded Green's function has to be chosen

$$\langle u(x_2, t_2) v(x_1, t_1) \rangle \propto \theta(t_2 - t_1). \quad (29)$$

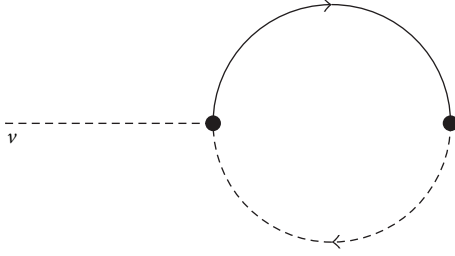
Consider the following.

- (ii)  $uvv$ -vertex:

$$\begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \text{---} \text{---} \end{array} \quad (30)$$

This vertex enters the following loop diagram (Figure 1).

As the retarded Green's function is proportional to  $\theta(t_2 - t_1)$ , this loop is proportional to the seemingly ambiguous quantity  $\theta(0)$ .


 FIGURE 1:  $u$ - $v$ -loop of the interaction term.

The appearance of  $\theta(0)$  is analogous to the It $\bar{o}$ -Stratanovich dilemma; see, for example, [7]. In our context,  $\theta(0)$  is fixed by the choice of discretisation of the time derivative inside the functional integral. For the symmetric (Stratanovich) derivative, one has  $\theta(0) = 1/2$ , while in the pure backward (It $\bar{o}$ ) case  $\theta(0) = 0$ .

To illustrate the contents of the functional integral aforementioned, we integrate out the nonphysical fields  $f$  and  $u$  by means of Gaussian integration, leading to

$$Z = \int \mathcal{D}v e^{-S_2[v]} \det \left\{ \frac{\delta \mathbf{N}_\alpha(v)(x)}{\delta v_\beta(x')} \right\} e^{(v,I)}, \quad (31)$$

with

$$S_2[v] = \frac{1}{2} (\mathbf{N}[v], F\mathbf{N}[v]). \quad (32)$$

This expression shows that field configurations not solving the NSE are admitted in the integral, but suppressed by a Gaussian weight. In principle, (31) can be used as a generating functional, and all correlation functions can be extracted from it using functional derivatives. But for the implementation of the RG, it is necessary to bring the determinant into a suitable form.

**3.2. Functional Determinant.** A straightforward way of writing the functional determinant is by using anticommuting ghost fields

$$\det \left\{ \frac{\delta \mathbf{N}_\alpha(v)(x)}{\delta v_\beta(y)} \right\} \propto \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-i(\psi^*, (\delta N/\delta v)\psi)} \quad (33)$$

$$= \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-i(\psi^*, (\partial_t P - v\nabla^2))\psi + i(\psi^*, (\delta I/\delta v)\psi)}, \quad (34)$$

where we dropped a field-independent term and defined  $I$  to be the nonlinear part of  $N$  as

$$I_\alpha[v] = v_\beta \partial_\beta v_\alpha. \quad (35)$$

This leads to the functional

$$Z = \int \mathcal{D}v \mathcal{D}u \mathcal{D}f \mathcal{D}\psi^* \mathcal{D}\psi e^{-S[v,u,f,\psi^*,\psi]} e^{(v,I)}, \quad (36)$$

where

$$S[u, v, f, \psi^*, \psi] = -i(u, \mathbf{N}[v] - Pf) - \frac{1}{2\epsilon} (f, \nabla^2 f) + i \left( \psi^*, \frac{\delta N}{\delta v} \psi \right). \quad (37)$$

It should be noted that the ghost fields, though anticommuting with each other, can be treated in the numerical procedure calculating the RG flow. In the algorithm the various contributions are generated according to some counting scheme. The terms generated by the ghost fields can be taken fully into account, order by order, as we have done in several runs of the RG flow equations.

The determinant has a simple graphical interpretation. It exactly cancels out the  $u$ - $v$ -loop shown in Figure 1. From (28) and (34), it can be seen that the  $\psi^* \psi$ - and the  $uv$ -propagators are identical. It is easily checked that  $\psi^*$ ,  $(\delta N/\delta v)\psi$  leads to two terms similar to  $u\mathbf{N}[v]$ , but with  $u$  replaced by  $\psi^*$  and one  $v$ -field replaced by  $\psi$ . When this vertex is closed to a loop by means of a  $\psi^* \psi$ -propagator, this is numerically identical to Figure 1.

This greatly simplifies the numerical calculations, as the program sorts and calculates contributions to the RG-flow according to their graphical representation. Rather than simulating two additional fields and calculating all the graphs, we are thus allowed to drop a certain class of graphs. The cancellation of certain averages can be proven even non-perturbatively, using the BRS-invariance of the action.

If the functional determinant is expressed in terms of ghost fields, this yields an extra symmetry, also called BRS-invariance [20]. The action (37) is indeed invariant under the infinitesimal transformation

$$\begin{aligned} \delta u &= 0, \\ \delta \psi^* &= 0, \\ \delta v &= \epsilon \psi^*, \\ \delta \psi &= i\epsilon u, \end{aligned} \quad (38)$$

which amounts to ‘‘half a super-symmetry.’’ From the Ward identities of this symmetry, the desired result follows on a nonperturbative level as follows:

$$\left\langle \psi \frac{\delta N}{\delta v} \psi^* \right\rangle = \langle uN \rangle. \quad (39)$$

The two sides of this equation can be interpreted as the sum of the corresponding graphs discussed in the previous section. For details we refer the reader to the explicit proof in [20].

**3.3. Incompressibility Condition.** As has been mentioned before, the incompressibility condition implies non-localities in the dynamics, which are relevant for the correlations in the fluid. This becomes apparent by considering models that only differ by the compressibility condition and give different statistics. An obvious example is Burgers’ equation, which models fully compressible fluids and shows bifractal scaling of the structure functions. It is therefore certainly inadequate to neglect condition (2) completely.

In [17] the incompressibility condition is considered to be implied in the functional integration measure. This measure is then, however, in combination with Gaussian integrands treated as a functional Gaussian measure, which effectively amounts to neglecting the incompressibility constraint.



In the context of direct numerical simulations it is sufficient to introduce the incompressibility condition through the initial conditions at time  $t = 0$ . Then the flow stays incompressible without enforcing it by a particular equation, because in the solenoidal form both the random force term and the former pressure term lead to incompressible contributions to the flow. Potential compressible perturbations of a given flow would die out due to the dissipation term. It is in fact rigorously known that in two dimensions the statistics of the Navier-Stokes equation converges to a unique steady state.

This argument does, however, not apply to our case, since the functional integral represents the equilibrium statistics and involves configurations in space and time, that is, it covers the statistics over whole histories of the fields for all times  $-\infty < t < \infty$ . Any possible compressible perturbation of the flow at a finite time  $t$  is going to be amplified in the negative  $t$ -direction. This poses a manifest problem for any numerical approach to the functional integral due to unavoidable numerical errors. The slightly compressible flows and the incompressible ones lie dense to each other in functional space, so that in any numerical application we would lose control of the boundary conditions completely.

We thus conclude that incompressibility should be taken care of explicitly in the functional integral. Writing the functional  $\delta$ -function as

$$\delta(\partial_\alpha v_\alpha) = \int \mathcal{D}\theta e^{i(\theta, \partial v)} \quad (40)$$

would be technically inconvenient in the RG equations. Therefore the  $\delta$ -functional is represented in a way familiar from the initial condition of the kernel of the diffusion equation

$$\delta(\partial_\alpha v_\alpha) \propto e^{(-1/2\kappa)(\partial_\alpha v_\alpha, \partial_\alpha v_\alpha)} \quad \text{in the limit } \kappa \rightarrow 0, \quad (41)$$

leading to

$$Z \propto \int \mathcal{D}u \mathcal{D}v \mathcal{D}f e^{-S[u, v, f]} \det \left\{ \frac{\delta \mathbf{N}_\alpha(v)(x)}{\delta v_\beta(y)} \right\} \quad (42)$$

with

$$S[u, v, f] = \frac{1}{2\kappa} (\partial v, \partial v) - i(u, \mathbf{N}[v] - Pf) - \frac{1}{2\epsilon} (f, \nabla^2 f), \quad (43)$$

where the limit  $\kappa \rightarrow 0$  is to be taken when results have been obtained, in order to enforce incompressibility strictly. In the functional integral (42) only the solenoidal part of the auxiliary field  $u$  is effective, being coupled to the velocity field  $v$ . Formally the functional integral implies an integration also over the longitudinal part of  $u$ , which would lead to a divergence. As this integration decouples completely from the remaining degrees of freedom, it contributes a constant to the generating functional  $W[\{J\}]$ , discussed in Section 4, and can therefore be neglected.

From now on we shall work with the functional (43), but formulated in wavenumber space, which is

$$\begin{aligned} S[u, v, f] &= \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \delta(p+q) dt \\ &\times \left\{ -\frac{1}{2\kappa} p_\alpha v_\alpha(p) q_\alpha v_\alpha(q) - i u_\alpha(p) \partial_t \bar{P}_{\alpha\beta}(q) v_\beta(q) \right. \\ &\quad \left. - i u_\alpha(p) v q^2 \bar{P}_{\alpha\beta} v_\beta(q) + i u_\alpha(p) \bar{P}_{\alpha\beta}(q) f_\beta(q) \right. \\ &\quad \left. + \frac{1}{2\epsilon} f_\alpha(p) q^2 f_\alpha(q) \right\} \\ &+ \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \left( \frac{d^D r}{(2\pi)^D} \right) \delta(p+q+r) dt \\ &\times u_\alpha(p) \bar{P}_{\alpha\beta}(p) (\bar{P}_{\gamma\delta}(q) v_\delta(q) r_\gamma \bar{P}_{\beta\epsilon}(r) v_\epsilon(r)). \end{aligned} \quad (44)$$

A remark concerning Galilean invariance, as analysed by Berera and Hochberg [21], is here in order. The path integral, like the NSE, is invariant under the transformations

$$\begin{aligned} v(x, t) &= v'(x', t') + c, \\ x &= x' + ct, \\ t &= t'. \end{aligned} \quad (45)$$

If averages or correlation functions of the field itself are considered, this would represent a problem that could be overcome by an application of the Faddeev-Popov method. In our case, however, it does not take effect, as we only consider averages of velocity differences, which are Galilean invariant.

**3.4. Nonlocal Interactions.** The derived action contains interaction terms of the type

$$u_\alpha P_{\alpha\beta} (P_{\gamma\delta} v_\delta \partial_\gamma P_{\beta\epsilon} v_\epsilon), \quad (46)$$

which are non-local in coordinate space, but of a very simple form in wavenumber space. In both cases we need to rewrite (46) in a local way. In coordinate space, non-local interactions are at best cumbersome; in wavenumber space, we are going to sort the terms of the action according to their power of momenta, so we try to avoid  $1/p^2$ -interactions.

We shall proceed in two steps; we will first redefine non-physical fields and then introduce new fields to remove the nonlocality of interactions. We will end up with a lengthy but local action that suits our needs for further analysis.

