Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 892146, 8 pages
http://dx.doi.org/10.1155/2014/892146

Research Article

Composition Operators from Certain μ-Bloch Spaces to $\mathcal{Q}_p$ Spaces

Chunyu Tan and Maofa Wang

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

Correspondence should be addressed to Maofa Wang; mfwang.math@whu.edu.cn

Received 26 August 2014; Revised 4 December 2014; Accepted 4 December 2014; Published 22 December 2014

Academic Editor: Narcisa C. Apreutesei

Some necessary and sufficient conditions are established for composition operators C_φ to be bounded or compact from μ-Bloch type spaces $\mathcal{B}^\mu$ to $\mathcal{Q}_p$ spaces. Moreover, the boundedness, compactness, and Fredholmness of composition operators on little spaces $\mathcal{Q}_{p,0}$ are also characterized.

1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ with the topology of uniform convergence on compact subsets of $\mathbb{D}$. If $f \in H(\mathbb{D})$, we let $f_r(z) = f(rz)$, $0 < r < 1$, be the dilation of f. The H^∞ space consists of all functions $f \in H(\mathbb{D})$ satisfying $\sup_{z\in\mathbb{D}}|f(z)| < \infty$. The Bloch space $\mathcal{B}$ consists of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)\left|f'(z)\right| < \infty. \quad (1)$$

$\mathcal{B}$ equipped with the norm $\|f\| := |f(0)| + \|f\|_{\mathcal{B}}$ becomes a Banach space (see [1, 2]). For $\alpha > 0$, the α-Bloch space $\mathcal{B}^\alpha$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}^\alpha} := \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty. \quad (2)$$

Refer to [3] for more details on α-Bloch spaces. Recently, many authors have studied different classes of Bloch type spaces, where the typical weight function $(1-|z|^2)^\alpha$ is replaced by a continuous positive function μ defined on $\mathbb{D}$. More precisely, let $\mu : \mathbb{D} \to (0,\infty)$ be a radial weight function; that is, $\mu(z) = \mu(|z|)$, $z \in \mathbb{D}$, which is decreasing in a neighborhood of 1, continuous and $\lim_{|z|\to 1^-}\mu(|z|) = 0$. The Bloch type space $\mathcal{B}^\mu$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\mu} := \sup_{z\in\mathbb{D}} \mu(z)\left|f'(z)\right| < \infty. \quad (3)$$

It is easy to check that $\|f\|_\mu := |f(0)| + \|f\|_{\mathcal{B}^\mu}$ is a norm on $\mathcal{B}^\mu$, and $\mathcal{B}^\mu$ is a Banach space equipped with this norm (see, e.g., [4]). Clearly, $\mathcal{B}^\mu$ includes $\mathcal{B}^\alpha$ as its special case. Indeed, if $\mu(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, $\mathcal{B}^\mu$ becomes α-Bloch space $\mathcal{B}^\alpha$. When $\alpha = 1$, $\mathcal{B}^\alpha$ is just the classical Bloch space $\mathcal{B}$. For $\mu(z) = (1-|z|^2)\ln(e/(1-|z|^2))$, $\mathcal{B}^\mu$ is logarithmic Bloch space, which first appeared in characterizing the multipliers of the Bloch spaces. The little Bloch-type space $\mathcal{B}_{\mu,0} = \mathcal{B}_{\mu,0}(\mathbb{D})$ consists of all $f \in \mathcal{B}^\mu$ such that

$$\lim_{|z|\to 1^-} \mu(z)\left|f'(z)\right| = 0. \quad (4)$$

For $a \in \mathbb{D}$, let $\sigma_a(z) = (a-z)/(1-\overline{a}z)$ be the involutive automorphism of the unit disc which interchanges a and 0. We recall that in [5], for $p \geq 0$, $f \in H(\mathbb{D})$ belongs to $\mathcal{Q}_p$ if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^p dm(z) < \infty. \quad (5)$$

$\mathcal{Q}_{p,0}$ is the subclass of $\mathcal{Q}_p$ consisting of all $f \in \mathcal{Q}_p$ such that