To make the non-local nature of the interactions more manifest, we consider functions in coordinate space within this paragraph. First we redefine the non-physical fields by

$$\begin{aligned} u &\longrightarrow \nabla^2 u, \\ \psi &\longrightarrow \nabla^2 \psi, \\ \psi^* &\longrightarrow \nabla^2 \psi^*. \end{aligned} \quad (47)$$

Introducing

$$Q_{\alpha\beta} := \nabla^2 \delta_{\alpha\beta} - \partial_\alpha \partial_\beta, \quad (48)$$

the action is written as

$$\begin{aligned} -iu_\alpha P_{\alpha\beta} (P_{\gamma\delta} v_\delta \partial_\gamma P_{\beta\epsilon} v_\epsilon) &\longrightarrow -iQ_{\alpha\beta} u_\alpha \partial_\gamma P_{\beta\epsilon} v_\epsilon P_{\gamma\delta} v_\delta, \\ \Psi_\alpha^* P_{\alpha\beta} (P_{\gamma\delta} \Psi_\delta \partial_\gamma P_{\beta\epsilon} v_\epsilon) &\longrightarrow -\partial_\beta Q_{\alpha\gamma} \Psi_\alpha^* Q_{\beta\epsilon} \Psi_\epsilon P_{\gamma\delta} v_\delta, \\ \Psi_\alpha^* P_{\alpha\beta} (P_{\gamma\delta} v_\delta \partial_\gamma P_{\beta\epsilon} v_\epsilon) &\longrightarrow Q_{\alpha\beta} \Psi_\alpha^* \partial_\gamma Q_{\beta\epsilon} \Psi_\epsilon P_{\gamma\delta} v_\delta. \end{aligned} \quad (49)$$

The functional determinant of these transformations is field independent and can thus be omitted.

The projector  $P$  contains the inverse Laplacian so that non-local terms of the general form

$$K \frac{1}{\nabla^2} L \quad (50)$$

are present. These can be removed by means of new auxiliary fields. They can be interpreted as transmitting fields that “carry” the non-local interaction from one place to another, thus replacing it by two local interactions and a propagator. Formally this is achieved by a Gaussian integral of the type

$$\int \mathcal{D}\widehat{M} e^{-(1/2)(\widehat{M}, \nabla^2 \widehat{M})}, \quad (51)$$

where  $\widehat{M}$  is the auxiliary field. This leads to a new kinetic term  $(1/2)(\widehat{M}, \nabla^2 \widehat{M})$  in the action, which is independent of all physical fields. Shifting the variables as

$$\widehat{M} := M + \lambda^{-1} \frac{1}{\nabla^2} K + \lambda \frac{1}{2} \frac{1}{\nabla^2} L \quad (52)$$

and noticing that

$$\begin{aligned} -\widehat{M} \nabla^2 \widehat{M} + K \frac{1}{\nabla^2} L &= -M \nabla^2 M - 2\lambda^{-1} MK - \lambda ML \\ &\quad - \lambda^{-2} K \frac{1}{\nabla^2} K - \frac{1}{4} \lambda^2 L \frac{1}{\nabla^2} L, \end{aligned} \quad (53)$$

we get rid of the original, non-local interaction (50) by replacing it by a new kinetic term for  $M$  and new interactions. Two of them are still non-local, but of the diagonal form

$$\lambda^2 L \frac{1}{\nabla^2} L. \quad (54)$$

They are treated by the same method to get a local action finally. We add again a Gaussian integral for a new field, say  $\widehat{N}$ , and define

$$\widehat{N} := N + \frac{1}{2} \lambda i \frac{1}{\nabla^2} L, \quad (55)$$

leading to

$$-\widehat{N} \nabla^2 \widehat{N} - \frac{1}{4} \lambda^2 L \frac{1}{\nabla^2} L = -N \nabla^2 N - i\lambda N L. \quad (56)$$

The constant  $\lambda$  is needed so that the new fields get a definite dimension.

Applying the method discussed earlier to action (44), we arrive at the local action

$$\begin{aligned} S_{\text{loc}} &= O_0 [v, u, f] + O_1 [\phi_1, \phi_2, \phi_3, f, v, u] \\ &\quad + O_2 [\phi_1, \phi_2, \phi_3, f, v, u], \end{aligned} \quad (57)$$

with

$$\begin{aligned} O_0 &= \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \delta(p+q) dt \\ &\quad \times (-iu_\alpha(p) \partial_t v_\alpha(q) + iu_\alpha(p) f_\alpha(q)), \\ O_1 &= \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \delta(p+q) dt \\ &\quad \times (-\lambda \phi_1(p) q_\alpha f_\alpha(q) + 2\lambda^{-1} \phi_2(p) q_\alpha u_\alpha(q) \\ &\quad + 2i\lambda^{-1} \phi_1(p) q_\alpha u_\alpha(q) - i\lambda \phi_3(p) q_\alpha f_\alpha(q)) \\ &\quad + \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \left( \frac{d^D r}{(2\pi)^D} \right) \\ &\quad \times \delta(p+q+r) dt u_\alpha(p) v_\beta(q) r_\beta v_\alpha(r), \\ O_2 &= \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \delta(p+q) dt \\ &\quad \times \left( -iv u_\alpha(p) q^2 v_\alpha(q) + \sum_{k=1}^3 \phi_k(p) q^2 \phi_k(q) \right. \\ &\quad \left. + \frac{1}{2\epsilon} f_\alpha(p) q^2 f_\alpha(q) \right) \\ &\quad + \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \left( \frac{d^D r}{(2\pi)^D} \right) \delta(p+q+r) dt \\ &\quad \times (-i\lambda p_\alpha \phi_1(p) v_\beta(q) r_\beta v_\alpha(r) \\ &\quad + \lambda p_\alpha \phi_3(p) v_\beta(q) r_\beta v_\alpha(r)). \end{aligned} \quad (58)$$

Here,  $\phi_1, \phi_2, \phi_3$  denote the auxiliary scalar fields. The terms are sorted according to their order of derivatives.

This is the result for the action  $S$ . Depending on how the determinant (33) is expressed, other non-local interactions may have to be rewritten in the same way.

**3.4.1. Discussion.** In this section we have shown how to transform non-local into local interactions: either by redefinition of unphysical fields, or by introduction of intermediate propagators. A drawback will be that we have to approximate this action to account for RG-transformations, and a common way is the derivative expansion. Due to our “localisation” of the action, the original terms have been mixed concerning the order of derivatives. Moreover, the number of derivatives has

increased for most interactions, which means that we would have to expand the RG-flow to a high order in the derivative expansion. Also, the number of fields involved increases the complexity of the numerical work, even to the lowest orders.

Nevertheless, this expansion is feasible to any order in derivatives, as shown in [22] for the first two orders. The results are rather lengthy rate equations, which will not be elaborated on here.

#### 4. Renormalisation Group

The Exact Renormalisation Group (ERG) originates in the work of [8, 9], based on Kadanoff's block-spin picture [23]. For introductions into the theory of the renormalisation group, see, for example, [24, 25]. It is surprising that some very basic questions, for example, concerning the renormalisation step and the anomalous dimension, are still being discussed. Therefore we shall consider this point in detail in Section 4.4, especially the anomalous dimension and the graphical representation of the flow.

In this section, we discuss the foundations of the ERG and of the flow equations. We shall not repeat the derivation of the equations, as this can be found in a number of articles, but we outline the graphical representation of the different terms, as it will lay the foundations for our numerical investigations that closely follow the loop expansion.

*4.1. Form of the Action.* We are looking for an RG-flow of a given theory defined by its generating functional  $Z$ . To be definite, let us work with the theory of a vector field  $v_i$  and write  $Z$  in the following way:

$$\begin{aligned} Z &= \exp \{-W[\{J\}]\} = \int \mathcal{D}v \exp(-S) \\ &= \int \mathcal{D}v \exp \left\{ -\frac{1}{2} (v_i, P_{v_{ij}}^{-1} v_j) - (J_{v_i}, Q_{v_{ij}}^{-1} v_j) \right. \\ &\quad \left. - S_{\text{int}}[v, \Lambda; \Lambda_0] - S_0[J, \Lambda; \Lambda_0] \right\}. \end{aligned} \quad (59)$$

The action depends on two momentum scales  $\Lambda$  and  $\Lambda_0$ . By  $\Lambda_0$  we denote the scale on which we impose the initial renormalisation condition—for example, the value of the four-point function is fixed to a certain value  $\lambda_4$  if all external momenta equal  $\Lambda_0$ .

The term  $S_0$  might look uncommon but is necessary to pick up terms nonlinear in  $J$  that will be generated by the RG-flow. As initial condition, we set  $S_0[\Lambda = \Lambda_0] = 0$ .

Starting from a renormalised action on scale  $\Lambda_0$ , the flow is going to generate the renormalised action on all lower scaled  $\Lambda$ , which is the second momentum scale involved. From the RG-perspective,  $S[\Lambda = \Lambda_0]$  plays the role of the initial condition of the flow.

It should be noted that the renormalised action  $S[\Lambda]$ , also called Wilsonian effective action, is not identical to the field theoretic effective action  $\Gamma$ , which generates the one-particle irreducible vertex functions. It will contain higher order terms even if the corresponding 1PI vertex functions vanish.

The flow equations depend on the choice of the kinetic action, so we will define the kinetic term to be

$$S_{\text{kin}} = \frac{1}{2} (v, P^{-1} v) \quad (60)$$

$$= \frac{1}{2(2\pi)^{2D}} \int d^D p d^D q \delta(p+q) v(p) P^{-1} \left( \frac{p^2}{\Lambda^2} \right) v(q), \quad (61)$$

where we define

$$P^{-1} \left( \frac{p^2}{\Lambda^2} \right) = \frac{p^2}{\Lambda^2} C^{-1} \left( \frac{p^2}{\Lambda^2} \right). \quad (62)$$

$C$  is the cutoff function, which has the following properties:

$$C^{-1} \left( \frac{p^2}{\Lambda^2} \right) \rightarrow 1 \quad \text{for } |p| \rightarrow 0, \quad (63)$$

$$C^{-1} \left( \frac{p^2}{\Lambda^2} \right) \rightarrow 0 \quad \text{for } |p| \rightarrow \infty, \quad (64)$$

$$C^{-1}(1) = \frac{1}{2}. \quad (65)$$

Though it is by no means necessary, one usually assumes that  $C^{-1}$  is monotonous and that it is a smooth approximation of the step function, thus suppressing degrees of freedom on scales bigger than  $\Lambda$ , while not effectively altering those on later scales. The last equation (65) is ambiguous, but we define a value for  $C^{-1}(1)$ , so that the role of  $\Lambda$  becomes definite. We will say that the degrees of freedom that are suppressed are “integrated out,” as this part of the involved integrals can be interpreted as already being performed. Apart from the properties (63)–(65), we are free in the definition of  $C$ . It follows that not even (61) is enforced; other definitions of the propagator have been tried. In practice, some propagators will lead to simpler numerical calculations than others. A very special choice of  $C$  is the sharp cutoff  $C_s$ :

$$C_s^{-1} \left( \frac{p^2}{\Lambda^2} \right) = 1 \quad \text{for } |p| < \Lambda, \quad (66)$$

$$C_s^{-1} \left( \frac{p^2}{\Lambda^2} \right) = 0 \quad \text{for } |p| \geq \Lambda, \quad (67)$$

which leads to the Wegner-Houghton equation and will be treated separately.

In a similar matter, we define the following kinetic terms for anticommuting Grassmann variables:

$$S_{\text{kin}} = \frac{1}{2} (\Psi_\mu, P_{\Psi_{\mu\nu}}^{-1} \Psi_\nu), \quad (68)$$

where  $P_{\Psi_{\mu\nu}}^{-1}$  is an antisymmetric matrix in the indices  $\mu$  and  $\nu$ .

For completeness, we already mention here that we will expand the interaction part of the action,  $S_{\text{int}}$ , in powers of the fields to illustrate some examples,

$$S_{\text{int}} = \sum_k S_{\text{int},k}, \quad (69)$$



where we will call

$$S_{\text{int},k} = \int \lambda_k \prod_{i=1}^k (d^D p_i v(p_i)) \delta \left( \sum_{j=1}^k p_j \right) \quad (70)$$

a  $k$ -vertex. The derivation of the flow equations does not depend on this expansion; but it is useful in some definite calculations.

#### 4.2. The Wilson Equation

**4.2.1. Integrating Out Degrees of Freedom.** The RGE can be derived by calculating the effect of a change of the cutoff on an action, keeping in mind that both the generating functional and the correlation functions may not change. A nice derivation of the Wilson-flow equation is, for example, found in [26]. Consider

$$\begin{aligned} -\dot{W} &:= \Lambda \frac{\partial W}{\partial \Lambda} \\ &= \left\langle \frac{1}{2} \left( v_i, \dot{P}_{v_{ij}}^{-1} v_j \right) + \left( J_{v_i}, \dot{Q}_{v_{ij}}^{-1} v_j \right) + \dot{S}_{\text{int}} + \dot{S}_0 \right\rangle = 0. \end{aligned} \quad (71)$$

In our case, we will lower the cutoff by lowering  $\Lambda$ , leaving  $\Lambda_0$  as a unit of measurement unchanged. Here and in the following the dot always denotes the RG flow and not a derivative with respect to physical time.

Applied to the vector theory, for example, we arrive at the following equation for the interaction term of the action:

$$\dot{S}_{\text{int}} = \frac{1}{2} \int_p \left\{ \frac{\delta S_{\text{int}}}{\delta v_j} \dot{P}_{v_{ji}} \frac{\delta S_{\text{int}}}{\delta v_i} - \frac{\delta}{\delta v_j} \dot{P}_{v_{ji}} \frac{\delta S_{\text{int}}}{\delta v_i} \right\}, \quad (72)$$

where we dropped a field-independent term. Taking the kinetic term into account, we find the simple equation

$$\begin{aligned} \dot{S} &= \dot{S}_{\text{int}} + \frac{1}{2} \left( v_i, \dot{P}_{v_{ij}}^{-1} v_j \right) \\ &= \frac{1}{2} \int_p \left\{ \frac{\delta S}{\delta v_j} \dot{P}_{v_{ji}} \frac{\delta S}{\delta v_i} - \frac{\delta}{\delta v_j} \dot{P}_{v_{ji}} \frac{\delta S}{\delta v_i} \right\} - \int_p \left\{ \frac{\delta S}{\delta v_i} \dot{P}_{v_{ik}} P_{v_{kj}}^{-1} v_j \right\}. \end{aligned} \quad (74)$$

The term  $(1/2) \int_p (\delta S / \delta v_j) \dot{P}_{v_{ji}} (\delta S / \delta v_i)$  will from now on be called the link-term of the flow-equation, while we will call  $(-1/2) \int_p (\delta / \delta v_j) \dot{P}_{v_{ji}} (\delta S / \delta v_i)$  the loop-term. These names will be justified in the following subsection. In complete analogy, equations for theories involving Grassmann variables  $\psi^*$  and  $\psi$  with propagator (68) can be derived as

$$\begin{aligned} \dot{S} &= -\frac{1}{2} \int_p \left\{ \frac{\delta S}{\delta \Psi_\nu} \dot{P}_{\Psi_\nu \mu} \frac{\delta S}{\delta \Psi_\mu} - \frac{\delta}{\delta \Psi_\nu} \dot{P}_{\Psi_\nu \mu} \frac{\delta S}{\delta \Psi_\mu} \right\} \\ &\quad + \int_p \left\{ \frac{\delta S}{\delta \Psi_\mu} \dot{P}_{\Psi_\mu \lambda} P_{\Psi_\lambda \nu}^{-1} \Psi_\nu \right\}. \end{aligned} \quad (75)$$

In case of anti-commuting fields it is important to keep track of all extra signs that arise.

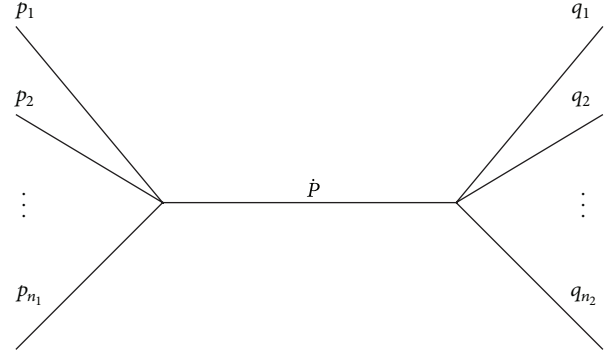


FIGURE 2: Link-term of the Wilson equation in its graphical representation.

These equations describe the lowering of the cutoff, or integrating out of degrees of freedom. Before we proceed, we shall discuss the graphical interpretation of the RG-equations.

**4.2.2. Graphical Representation.** Let us begin with the interpretation of the link-term. For the time being, we assume that it is applied to a part of the interaction term of the form (70), a vertex with  $n_1 + 1$  attached lines. Then the functional derivative of this gives us a vertex with  $n_1$  lines; the missing line is linked by the part of the propagator that is integrated out,  $\dot{P}$ , to a second, similar vertex with, say,  $n_2 + 1$  lines. The graphical result is shown in Figure 2. Observe that the functional derivatives automatically lead to the correct symmetry factor of the graph.

This graph gives a contribution to the  $(n_1 + n_2)$ -vertex, proportional to

$$\int \lambda_{n_1+1} \lambda_{n_2+1} \dot{P} \left( \frac{p^2}{\Lambda^2} \right) \delta \left( \sum_i p_i + p \right) \delta \left( \sum_j q_j - p \right) d^D p. \quad (76)$$

Since

$$\begin{aligned} &\delta \left( \sum_i p_i + p \right) \delta \left( \sum_j q_j - p \right) \\ &= \delta \left( \sum_i p_i + p \right) \delta \left( \sum_i p_i + \sum_j q_j \right), \end{aligned} \quad (77)$$

one of the two  $\delta$ -functions just implies the overall conservation of momentum and can be eliminated. The other  $\delta$ -function will have to be approximated in order to be suitable for a derivative expansion.

The loop-term is equally easy to understand. From a vertex with  $n+2$  attached lines, two are joined by a propagator  $\dot{P}$  (Figure 3). Again, the symmetry factor is given correctly.

This graph gives a contribution to the  $(n)$ -vertex as

$$\int \lambda_{n+2} \dot{P} \left( \frac{p^2}{\Lambda^2} \right) \delta \left( \sum_{i=1}^N p_i \right) d^D p. \quad (78)$$

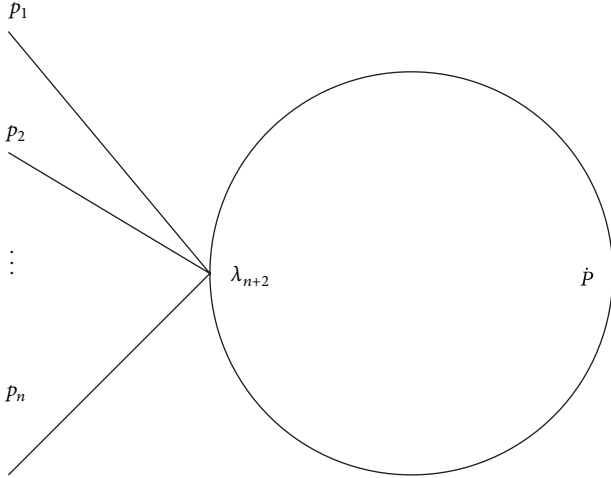


FIGURE 3: Loop calculated in the Wilson equation.

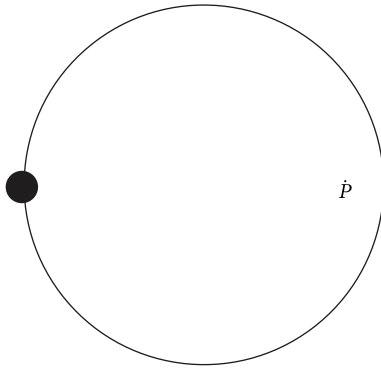


FIGURE 4: Field-independent loop that is constructed from the kinetic term of the action.

So far, we explained the effect of the flow equation as only  $S_{\text{int}}$  is concerned. Let us now investigate the contributions of the kinetic term.

The loop-term generated from the kinetic term (Figure 4) is trivial, as it is field independent and can be dropped.

Let us consider the terms arising when one field derivative in the link-term acts on the interaction, and the other one on the kinetic term; this one is compensated by another term in the RG-equation as follows:

$$\frac{1}{2} \int_p \left\{ \frac{\delta S_{\text{int}}}{\delta v_j} \dot{P}_{v_{ji}} \frac{\delta S_{\text{kin}}}{\delta v_i} + \frac{\delta S_{\text{kin}}}{\delta v_j} \dot{P}_{v_{ji}} \frac{\delta S_{\text{int}}}{\delta v_i} \right\} - \int_p \left\{ \frac{\delta S_{\text{int}}}{\delta v_i} \dot{P}_{v_{ik}} P_{v_{kj}}^{-1} v_j \right\} = 0. \tag{79}$$

The remaining term to be considered is

$$\frac{1}{2} \int_p \frac{\delta S_{\text{kin}}}{\delta v} \dot{P} \frac{\delta S_{\text{kin}}}{\delta v} - \int_p \frac{\delta S_{\text{kin}}}{\delta} \dot{P} P^{-1} v = -\frac{1}{2} \int_p v P^{-1} \dot{P} P^{-1} v. \tag{80}$$

From

$$0 = -\Lambda \frac{d}{d\Lambda} (P P^{-1}) = \dot{P} P^{-1} + P \dot{P}^{-1} \tag{81}$$

we get

$$-\frac{1}{2} \int_p v P^{-1} \dot{P} P^{-1} v = \frac{1}{2} \int_p v \dot{P}^{-1} v, \tag{82}$$

and this is, as defined in (61), the change in the kinetic term.