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^p dm(z) = 0. \quad (6)$$

With the norm,

$$\|f\|_{\mathcal{Q}_p} := |f(0)| + \sup_{a\in\mathbb{D}}\left(\int_{\mathbb{D}}|f'(z)|^2\left(1-|\sigma_a(z)|^2\right)^p dm(z)\right)^{1/2}, \tag{7}$$

$\mathcal{Q}_p$ is a Banach space, and $\mathcal{Q}_{p,0}$ is the closure of all polynomials in $\mathcal{Q}_p$. It is well known that $\mathcal{Q}_1 = \text{BMOA}$, the space of all analytic functions of the bounded mean oscillation on $\mathbb{D}$. $\mathcal{Q}_0$ is the classical Dirichlet space $\mathcal{D}$. For all $1 < p < \infty$, $\mathcal{Q}_p$ is the Bloch space $\mathcal{B}$. Also, $\mathcal{Q}_{1,0} = \text{VMOA}$, the subspace of BMOA consisting of all analytic functions with vanishing mean oscillation, and for $p > 1$, $\mathcal{Q}_{p,0} = \mathcal{B}_0$; see [5, 6] for more details on those spaces.

Let H_1 and H_2 be two linear subspaces of $H(\mathbb{D})$. If φ is an analytic self-map of $\mathbb{D}$, then φ induces a composition operator $C_\varphi : H_1 \to H_2$ defined by

$$C_\varphi(f) := f\circ\varphi. \tag{8}$$

Composition operators have been studied by numerous authors in many subspaces of $H(\mathbb{D})$. Among others, Madigan and Matheson characterized the continuity and compactness of composition operators on the classical Bloch space $\mathcal{B}$ in [7]. Lou studied composition operators on $\mathcal{Q}_p$ spaces in [8]. Composition operators between the logarithmic Bloch-type space and $\mathcal{Q}^p_{\log}$ are studied in [9–11].

This paper studies composition operators from μ-Bloch type spaces $\mathcal{B}^\mu$ to $\mathcal{Q}_p$ spaces. After some necessary background materials, Section 2 gives some function-theoretic characterizations of bounded and compact composition operators $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_p$ by using the Hadamard gap series technique. Section 3 characterizes the continuity, compactness of $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$, and the Fredholmness of C_φ on $\mathcal{Q}_{p,0}$.

Throughout the paper we use the same letter C to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants C will be often specified in the parenthesis. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \le CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Composition Operators from $\mathcal{B}^\mu$ to $\mathcal{Q}_p$

We recall that an analytic function f on the unit disk $\mathbb{D}$ has Hadamard gaps if

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \tag{9}$$

with $n_{k+1}/n_k \ge \lambda > 1$ for all $k \in \mathbb{N}$. The following results are cited from [12].

Theorem A. *Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}^\mu$. Then*

$$\limsup_{n\to\infty} n\mu\left(1-\frac{1}{n}\right)|a_n| < \infty. \tag{10}$$

Theorem B. *Assume that μ is a nonincreasing radial weight satisfying*

$$\liminf_{k\to\infty}\frac{\mu(1-(1/n_k))}{\mu(1-(1/n_{k+1}))} = q > 1,$$
$$\mu\left(1-\ln\frac{1}{|z|}\right) \approx \mu(|z|), \quad as\ |z| \longrightarrow 1^-, \tag{11}$$

and such that $F(t) = 1/t\mu(1-1/t)$ is a positive nonincreasing absolutely continuous function on the interval $[1,\infty)$ satisfying $\lim_{t\to\infty}(tF'(t)/F(t)) = 0$ and $\lim_{t\to\infty} t^2F(t) = \infty$, where $\{n_k\}$ is a sequence such that $n_{k+1}/n_k = p > 1$, $k \in \mathbb{N}$. Let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in H(\mathbb{D})$. If

$$\limsup_{k\to\infty} n_k\mu\left(1-\frac{1}{n_k}\right)|a_k| < \infty, \tag{12}$$

then $f \in \mathcal{B}^\mu(\mathbb{D})$.

In the sequel, we always suppose that μ is as in Theorem B. The next lemma will play a key role in our main results.