Let us summarise. We have seen that the RG-flow can be expressed graphically. Iteratively, we calculate the contributions from all graphs with one propagator  $\dot{P}$  and all other propagators  $P$ , that are inner propagators which have been generated in RG-steps before. This is simply an application of the product rule

$$-\Lambda \frac{d}{d\Lambda} \prod_i P \left( \frac{p_i^2}{\Lambda^2} \right) = \sum_i \dot{P} \left( \frac{p_i^2}{\Lambda^2} \right) \prod_{j \neq i} P \left( \frac{p_j^2}{\Lambda^2} \right). \tag{83}$$

Let  $\mathcal{E}[f(p)]$  formally denote the sum of all possible Feynman-graphs of the theory with inner propagators  $f(p)$ . The formal solution to the Wilson equation is

$$\begin{aligned} S[\Lambda; \Lambda_0] &= S[\Lambda = \Lambda_0] - \int_{\Lambda_0}^{\Lambda} \mathcal{E} \left[ \dot{P} \left( \frac{p^2}{\Lambda^2} \right) \right] \frac{d\bar{\Lambda}}{\bar{\Lambda}} \\ &= S[\Lambda = \Lambda_0] - \int_{\Lambda}^{\Lambda_0} \mathcal{E} \left[ \frac{d}{d\bar{\Lambda}} P \left( \frac{p^2}{\bar{\Lambda}^2} \right) \right] d\bar{\Lambda} \\ &= S[\Lambda = \Lambda_0] - \mathcal{E} \left[ P \left( \frac{p^2}{\Lambda_0^2} \right) - P \left( \frac{p^2}{\Lambda^2} \right) \right]. \end{aligned} \tag{84}$$

In the limit  $\Lambda_0 \rightarrow \infty$  this becomes

$$S[\Lambda; \Lambda_0] = S[\Lambda = \Lambda_0 = \infty] - \mathcal{E} \left[ 1 - P \left( \frac{p^2}{\Lambda^2} \right) \right]. \tag{85}$$

This reveals the meaning of the changes to the action. The RG-flow sums up the part of the propagator that is cut off iteratively. Notice that we seem to subtract all graphs—this is because we are working with  $e^{-S}$  rather than  $e^S$ .

Of course the derivation and graphical interpretation of the flow-equation does not apply directly to the case of a sharp cutoff as defined in (66) and (67). As our numerical approach favours the Wegner-Houghton equation, we discuss it in the following paragraph.

**4.3. The Wegner-Houghton Equation.** For numerical purposes, it is easiest to work with the sharp cutoff function  $C_s$  defined in (66) and (67). This changes the form of the flow equation drastically. Again, we shall not present a derivation here as it is found in the original literature [27], but only present a short overview. We start from dividing degrees of freedom into a high-momentum part that is to be integrated out and a low-momentum part that is kept. Expressing

the integrated part of the functional as a change  $\Delta S$  to the action, one finds

$$e^{-\Delta S} = \int' \mathcal{D}v \exp \left\{ - \int' L \right\}, \quad (86)$$

where the prime denotes integration over the momentum shell between  $\Lambda - d\Lambda$  and  $\Lambda$ . Expanding the action to second order in the fields gives

$$\begin{aligned} e^{-\Delta S} &= \int' \mathcal{D}v \exp \left\{ - \int' \left\{ v \frac{\delta S}{\delta v} + \frac{1}{2} v \frac{\delta^2 S}{\delta v^2} v \right\} \right\} \\ &= \int' \mathcal{D}v \exp \left\{ - \int' \left[ \frac{1}{2} \left( v \frac{\delta^2 S}{\delta v^2} v + 2v \frac{\delta S}{\delta v} + \frac{\delta S}{\delta v} \right) \right. \right. \\ &\quad \times \left( \frac{\delta^2 S}{\delta v^2} \right)^{-1} \frac{\delta S}{\delta v} \\ &\quad \left. \left. - \frac{1}{2} \frac{\delta S}{\delta v} \left( \frac{\delta^2 S}{\delta v^2} \right)^{-1} \frac{\delta S}{\delta v} \right] \right\}, \end{aligned} \quad (87)$$

where the square has been completed. Integrating out the field  $v$  in the shell of momentum, which is assumed to be of thickness  $(\Delta\Lambda/\Lambda) \ll 1$ , we get

$$e^{-\Delta S} \propto \frac{\Delta\Lambda}{\Lambda} \left( \det \frac{\delta^2 S}{\delta v^2} \right)^{-1/2} \exp \left\{ \int' \frac{1}{2} \frac{\delta S}{\delta v} \left( \frac{\delta^2 S}{\delta v^2} \right)^{-1} \frac{\delta S}{\delta v} \right\}, \quad (88)$$

and with the aid of

$$(\det A)^\alpha = \exp \{ \alpha \text{Tr} \ln A \} \quad (89)$$

we arrive at the Wegner-Houghton equation

$$\dot{S} = -\frac{1}{2} \int_p \left\{ \frac{\delta S}{\delta v} \frac{\delta S}{\delta v} \left( \frac{\delta^2 S}{\delta v^2} \right)^{-1} - \text{Tr} \ln \left( \frac{\delta^2 S}{\delta v^2} \right) \right\}. \quad (90)$$

In the derivation it is used that it is sufficient to work in one-loop order, as higher order contributions are also of higher order in  $d\Lambda$ . A proof of this is found in the original work [27].

Notice that (90) seems to differ from the Wilson equation (74) by an overall sign; but this is explained as  $\dot{P}$  is negative.

Equation (90) is especially convenient for numerical applications, as the contributions to the integrals can be calculated explicitly. At first glance the logarithm looks problematic, but it will be shown in the next paragraph that it has a very simple graphical interpretation and is thus favourable for our graphically based program.

**4.3.1. Graphical Representation.** For the Wegner-Houghton equation link-term does not involve the bare propagator alone, but the quantity  $(\delta^2 S/\delta v^2)^{-1}$ , which is more than just

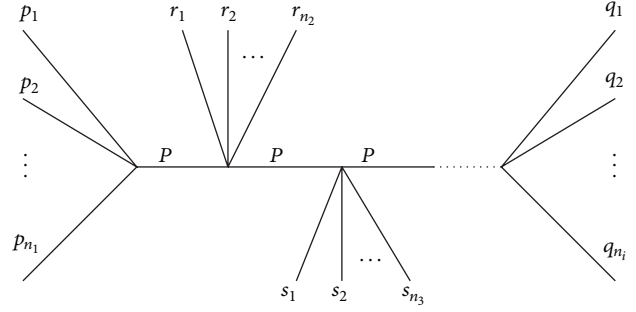


FIGURE 5: Link to be calculated using the Wegner-Houghton equation. This graph gives a contribution to the vertex with  $\sum_i n_i$  outer fields.

the inverse of the kinetic term. Using the geometric series, we can write

$$\begin{aligned} \left( \frac{\delta^2 S}{\delta v^2} \right)^{-1} &= \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} + \frac{\delta^2 S_{\text{int}}}{\delta v^2} \right)^{-1} \\ &= \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \right)^{-1} \left( 1 + \frac{\delta^2 S_{\text{int}}}{\delta v^2} \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \right)^{-1} \right)^{-1} \\ &= \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \right)^{-1} \sum_i \left\{ \frac{\delta^2 S_{\text{int}}}{\delta v^2} \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \right)^{-1} \right\}^i. \end{aligned} \quad (91)$$

This is the sum of all graphs with  $i$  vertices, linked into a line by propagators. The first and the last vertexes in the line are also attached to propagators, which link them to terms  $\delta S/\delta v$ . Again, let us assume first that these act on the interaction part of the action, and thus the chain described earlier is linked to other vertexes. We therefore find the graphical representation Figure 5.

In a similar way the graphical representation of the loop-term is derived. The logarithm is rewritten as

$$\begin{aligned} \text{Tr} \ln \left( \frac{\delta^2 S}{\delta v^2} \right) &= \text{Tr} \ln \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} + \frac{\delta^2 S_{\text{int}}}{\delta v^2} \right) \\ &= \text{Tr} \ln \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \left( 1 + \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \right)^{-1} \frac{\delta^2 S_{\text{int}}}{\delta v^2} \right) \right) \\ &= \text{Tr} \ln \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \right) + \text{Tr} \ln \left( 1 + \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \right)^{-1} \frac{\delta^2 S_{\text{int}}}{\delta v^2} \right). \end{aligned} \quad (92)$$

The first term is field independent and is dropped. The second logarithm is expanded as a Taylor-series, reading

$$\text{Tr} \sum_i \frac{(-1)^{i+1}}{i} \left( \left( \frac{\delta^2 S_{\text{kin}}}{\delta v^2} \right)^{-1} \frac{\delta^2 S_{\text{int}}}{\delta v^2} \right)^i. \quad (93)$$

The corresponding graph is shown in Figure 6. It is similar to the link-term before, but closed to a loop by the

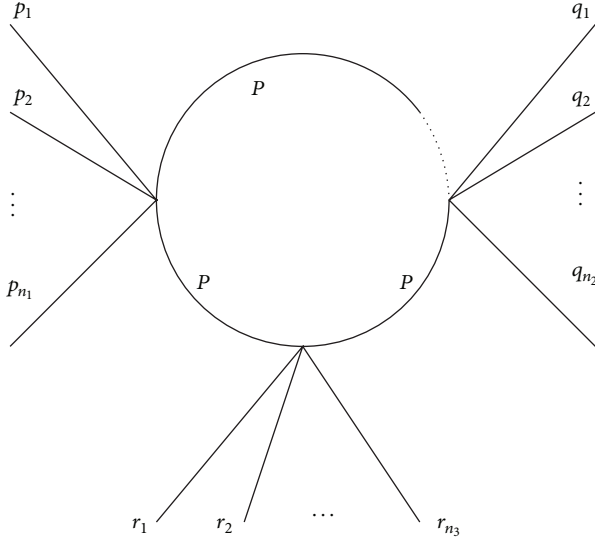


FIGURE 6: Loop calculated in the Wegner-Houghton equation.

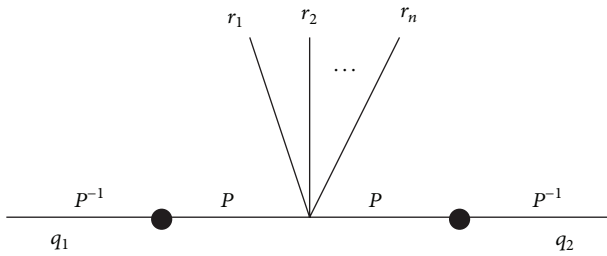


FIGURE 7: Graph of the Wegner-Houghton flow, linking a vertex to two outer propagators in this case.

trace. The factor  $1/n$  compensates the rotational symmetry of the graph.

As before, we still have to sort out the link-terms involving the kinetic action. Let us start with an example.

The graph in Figure 7 is obviously one of those that arise from the link-term; the reader may focus his attention to one of the  $PP^{-1}$  legs. Integration is again over the momentum shell, so  $PP^{-1} = 1$ . The result looks like the vertex, but with the difference that fields  $q_1$  and  $q_2$  are depending only on momenta less than  $\Lambda - d\Lambda$ . These terms can thus be interpreted as integrating out the momenta on remaining fields. Integrating out more outer fields at the same time would again be of higher order in  $d\Lambda$  and can be omitted.

The change in the kinetic term itself is again simple, and not even a sign problem arises as in the Wilson case. The corresponding graph is shown in Figure 8, and as  $P^{-1}PP^{-1} = P^{-1}$ , this is exactly

$$\int' dpvP^{-1}v. \tag{94}$$

Again, this is precisely the change of the kinetic action, as expected from (61).

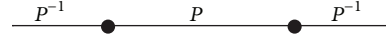


FIGURE 8: Graphical representation of the change of the propagator in the Wegner-Houghton flow.

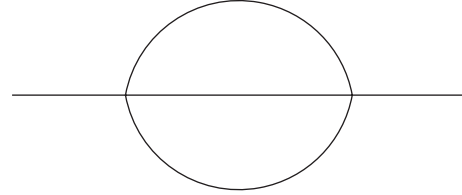


FIGURE 9: Sunset graph, leading to the simplest contribution to the field strength renormalisation of  $\phi^4$ -theory.

As in the case of the Wilson equation, we are now able to give a formal solution to the Wegner-Houghton equation. The final result reads

$$S[\Lambda; \Lambda_0] = S[\Lambda = \Lambda_0] - \int_{\Lambda}^{\Lambda_0} \mathcal{L} \left[ \frac{1}{p^2} \right] d^D p. \tag{95}$$

**4.4. Renormalisation and Rescaling.** The renormalisation of the field, also called “field strength renormalisation” or “wave function renormalisation,” is not required in an RG step but is usually implemented for convenience. As it is related to the anomalous dimension of the field, it is appropriate to discuss this point here. We shall demonstrate the concept using  $\phi^4$ -theory in  $D$ -dimensions; for other theories the procedure works in exactly the same way. Let us emphasise that this step is not unique to the ERG but also applied in perturbative renormalisation.

**4.4.1. Field Strength Renormalisation.** We started our integration step with the kinetic term

$$S_{\text{kin}} = \frac{1}{2} \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \times C^{-1} \left( \frac{p^2}{\Lambda_0^2} \right) p^2 \phi(p) \phi(q) \delta(p+q). \tag{96}$$

$S_{\text{kin}}$  is defined to be the only term quadratic in fields and quadratic in the momenta in the limit  $p \rightarrow 0$ . After the integration step (lowering the cutoff from  $\Lambda_0$  to  $\Lambda$ ), new terms are generated in the interaction part of the action that, according to the previous definition, should belong to the kinetic term. Such terms have then to be included in the kinetic term, which changes to some  $S'_{\text{kin}}$ . In practice, the first contribution to the kinetic term arises in the second step of the RG-flow, as it is of two-loop order. The simplest graph contributing to field strength renormalisation is the so-called sunset graph, Figure 9.

In our case, a graph analogous to Figure 9 is to be computed by our numerical approach in an iterative way later on, summing up contributions from every infinitesimal integration. The result depends on the used renormalisation

scheme; by means of a Taylor expansion, one can always identify the contribution to the kinetic action; let us denote it as

$$\frac{\eta}{2} \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \times C^{-1} \left( \frac{p^2}{\Lambda^2} \right) p^2 \phi(p) \phi(q) \delta(p+q). \quad (97)$$

As a renormalisation condition for the field strength, it is commonly required that the coefficient of the kinetic action in the limit  $p \rightarrow 0$  is equal to  $1/2$ . According to the definition of the cutoff properties, we introduce the field-strength renormalisation factor  $Z$  in a way that compensates for the new term in  $S'_{\text{kin}}$ . If we write for the original action (96) as

$$S_{\text{kin}} = Z \frac{1}{2} \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \times C^{-1} \left( \frac{p^2}{\Lambda^2} \right) p^2 \phi(p) \phi(q) \delta(p+q), \quad (98)$$

we conclude that  $Z$  transforms as

$$\Lambda \frac{\partial}{\partial \Lambda} \ln Z = -\eta. \quad (99)$$

(The sign is negative as in the integration step we actually lower  $\Lambda$ .) The initial condition has to be

$$Z[\Lambda = \Lambda_0] = 1, \quad (100)$$

so that (96) is fulfilled. Equation (99) is easily integrated to

$$Z = \left( \frac{\Lambda_0}{\Lambda} \right)^\eta. \quad (101)$$

$Z$  is now absorbed into the fields in the following way, which explains the name of field strength renormalisation:

$$\phi \longrightarrow \phi' = \left( \frac{\Lambda_0}{\Lambda} \right)^{\eta/2} \phi. \quad (102)$$

We will use this in the next paragraph to determine the scaling of the field. This is in complete accordance with renormalisation conditions met in perturbative renormalisation; see, for example, [28, 29].

The kinetic term now reads

$$S'_{\text{kin}} = \frac{1}{2} \int \left( \frac{d^D p}{(2\pi)^D} \right) \left( \frac{d^D q}{(2\pi)^D} \right) \times C^{-1} \left( \frac{p^2}{\Lambda^2} \right) p^2 \phi'(p) \phi'(q) \delta(p+q), \quad (103)$$

which, expressed in renormalised fields, is exactly of the same form as (96).

It has been stressed by Golner [30] and Bervillier [31] that the rescaling step has to be regarded carefully, to account for the renormalisation step consistently.

4.4.2. *Rescaling.* The last step in the renormalisation group process is the rescaling of the momenta and the functions thereof. Define new momenta  $\tilde{p}$  by

$$\tilde{p} = \left( \frac{\Lambda_0}{\Lambda} \right) p. \quad (104)$$

The replacement

$$p \longrightarrow \left( \frac{\Lambda}{\Lambda_0} \right) \tilde{p} \quad (105)$$

changes the cutoff function in the expected way as follows:

$$C^{-1} \left( \frac{p^2}{\Lambda^2} \right) \longrightarrow \tilde{C}^{-1} \left( \frac{\tilde{p}^2}{\Lambda_0^2} \right), \quad (106)$$

so we are ready to identify the new with the old cutoff.

From the renormalisation step, it is now easy to deduce the scaling of a field. Beginning with the original kinetic term (96), we conclude that, as  $S_{\text{kin}}$  does not scale at all, the rescaled field  $\tilde{\phi}$  has to be scaled as

$$\tilde{\phi}(\tilde{p}) = \left( \frac{\Lambda_0}{\Lambda} \right)^{-(D-2)/2} \phi(p), \quad (107)$$

where the canonical dimension  $D_{\phi, \text{can}} = (D-2)/2$  appears. Equation (102) then fixes the anomalous exponent

$$\tilde{\phi}'(\tilde{p}) = \left( \frac{\Lambda_0}{\Lambda} \right)^{-((D+2-\eta)/2)} \phi(p). \quad (108)$$

One can now see that due to the effect of the rescaling of the cutoff function, the renormalised kinetic action indeed does not scale. As the cutoff is a function of the ratio  $p^2/\Lambda^2$ , a change in  $p$  has the inverse effect as the same change in  $\Lambda$ . By this, we reverse the effect of the integration step. As a part of this, terms are redistributed back to the interaction of the theory. Now, in the kinetic term the canonical scaling of the fields is compensated by the integration measure and the anomalous scaling by the renormalisation step, so that finally

$$\tilde{S}'_{\text{kin}} = \frac{1}{2} \int \left( \frac{d^D \tilde{p}}{(2\pi)^D} \right) \left( \frac{d^D \tilde{q}}{(2\pi)^D} \right) \times \tilde{C}^{-1} \left( \frac{\tilde{p}^2}{\Lambda_0^2} \right) \tilde{p}^2 \tilde{\phi}'(\tilde{p}) \tilde{\phi}'(\tilde{q}) \delta(\tilde{p} + \tilde{q}) \quad (109)$$

is identical to (96) as a function of the rescaled quantities, as desired. We will drop the tildes and primes, formally getting back to (96).

4.5. *The Interaction Terms.* The steps discussed for the kinetic terms have to be applied to the interaction terms, too. As an example, let us consider the four-field interaction

$$\int d^D p_1 \int d^D p_2 \int d^D p_3 \lambda_4 \phi(p_1) \phi(p_2) \phi(p_3) \times \phi(p_4) \delta(p_1 + p_2 + p_3 + p_4). \quad (110)$$



4.5.1. *Renormalisation.* When  $\phi$  is changed to the renormalised field  $\phi'$ , without changing the interaction term, the coupling has to be renormalised as follows:

$$\begin{aligned} & \int d^D p_1 \int d^D p_2 \int d^D p_3 \lambda_4 \phi(p_1) \phi(p_2) \\ & \quad \times \phi(p_3) \phi(-p_1 - p_2 - p_3) \\ & \rightarrow \int d^D p_1 \int d^D p_2 \int d^D p_3 \lambda_4 \left(\frac{\Lambda_0}{\Lambda}\right)^{-4\eta/2} \\ & \quad \times \phi'(p_1) \phi'(p_2) \phi' \\ & \quad \times (p_3) \phi'(-p_1 - p_2 - p_3) \quad (111) \\ & \Rightarrow \lambda'_4 = \lambda_4 \left(\frac{\Lambda_0}{\Lambda}\right)^{-4\eta/2} \\ & \Leftrightarrow \lambda_4 \rightarrow \left(\frac{\Lambda_0}{\Lambda}\right)^{-4\eta/2} \lambda'_4. \end{aligned}$$

For an infinitesimal integration step this amounts to

$$\dot{\lambda}_4 = -4 \frac{\eta}{2} \lambda_4. \quad (112)$$

For a general interaction  $S_{\text{int}}$  with any number of vertices, this generalises to

$$\dot{S}_{\text{int, Ren}} = \frac{\eta}{2} \int \phi \frac{\delta S_{\text{int}}}{\delta \phi}, \quad (113)$$

as the operator  $\int \phi(\delta/\delta\phi)$  counts the number of fields in a vertex.

4.5.2. *Rescaling.* The contributions from the rescaling step are the following.