Lemma 1. *There exist two functions $f, g \in \mathcal{B}^\mu$ such that*

$$|f'(z)| + |g'(z)| \gtrsim \frac{1}{\mu(z)}, \quad z \in \mathbb{D}. \tag{13}$$

Proof. We consider the function:

$$f(z) = \varepsilon z + \sum_{j=1}^{\infty}\frac{(q^j)^{-1}}{\mu(1-1/q^j)} z^{q^j}, \quad z \in \mathbb{D}, \tag{14}$$

where q is an appropriately large integer, and $\varepsilon > 0$ is sufficiently small. It follows from Theorem B that $f \in \mathcal{B}^\mu$. We claim that

$$\mu(z)|f'(z)| \gtrsim 1 \tag{15}$$

for $1-q^{-l} \le |z| \le 1-q^{-(l+1/2)}$, $l \in \mathbb{N}$. Indeed,

$$\begin{aligned}|f'(z)| &= \left|\varepsilon + \sum_{j=1}^{\infty}\frac{1}{\mu(1-1/q^j)} z^{q^j-1}\right| \\ &> \frac{1}{\mu(1-1/q^l)}|z|^{q^l} - \left(\varepsilon + \sum_{j=1}^{l-1}\frac{1}{\mu(1-1/q^j)}|z|^{q^j}\right) \\ &\quad - \left(\sum_{j=l+1}^{\infty}\frac{1}{\mu(1-1/q^j)}|z|^{q^j}\right) \\ &=: I_1 - I_2 - I_3.\end{aligned} \tag{16}$$

For q large enough, since

$$\left(1-q^{-l}\right)^{q^l} \le |z|^{q^l} \le \left(\left(1-q^{-(l+1/2)}\right)^{q^{l+1/2}}\right)^{q^{-1/2}}, \tag{17}$$

then

$$\frac{1}{3} \le |z|^{q^l} \le \left(\frac{1}{2}\right)^{q^{-1/2}}. \tag{18}$$

Hence

$$I_1 \ge \frac{1}{3}\frac{1}{\mu(1-1/q^l)}. \tag{19}$$

On the other hand, for q large enough,

$$\begin{aligned} I_2 &\le \varepsilon + \sum_{j=1}^{l-1} \frac{1}{\mu(1-1/q^j)} \\ &\le \frac{1}{\mu(1-1/q^l)}\left(\varepsilon + \frac{1}{q-1}\right), \\ I_3 &\le \frac{|z|^{q^{l+1}}}{q^l \mu(1-1/q^l)} \sum_{j=l+1}^{\infty} q^j |z|^{q^j - q^{l+1}} \\ &= \frac{|z|^{q^{l+1}}}{q^l \mu(1-1/q^l)} \sum_{s=0}^{\infty} q^{s+l+1} |z|^{(q^{l+2}-q^{l+1})s} \\ &\le \frac{\left(|z|^{q^l}\right)^q}{\mu(1-1/q^l)} \frac{q}{1-q|z|^{(q^{l+2}-q^{l+1})}} \\ &\le \frac{q2^{-q^{1/2}}}{\mu(1-1/q^l)\left(1-q2^{-(q^{3/2}-q^{1/2})}\right)}. \end{aligned} \tag{20}$$

It follows from (19) and (20) that

$$\begin{aligned} \left|f'(z)\right| &\ge \frac{1}{\mu(1-1/q^l)} \\ &\quad \times \left[\frac{1}{3} - \left(\varepsilon + \frac{1}{q-1}\right) - \frac{q2^{-q^{1/2}}}{\left(1-q2^{-(q^{3/2}-q^{1/2})}\right)}\right]. \end{aligned} \tag{21}$$

Since $\mu(1-1/q^{l+1/2}) \approx \mu(1-1/q^l)$ for sufficient large q,

$$\left|f'(z)\right| \gtrsim \frac{1}{\mu(1-1/q^l)} \gtrsim \frac{1}{\mu(1-1/q^{l+1/2})} \gtrsim \frac{1}{\mu(z)} \tag{22}$$

for $1/q^{l+1/2} \le 1-|z| \le 1/q^l$. That is (15).