(i) *Integral.* For each integration measure, we get a factor  $D$ , there is one integration measure less than the number of fields (because of the  $\delta$ -function), so we get a contribution

$$\dot{S}_{\text{int}, dp} = -D \int \phi \frac{\delta S_{\text{int}}}{\delta \phi} + DS_{\text{int}}. \quad (114)$$

(ii) *Momentum.* The vertex will depend explicitly on the momentum, so we introduce another operator  $\int \phi(p) p(\partial/\partial p)'(\delta/\delta\phi(p))$  that counts the powers of momenta in each vertex. The prime at the derivative indicates that it is not acting upon the momentum conserving  $\delta$ -function. We get the contribution

$$\dot{S}_{\text{int}, p} = - \int \phi(p) p \left(\frac{\partial}{\partial p}\right)' \frac{\delta S_{\text{int}}}{\delta \phi(p)}. \quad (115)$$

(iii) *Fields.* As derived earlier, each field brings a contribution proportional to  $-((D+2-\eta)/2)$ , so in total we find

$$\dot{S}_{\text{int}, \phi} = \frac{D+2-\eta}{2} \int \phi \frac{\delta S_{\text{int}}}{\delta \phi}. \quad (116)$$

(iv) *Renormalised Coupling.* Any coupling is renormalised according to (113), so it scales itself anomalously, exactly compensating the anomalous scaling of the fields

$$\dot{S}_{\text{int}, \lambda} = \frac{\eta}{2} \int \phi \frac{\delta S_{\text{int}}}{\delta \phi}. \quad (117)$$

Summing up all contributions yields the rescaling term

$$\dot{S}_{\text{int, Rescaling}} = DS_{\text{int}} - \int \phi \left( \frac{D-2}{2} + p \left(\frac{\partial}{\partial p}\right)' \right) \frac{\delta S_{\text{int}}}{\delta \phi}. \quad (118)$$

4.6. *The RG-Equation.* From the previous discussion, the resulting flow equation for the interaction term is

$$\begin{aligned} \dot{S}_{\text{int}} = & \frac{1}{2} \int_p \left\{ \frac{\delta S_{\text{int}}}{\delta v_j} \dot{p}_{v_{ji}} \frac{\delta S_{\text{int}}}{\delta v_i} - \frac{\delta}{\delta v_j} \dot{p}_{v_{ji}} \frac{\delta S_{\text{int}}}{\delta v_i} \right\} \\ & - \int \phi \left( \frac{D-2-\eta}{2} + p \left(\frac{\partial}{\partial p}\right)' \right) \frac{\delta S_{\text{int}}}{\delta \phi} + DS_{\text{int}}. \end{aligned} \quad (119)$$

This equation depends on the choice of propagator (62). As an example consider the inverse propagator to be given by

$$\tilde{P}^{-1} = \Lambda^k C^{-1} \left( \frac{p^2}{\Lambda^2} \right), \quad k = 2, \quad (120)$$

as often found in the literature. Then the resulting equation is the one given by Bervillier [31] or Golner [30] as

$$\begin{aligned} \dot{S}_{\text{int}} = & \frac{1}{2} \int_p \left\{ \frac{\delta S_{\text{int}}}{\delta v_j} \dot{\tilde{P}}_{v_{ji}} \frac{\delta S_{\text{int}}}{\delta v_i} - \frac{\delta}{\delta v_j} \dot{\tilde{P}}_{v_{ji}} \frac{\delta S_{\text{int}}}{\delta v_i} \right\} \\ & - \int \phi \left( \frac{D+2-\eta}{2} + p \left(\frac{\partial}{\partial p}\right)' \right) \frac{\delta S_{\text{int}}}{\delta \phi} + DS_{\text{int}}, \end{aligned} \quad (121)$$

which in turn is equivalent to Wilson's equation.

Let us point out that both (119) and (121) are correct, even though they seem to differ by a sign. The equivalence is obscured by a different choice of propagator functions, taking advantage of the reparameterisation invariance of the equation.

The propagator (120) indeed has some advantages, as in principle other values for  $k$  are also possible and simplify the implementation of K41, as we shall see. On the other hand, the derivative of (120) is more complicated and is especially inconvenient if the derivative expansion is applied.

The complete RG-equation, including vectorial and Grassmannian fields, finally reads

$$\begin{aligned} \dot{S} = & \frac{1}{2} \int_p \dot{P}_v \left\{ \frac{\delta S}{\delta v_i} \frac{\delta S}{\delta v_i} - \frac{\delta}{\delta v_i} \frac{\delta S}{\delta v_i} - 2P_v^{-1} \frac{\delta S}{\delta v_i} v_i \right\} \\ & + \int_p \dot{P}_\Psi \left\{ \frac{\delta S}{\delta \psi_i^*} \frac{\delta S}{\delta \psi_i} - \frac{\delta}{\delta \psi_i^*} \frac{\delta S}{\delta \psi_i} \right. \\ & \quad \left. + P_\Psi^{-1} \left( \frac{\delta S}{\delta \psi_i^*} \psi_i^* + \frac{\delta S}{\delta \psi_i} \psi_i \right) \right\} \\ & - \left( D + D_{v_i, \text{kan}} - \frac{\eta_{v_i}}{2} \right) \int_p v_i \frac{\delta S}{\delta v_i} \\ & - \left( D + D_{\psi_i^*, \text{kan}} - \frac{\eta_{\psi_i^*}}{2} \right) \int_p \psi_i^* \frac{\delta S}{\delta \psi_i^*} \\ & - \left( D + D_{\psi_i, \text{kan}} - \frac{\eta_{\psi_i}}{2} \right) \int_p \psi_i \frac{\delta S}{\delta \psi_i} \\ & - \int_p v_i P \frac{\partial'}{\partial P} \frac{\delta S}{\delta v_i} - \int_p \psi_i^* P \frac{\partial'}{\partial P} \frac{\delta S}{\delta \psi_i^*} - \int_p \psi_i P \frac{\partial'}{\partial P} \frac{\delta S}{\delta \psi_i} + DS. \end{aligned} \quad (122)$$

This equation is general enough to cover our intended applications, including different ways of considering the functional determinant (33).

### 5. Derivative Expansion

The actions involved in the RG flow represent infinitely many degrees of freedom and have to be approximated in the context of numerical investigations. A common way of approximation is the derivative expansion; see, for example, [32, 33]. Applied to the scalar theory, it amounts to expanding the action in powers of derivatives

$$S = \frac{1}{2} \int_x Z(\phi(x)) (\partial\phi)^2 + V(\phi(x)) + \mathcal{O}(\partial^4), \quad (123)$$

in contrast to an expansion in powers of fields, which can be seen as expansion around a weak field. As a special case, in the Local Potential Approximation (LPA) the action is reduced to an interaction term depending only locally on the field  $\phi(x)$  (and not on its derivatives) and a kinetic term whose coefficient  $Z$  is held constant throughout the flow as

$$S_{\text{LPA}} = \frac{1}{2} \int_x (\partial\phi)^2 + V(\phi(x)). \quad (124)$$

Applying the loop- and link-terms to the action expanded in powers of fields leads to rate equations for the coefficients. Let us, as an example, apply the Local Potential Approximation (LPA) to (76) and (78), expanded in powers of fields.

Starting with (76) for the link-term, the nontrivial part of graph 2 is proportional to

$$\begin{aligned} \dot{\lambda}_{n+m} \propto & \int \dot{P} \left( \frac{p^2}{\Lambda^2} \right) \lambda_{n+1} \\ & \times (p_1, \dots, p_n, p) \lambda_{m+1}(q_1, \dots, q_m, p) \\ & \times \delta(p_1 + p_2 + \dots + p_n + p) \\ & \times \delta(q_1 + q_2 + \dots + q_m - p) d^D p \\ & = \dot{P} \left( \frac{\sum_i P_i^2}{\Lambda^2} \right) \lambda_{n+1} \left( p_1, \dots, p_n, -\sum_i p_i \right) \lambda_{m+1} \\ & \times \left( q_1, \dots, q_m, \sum_j q_j \right) \delta \left( \sum_i p_i + \sum_j q_j \right). \end{aligned} \quad (125)$$

In the LPA, the couplings are approximated to be momentum independent, and developing the cutoff to zeroth order in the momenta gives

$$\dot{\lambda}_{n+m, \text{LPA}} = \lim_{p \rightarrow 0} \dot{P} \left( \frac{\sum_i P_i^2}{\Lambda^2} \right) \lambda_{n+1, \text{LPA}} \lambda_{m+1, \text{LPA}}. \quad (126)$$

A difficulty is the momentum dependence of the factor  $\dot{P}(\sum_i p_i^2/\Lambda^2)$  which we need to expand, according to the derivative expansion. The result obviously depends on the choice of the cutoff; if we apply it to the LPA, we can subsume the result into the constant

$$\tilde{P}_1 := \lim_{p \rightarrow 0} \dot{P} \left( \frac{\sum_i P_i^2}{\Lambda^2} \right). \quad (127)$$

If the cutoff is an approximation of the step function, the limit is expected to converge, and  $\tilde{P}_1 = 0$ . This clearly is not an option, as it would suppress the non-trivial character of the RG-flow.

On the other hand, the loop-equation (78) leads in the LPA to

$$\begin{aligned} \dot{\lambda}_n = & \int \dot{P} \left( \frac{p^2}{\Lambda^2} \right) \lambda_{n+2}(p_1, \dots, p_n, p, -p) \\ & \times \delta(p_1 + p_2 + \dots + p_n) d^D p \\ \implies \dot{\lambda}_{n, \text{LPA}} = & \lambda_{n+2, \text{LPA}} \int \dot{P} \left( \frac{p^2}{\Lambda^2} \right) d^D p \\ = & \lambda_{n+2, \text{LPA}} \Omega_{D-1} \int \left( \frac{d}{dp} P \left( \frac{p^2}{\Lambda^2} \right) \right) p^D dp. \end{aligned} \quad (128)$$

Again, the integral depends on the choice of the cutoff; for the LPA we write

$$\Omega_{D-1} \int \left( \frac{d}{dp} P \left( \frac{p^2}{\Lambda^2} \right) \right) p^D dp \longrightarrow \tilde{P}_0. \quad (129)$$

Rather than to specify a cutoff, in the LPA it is sufficient to define the constants  $\tilde{P}_1$  and  $\tilde{P}_0$ . In higher orders of

the derivative expansion, additional information concerning the cutoff will be required.

In the case of a vector theory in three dimensions, the situation is not that simple, as products of the type  $v_i v_j$  or any contraction with other three-component fields will be present. We need to keep track of this to calculate the contributions to a renormalisation group flow, so we propose to expand the terms of the action in powers of fields and momenta in the following way:

$$\begin{aligned}
 V = & \sum_{x_i, r_i, q, A} q V_{(x_1, x_2, x_3)}^{(A_1, A_2, \dots, A_6)}(r_1, r_2, \dots, r_6) \\
 & \times (v\nu)^{A_1} (uu)^{A_2} (ff)^{A_3} (vu)^{A_4} (vf)^{A_5} (uf)^{A_6} \\
 & \times (v\psi^*)^{r_1} (v\psi^*)^{r_2} (u\psi^*)^{r_3} (u\psi^*)^{r_4} (f\psi^*)^{r_5} \\
 & \times (f\psi)^{r_6} (\psi^* \psi)^q (\phi^1)^{x_1} (\phi^2)^{x_2} (\phi^3)^{x_3}.
 \end{aligned} \quad (130)$$

From now on, we will work with the coefficients

$$q V_{(x_1, x_2, x_3)}^{(A_1, A_2, \dots, A_6)}(r_1, r_2, \dots, r_6). \quad (131)$$

In first order, the terms of the derivative expansion are even more complicated, as we also have to keep track of terms like  $p_i v_j(q)$ .

As the overall number of momenta is fixed for each term and the action itself is scalar, we get the following possible values for the indices of  $V$ :

$$\begin{aligned}
 x_i \in \mathbf{N}_0, \quad A_i \in \mathbf{N}_0, \\
 r_i \in \{0, 1\}, \quad q \in \{0, \dots, D - r_1 - r_3 - r_5\}, \\
 r_1 + r_3 + r_5 = r_2 + r_4 + r_6,
 \end{aligned} \quad (132)$$

As well as for  $Z$  equivalently.

## 6. Application to Turbulence

Applying the RG to turbulence, a point of central importance is to specify how the RG transformations should act on the degrees of freedom contained in the action. In the case at hand we decide to consider transformations that describe pure spatial rescalings, while physical times are not being rescaled. In the language of the block spin RG, this represents a block spin transformation highly anisotropic in the coordinates  $(t, x)$ , in which the blocking is applied to the three spatial coordinates  $x$  only. In Fourier space, the RG transformation acts on three-dimensional momenta, but not on frequencies. The reason for this approach is twofold. First, the goal of the RG calculations is to study the scaling behaviour of the structure functions, which are spatial correlation functions and do not involve physical time  $t$ . The RG transformations relevant for this are spatial ones. Secondly, this allows to apply the formalism discussed in the previous section without fundamental modifications, because pure spatial scalings are being considered there. As a consequence, the loop integrals contributing to the flow equations are momentum space integrals and do not involve frequencies.

The complete correlation functions of a given theory do of course not depend on how the action is divided into a kinetic part and interaction terms. In order to implement RG transformations it is, however, crucial to specify the kinetic part of the action, because it contains the cutoff function, which is the primary source of the dependence of the action on the cutoff  $\Lambda$ . The kinetic part appropriate for the kind of RG transformations intended here consists of the terms quadratic in the fields and in the spatial derivatives in the action corresponding to the Navier-Stokes equation, (43) and (44). Consequently, terms linear in  $\partial_t$  are treated as parts of the interaction. The RG transformations will thus involve momentum/space integrals but not frequency/time integrals. This does, however, not mean that the time dependence of the theory is eliminated; it just does not enter the integrals effecting the RG transformations.

Fields with time derivatives are to be tracked in the book keeping as they will be generated by the RG flow. We denote the number of time derivatives in a term by  $\text{Der}$ . In the derivative expansion the coefficients are correspondingly labelled

$$q V_{(x)}^{(A)}(r) [\text{Der}]. \quad (133)$$

For the final assembly of the rate equations, we need the scaling dimensions and exponents  $k$  for the involved fields. The canonical dimension for the velocity field  $v$  is derived from the energy flow  $\Pi'$ , see [14] as follows:

$$\Pi'_l \propto \frac{\sqrt{\langle v^2(l) \rangle^3}}{l} \propto \epsilon, \quad (134)$$

from which we see that  $\sqrt{\langle v^2(l) \rangle} \propto l^{1/3}$ . In wavenumber space this implies

$$\begin{aligned}
 [\partial_t] = \frac{2}{3}, \quad [v] = -\frac{4}{3}, \\
 [v] = -D - \frac{1}{3}.
 \end{aligned} \quad (135)$$

The scaling dimensions of the non-physical fields and constants are then

$$\begin{aligned}
 [f] = -D + \frac{1}{3}, \quad [u] = -\frac{1}{3}, \\
 [\rho] = -D + \frac{8}{3}, \quad [\psi^*] = [\psi] = -\frac{D}{2} - \frac{1}{3}, \\
 [\lambda] = \frac{D}{2} - \frac{1}{3}, \quad [\phi^i] = -\frac{D}{2} - 1 \quad \forall i,
 \end{aligned} \quad (136)$$

and the exponents  $k$  are

$$\begin{aligned}
 k_v = D + \frac{2}{3}, \quad k_{\phi^i} = 2 \quad \forall i, \\
 k_f = D - \frac{2}{3}, \quad k_u = -D + \frac{2}{3}, \\
 k_\psi = \frac{2}{3}.
 \end{aligned} \quad (137)$$

In this way we arrive at the rate equations that we simulated numerically. These equations are quite lengthy, and details are presented in [22]. Here we only present the LPA as follows:

$$\begin{aligned}
 & -\Lambda \frac{d}{d\Lambda} \left( {}^q V_{(r)}^{(A)} \text{[Der]} \right) \\
 &= \frac{-D - (2/3) + \eta_v}{2} \left( \bar{P}_{v,1} \left( \frac{\partial V}{\partial v} \frac{\partial V}{\partial v} \right)_{q,(r)}^{(x),(A)} \right. \\
 &\quad \left. - \bar{P}_{v,0} \left( \frac{\partial^2 V}{\partial v^2} \right)_{q,(r)}^{(x),(A)} \right) \\
 &+ \frac{D - 2/3 + \eta_u}{2} \left( \bar{P}_{u,1} \left( \frac{\partial V}{\partial u} \frac{\partial V}{\partial u} \right)_{q,(r)}^{(x),(A)} \right. \\
 &\quad \left. - \bar{P}_{u,0} \left( \frac{\partial^2 V}{\partial u^2} \right)_{q,(r)}^{(x),(A)} \right) \\
 &+ \frac{-D + 2/3 + \eta_f}{2} \left( \bar{P}_{f,1} \left( \frac{\partial V}{\partial f} \frac{\partial V}{\partial f} \right)_{q,(r)}^{(x),(A)} \right. \\
 &\quad \left. - \bar{P}_{f,0} \left( \frac{\partial^2 V}{\partial f^2} \right)_{q,(r)}^{(x),(A)} \right) \\
 &+ \left( -\frac{2}{3} + \eta_\Psi \right) \left( \bar{P}_{\Psi,1} \left( \frac{\partial V}{\partial \Psi^*} \frac{\partial V}{\partial \Psi} \right)_{q,(r)}^{(x),(A)} \right. \\
 &\quad \left. - \bar{P}_{\Psi,0} \left( \frac{\partial^2 V}{\partial \Psi^* \partial \Psi} \right)_{q,(r)}^{(x),(A)} \right) \\
 &+ \sum_{i=1}^3 \frac{-2 + \eta_{\phi^i}}{2} \left( \bar{P}_{\phi,1} \left( \frac{\partial V}{\partial \phi^i} \frac{\partial V}{\partial \phi^i} \right)_{q,(r)}^{(x),(A)} \right. \\
 &\quad \left. - \bar{P}_{\phi,0} \left( \frac{\partial^2 V}{\partial \phi^i \partial \phi^i} \right)_{q,(r)}^{(x),(A)} \right) \\
 &+ \left\{ \left( \frac{1}{3} - \frac{\eta_v}{2} \right) (2A_1 + A_4 + A_5 + r_1 + r_2) \right. \\
 &\quad + \left( -D + \frac{1}{3} - \frac{\eta_u}{2} \right) (2A_2 + A_4 + A_6 + r_3 + r_4) \\
 &\quad + \left( -\frac{1}{3} - \frac{\eta_f}{2} \right) (2A_3 + A_5 + A_6 + r_5 + r_6) \\
 &\quad + \left( -\frac{D}{2} + \frac{1}{3} - \frac{\eta_\Psi}{2} \right) (r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + 2q) \\
 &\quad \left. + \sum_{i=1}^3 \left( -\frac{D}{2} + 1 - \frac{\eta_{\phi^i}}{2} \right) x_i - \frac{2}{3} \text{Der} + D \right\} \\
 &\times {}^q V_{(r)}^{(A)} \text{[Der]}.
 \end{aligned} \tag{138}$$

Here we defined

$$\left( \frac{\partial V}{\partial v_i} \frac{\partial V}{\partial v_i} \right)_{q,(r)}^{(x),(A)} \tag{139}$$

as the contribution of  $(\partial V/\partial v_i)(\partial V/\partial v_i)$  with the indicated field expansion, and similarly for the other terms.

If the determinant (33) is taken into account in a way that implies additional fields, these have to be included into the RG-equation in the same way.

In the theory described by the effective actions (37) or (57) the 1-particle irreducible Green functions of the velocity field  $v$  vanish as a consequence of the fact that in the effective action there is no  $v$ -propagator and there are no vertices with the field  $v$  only. On the other hand, integrating the auxiliary fields out would produce an action containing a  $v$ -propagator and  $v$ -vertices, leading to Green functions that are 1-particle irreducible within this theory. Even though these properties have important consequences for studies of the perturbation expansions of these actions, they do not influence our numeric approach based on (138), as will become clear in a subsequent paragraph.

Of course, it is a drawback to expand the action in powers of fields and momenta. For a numerical implementation of the RG flow, however, some approximation scheme has to be chosen. This one enables us to work with a very simple and fast numerical algorithm, which is described in the next section.

At this point, it is possible to examine the scaling of the two- and four-point functions  $\langle vv \rangle$  and  $\langle vvvv \rangle$  near the free fixed point. The flow equations are

$$\begin{aligned}
 -\Lambda \frac{d}{d\Lambda} \langle vv \rangle &= -\frac{2}{3} \langle vv \rangle + 44 \bar{P}_{v,0} \frac{\langle vvvv \rangle}{\langle vv \rangle}, \\
 -\Lambda \frac{d}{d\Lambda} \langle vvvv \rangle &= -\frac{4}{3} \langle vvvv \rangle + 55 \bar{P}_{v,0} \frac{\langle vvvvvv \rangle}{\langle vv \rangle}.
 \end{aligned} \tag{140}$$

In the limit of small couplings, this reduces to

$$\begin{aligned}
 -\Lambda \frac{d}{d\Lambda} \langle vv \rangle &= -\frac{2}{3} \langle vv \rangle, \\
 -\Lambda \frac{d}{d\Lambda} \langle vvvv \rangle &= -\frac{4}{3} \langle vvvv \rangle.
 \end{aligned} \tag{141}$$

From this we obtain the scaling of the two- and four-point function interaction as

$$\begin{aligned}
 \langle vv \rangle &\sim (x)^{2/3}, \\
 \langle vvvv \rangle &\sim (x)^{4/3}.
 \end{aligned} \tag{142}$$

This is precisely the K41 scaling predicted by Kolmogorov.  $\bar{P}_{v,0}$  can be interpreted as a measure for the coupling, defining the meaning of *being near the free fixed point*.

## 7. Numerical Analysis

*7.1. Choice of Renormalisation Group Equation.* In developing the numerical algorithm, we tried different choices for the

cutoff, including the sharp cutoff of the Wegner-Houghton equation, (90). We found that this choice is particularly suitable, as it allows us to compute the contribution for an infinitesimal integration step independently of the couplings involved.

The algorithm calculates the RG flow in terms of the coefficients (133) of the derivative expansion. The flow equations are a set of coupled ordinary differential equations for these coefficients, which are solved numerically with given initial conditions. For the calculation of the RG flow we worked with a predictor-corrector, as well as a Runge-Kutta integration algorithm, both with self-adjusting step width. We used two sets of algorithms—one of them involves explicitly programmed versions of the rate equations, while the other worked out the loop- and link-graphs automatically, only needing the parameters of the physical system.

Apart from the algorithm for the calculation of the flow, we developed a number of tools for the analysis of the resulting data. As the coupling space, in which we are working, is very abstract and high dimensional, it is helpful to start with explorative studies of unphysical toy systems, that is, simple and solvable physical systems, and of reduced turbulent systems (Burgulence), to gain confidence in the correct working of the algorithm and to develop some intuition for the work with renormalised couplings.

**7.2. Nonturbulent Systems.** We started our investigations by analysing unphysical (toy) systems with arbitrary constants and dimensionality of space, to learn more about the detection and features of different sorts of fixed points. A main question was how structures in coupling space can be recognised, if the dimensionality of the coupling space is high, and whether terms of higher order in the field expansion contribute as corrections.

In a second step, we applied the algorithm to physical systems with known properties, such as the scalar and the  $O(3)$ -symmetric field theory, in order to check that the algorithm works correctly and to see how closely we can reproduce analytic values for fixed point scalings and on the other hand to approach turbulent hydrodynamics in a stepwise manner, interpreting it as a special case of the general 3-vector-model.