Now with a similar argument for $1-q^{-(l+1/2)} \le |z| \le 1-q^{-(l+1)}$, $l \in \mathbb{N}$ and q large enough, we have

$$\mu(z)\left|g'(z)\right| \gtrsim 1, \tag{23}$$

where

$$g(z) = \sum_{j=1}^{\infty} \frac{\left(q^{j+1/2}\right)^{-1}}{\mu(1-1/q^{j+1/2})} z^{q^{j+1/2}}. \tag{24}$$

Now inequality (13) follows immediately from (15) and (23) on the annulus $1-q^{-1} < z < 1$.

On the other hand, in the disc $|z| \le 1-q^{-1}$, we have that $g'(0) = 0$, $f'(0) \ne 0$, and f' and g' have a finite number of zeros in the disc. Hence if f' and g' have common zeros in the disc $|z| \le 1-q^{-1}$, then one can replace g by the function $g_0(z) = g(e^{i\theta}z)$ for an appropriate θ and obtain a pair of functions which satisfy inequality (13). □

We now characterize the boundedness of the composition operator $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_p$.

Theorem 2. *Let $p > 0$ and φ be an analytic self-map of the unit disc. Then $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_p$ is bounded if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^p}{\mu(|\varphi(z)|)^2} dm(z) < \infty. \tag{25}$$

Proof. Assume that $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_p$ is bounded; then $C_\varphi f \in \mathcal{Q}_p$, for $f \in \mathcal{B}^\mu$. By Lemma 1, there exist $f, g \in \mathcal{B}^\mu$ such that $|f'(z)| + |g'(z)| \gtrsim 1/\mu(z)$. So

$$\begin{aligned} \infty &> \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[\left|(f\circ\varphi)'(z)\right|^2 + \left|(g\circ\varphi)'(z)\right|^2\right] \\ &\qquad \times \left(1-\left|\sigma_a(z)\right|^2\right)^p dm(z) \\ &\gtrsim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[\left|(f\circ\varphi)'(z)\right| + \left|(g\circ\varphi)'(z)\right|\right]^2 \\ &\qquad \times \left(1-\left|\sigma_a(z)\right|^2\right)^p dm(z) \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[\left|f'(\varphi(z))\right| + \left|g'(\varphi(z))\right|\right]^2 \left|\varphi'(z)\right|^2 \\ &\qquad \times \left(1-\left|\sigma_a(z)\right|^2\right)^p dm(z) \\ &\gtrsim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^p}{\mu(|\varphi(z)|)^2} dm(z), \end{aligned} \tag{26}$$

which implies (25).

Conversely, for any $f \in \mathcal{B}^\mu$, it is clear to see that

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|(f\circ\varphi)'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^p dm(z) \\ &\quad = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(\varphi(z))\right|^2 \left|\varphi'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^p dm(z) \\ &\quad \le \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^2 \left(1-\left|\sigma_a(z)\right|^2\right)^p}{\mu(|\varphi(z)|)^2} dm(z) \cdot \|f\|_{\mathcal{B}^\mu}^2. \end{aligned} \tag{27}$$

By (25), $C_\varphi f \in \mathcal{Q}_p$. Then $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_p$ is bounded by the closed graph theorem. □

Now, we are going to characterize the compactness of composition operators $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_p$. In [13], Tjani showed the following result.

Lemma 3. *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose the following:*

(1) *The point evaluation functions on Y are continuous.*

(2) *The closed unit ball of X is compact subset of X in the topology of uniform convergence on compact sets.*

(3) *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if, given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Observe that for any fixed $z \in \mathbb{D}$ we have

$$\begin{aligned} |f(z)| &\lesssim |f(0)| + \log \frac{1}{1-|z|} \|f\|_{\mathscr{B}} \\ &\lesssim |f(0)| + \log \frac{1}{1-|z|} \|f\|_{\mathcal{Q}_p}, \end{aligned} \tag{28}$$

so the point evaluation functionals on $\mathcal{Q}_p$ are continuous. Thus, as a consequence of Lemma 3, we have the following result.