**7.2.1. Toy Systems.** We worked with a number of unphysical systems for testing the algorithm and analysis tools, thus merely looking for nontrivial structures. These systems were defined by an action consisting of a propagator, a two-field, and a four-field interaction, where the field was a 3-vector field. Parameters were deliberately adjusted to allow the presence of different fixed points.

Investigations of the coupling space were mainly done using the shooting method, which is especially useful for finding fixed points. In practice, one initiates a number of RG-flows, starting from initial conditions sufficiently close to each other, and searches the topology of the flow for interesting structures. To identify the location of the fixed point more precisely, one repeats the method with initial conditions closer to the estimated fixed point couplings,

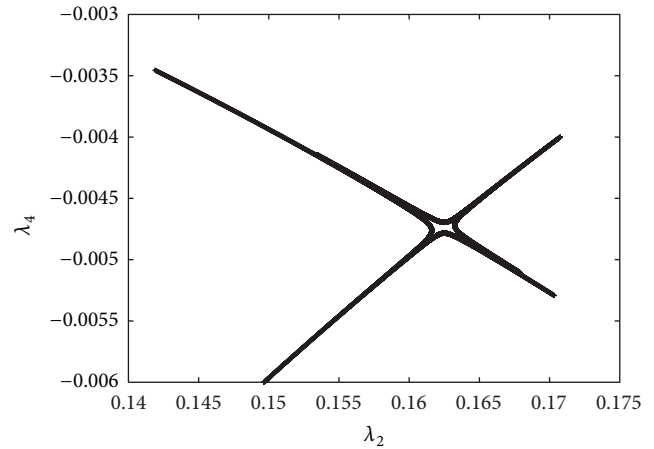


FIGURE 10: Fixed point of an unphysical model system, as found by use of the shooting method. Shown is the flow in the two-field interaction  $\lambda_2$  and the four-field interaction  $\lambda_4$ . The attractive direction goes to the upper right and lower left corners of the diagram, and the other two directions are repulsive.

leading to a picture like Figure 10. In this way, one approaches the fixed point iteratively. Following this procedure, the simulated trajectories approach the ideal trajectories, that is, the flows directly running into or out of the fixed point.

The shooting method is limited by the numerical accuracy of the computer program and the stepsize adjustment of the flow integration, as the algorithm slows down drastically when a fixed point is approached.

In a simulation involving more than two couplings, as is usually the case, the projection of the flow onto a two-dimensional subspace will in general not look so evident, but quite similar if the fixed point is approached closely enough.

**7.2.2. Simple Physical Systems.** Using our algorithm, the renormalisation group flows of the scalar field theory and the  $O(3)$ -symmetric theory in  $D$  dimensions have been analysed by Düben [34]. By reproducing known values of these theories like fixed point locations and scaling (also in the  $\epsilon$ -expansion), we went a step further towards the much more divert general three-vector theory and again checked the correctness of our algorithms. We found that we are able to accurately reproduce the values known from the literature, to a given order of the  $\epsilon$ -expansion. These results will be published in a forthcoming paper.

**7.3. Hydrodynamics Near the Local Potential Approximation.** Now we return to the analysis of the action for hydrodynamics derived previously in the LPA. The system is specified by the dimensionality of space and symmetries of the fields; the action in the LPA (57) serves as the initial condition of the flow.

In calculations of the RG flow it is generally preferable to calculate  $\eta$ , rather than to search for it by means of the shooting method. In the strict version of the LPA, on the other hand, one has  $\eta = 0$  as no field renormalisation is performed. We can extend the LPA by rescaling the field such



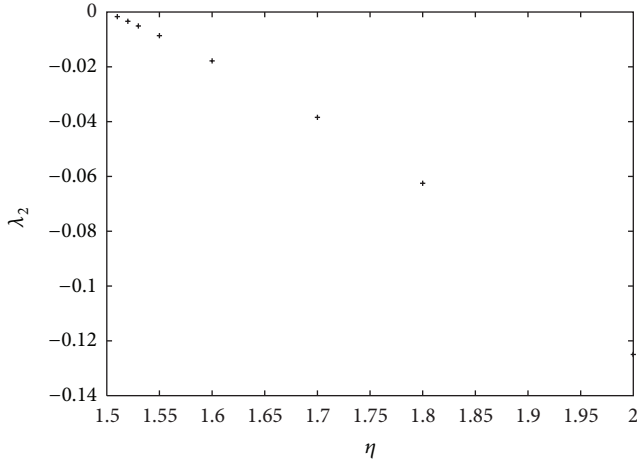


FIGURE 11: Fixed point of hydrodynamics in the Local Potential Approximation, as found by use of the shooting method. The plot shows the fixed point value of the  $\langle \nu\nu \rangle$ -coupling  $\lambda_2$ , depending on the anomalous exponent  $\eta$ .

that the corresponding anomalous dimension  $\eta$  equals some prescribed value.

The calculations of the RG flow were performed using two distinct algorithms: the first one iterating the rate equations derived in the previous sections and doing the book-keeping of the terms involved explicitly and the second one finding the graphs to be computed automatically. The second formulation turned out to be not only more elegant, but a great deal faster than the cumbersome implementation of the book-keeping.

The advantage of this approach is the fast integration of a large number of couplings, in that way evading the drawbacks of the expansions. Calculations were done with up to 100 couplings, though it has to be said that the identification of fixed points becomes nearly impossible in these high-dimensional spaces. Working with such a number of terms can only be done iteratively, meaning that one starts with a low number of couplings, identifies the fixed point, and then changes to more and more terms, hoping that these act as corrections to the overall behaviour.

It is not difficult to show that for values  $\eta > 1.5$  of the anomalous exponent, a non-trivial fixed point exists in the vicinity of the trivial one. We used the shooting method to determine the position of this non-trivial fixed point, depending on the anomalous exponent, as can be seen in Figures 11 and 12. The distance to the origin of coupling space can be seen to grow linearly with  $\eta$ ; we can, however, not relate this fixed point to any physical property. For  $\eta < 1.5$  this fixed point does not exist.

**7.4. Scaling of the Trivial Fixed Point.** It is straightforward to analyse the scaling of the trivial fixed point. The correlation functions of even orders are directly computed by the RG-flow; after Fourier transformation to physical space we can read off the scaling and find the results given in Table 1.

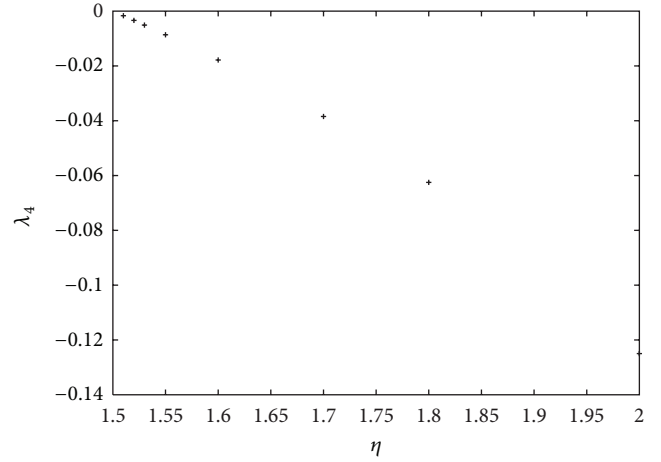


FIGURE 12: Fixed point of hydrodynamics in the Local Potential Approximation, as found by use of the shooting method. The plot shows the fixed point value of the  $\langle \nu\nu\nu\nu \rangle$ -coupling  $\lambda_4$ , depending on the anomalous exponent  $\eta$ .

TABLE 1: Scaling exponents at the trivial fixed point.

Order of the correlation function	Scaling exponent
2	$0.666 \pm 0.017$
4	$1.338 \pm 0.035$
6	$1.999 \pm 0.052$

TABLE 2: Scaling exponents at the trivial fixed point.

Order of the correlation function	Scaling exponent
1	$0.3334 \pm 0.0018$
3	$1.0004 \pm 0.0012$
5	$1.6681 \pm 0.0012$
7	$2.3348 \pm 0.0012$

The correlation functions of odd orders are not explicit terms of the action and so have to be measured indirectly. The correlation function of order  $n$  can, for example, be derived from the term  $\langle uv^n \rangle$ , if the scaling of the field  $u$  is known. We chose to measure the scaling of  $u$  from the two-point function  $\langle uu \rangle$  and subtract it from  $\langle uv^n \rangle$ . In this way the exponents given in Table 2 can be measured.

These numbers demonstrate that the trivial fixed point represents the scaling of Kolmogorov's K41-theory.

## 8. Conclusions and Outlook

We have shown how to define a generating functional for hydrodynamic turbulence, including a strict treatment of the incompressibility condition. The non-local interactions have been transformed into local ones by means of auxiliary fields. In addition, we have applied a derivative expansion to approximate the resulting action.

Concerning the renormalisation group, we discussed the procedure of renormalisation and rescaling in some detail. We obtained an RG-equation for a general multicomponent

action, including the turbulent action, and a set of rate equations after application of the derivative expansion.

Our numerical algorithm allows to compute the RG flow in this setting, including products of Grassmannian variables. We tested the numerical algorithm by reproducing known values for non-trivial scalings of the scalar theory in  $4 - \epsilon$  dimensions and the  $O(3)$ -symmetric field theory. The results are in agreement with values found in the literature, giving us confidence in the reliability of the numerical algorithm.

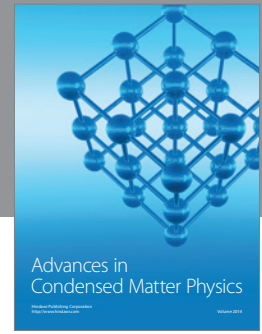
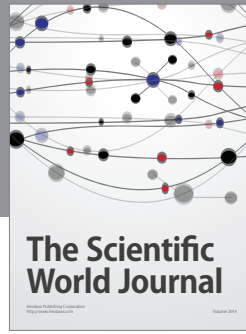
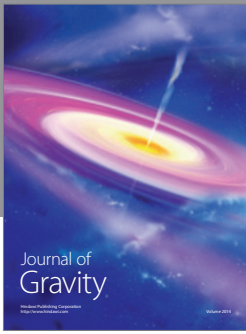
In the context of turbulence we were able to identify the trivial fixed point with the scaling exponents predicted by the K41 theory.

So far we have not been able to reproduce the intermittent exponents for the structure functions of fully developed turbulence that would agree with the experimental values. The reason for this deficit lies in the complexity of the general 3-vector model, including all theories that are based on hydrodynamics. Although the basic foundations of these theories are well understood, all of them (including Navier-Stokes and Burgers turbulence) involve the same dimensionality of space and symmetry of the fields, while leading to different predictions for the intermittent exponents.

Finally, it should be noted that it is not clear whether the analysis of a fixed point will eventually lead to an understanding of intermittency. Available data on turbulence show that the probability distribution of the velocity increment looks, for small distances, like a Lévy distribution—on large scales like normally distributed [35]. This could be an indication for a crossover between two fixed points. It would be interesting to test this conjecture by future flow calculations with our algorithm.

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