Lemma 4. *The composition operator $C_\varphi : \mathscr{B}^\mu \to \mathcal{Q}_p$ is compact if and only if for every bounded sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathscr{B}^\mu$, which converges uniformly to zero on any compact subset of the unit disk, $\|C_\varphi(f_n)\|_{\mathcal{Q}_p} \to 0$ as $n \to \infty$.*

We now use Lemma 4 above to give a characterization of compact composition operator $C_\varphi : \mathscr{B}^\mu \to \mathcal{Q}_p$.

Theorem 5. *Let $p > 0$ and φ be an analytic self-map of the unit disc. Then $C_\varphi : \mathscr{B}^\mu \to \mathcal{Q}_p$ is compact if and only if $\varphi \in \mathcal{Q}_p$ and*

$$\lim_{t\to 1} \sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} \frac{|\varphi'(z)|^2 \left(1-|\sigma_a(z)|^2\right)^p}{\mu(|\varphi(z)|)^2} dm(z) = 0. \tag{29}$$

Proof. We first assume that $C_\varphi : \mathscr{B}^\mu \to \mathcal{Q}_p$ is compact; then $\varphi \in \mathcal{Q}_p$. Since $\|z^n/n\|_{\mathscr{B}^\mu} \lesssim 1$ and $z^n/n \to 0$ as $n \to \infty$, uniformly on any compact subsets of the unit disk, then by Lemma 4, $\|C_\varphi(z^n/n)\|_{\mathcal{Q}_p} \to 0$ as $n \to \infty$. So for each $t \in (0,1)$ and each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$t^{2(n_0-1)} \sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} |\varphi'(z)|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) < \varepsilon. \tag{30}$$

If we choose $t \geq 2^{-(1/2(n_0-1))}$, then

$$\sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} |\varphi'(z)|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) < 2\varepsilon. \tag{31}$$

We now consider the functions $f_r(z) = f(rz)$ and $r \in (0,1)$ for f with $\|f\|_{\mathscr{B}^\mu} \leq 1$. Since $\|f_r\|_{\mathscr{B}^\mu} \lesssim 1$ and f_r uniformly to f on any compact subsets of the unit disk, for $\varepsilon > 0$ there exists $r_0 \in (0,1)$ such that, for all $r \geq r_0$,

$$\begin{aligned} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} &\left|(f\circ\varphi)'(z) - (f_r\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &< \varepsilon. \end{aligned} \tag{32}$$

Note that, for $t \geq 2^{-(1/2(n_0-1))}$,

$$\begin{aligned} \sup_{a\in\mathbb{D}} &\int_{\{|\varphi(z)|>t\}} \left|(f\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &\leq 2\sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} \left|(f\circ\varphi)'(z) - (f_{r_0}\circ\varphi)'(z)\right|^2 \\ &\qquad \times \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &\quad + 2\sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} \left|(f_{r_0}\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &\leq 2\varepsilon + 2\left\|f'_{r_0}\right\|^2_{H^\infty} \\ &\quad \times \sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} |\varphi'(z)|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &\leq 4\varepsilon\left(1 + \left\|f'_{r_0}\right\|^2_{H^\infty}\right). \end{aligned} \tag{33}$$

Namely, for each $\|f\|_{\mathscr{B}^\mu} \leq 1$ and $\varepsilon > 0$, there is $0 < \delta < 1$ and some constant $C(f)$ depending only on f such that, for $t \in [\delta, 1)$,

$$\sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} \left|(f\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) < C(f)\,\varepsilon. \tag{34}$$

Since C_φ is compact, it maps the unit ball of $\mathscr{B}^\mu$ into a relative compact subset of $\mathcal{Q}_p$. Thus for each $\varepsilon > 0$, there exists a finite collection of functions $f_1, f_2, \ldots, f_N$ in the unit ball of $\mathscr{B}^\mu$, such that for each $\|f\|_{\mathscr{B}^\mu} \leq 1$ there is a $k \in \{1, 2, \ldots, N\}$ with

$$\begin{aligned} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} &\left|(f\circ\varphi)'(z) - (f_k\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &< \varepsilon. \end{aligned} \tag{35}$$

If we take $C = \max_{1\leq k\leq N} C(f_k)$, then for $t \in [\delta, 1)$

$$\sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} \left|(f_k\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) < C\varepsilon. \tag{36}$$

Then

$$\begin{aligned} \sup_{\|f\|_{\mathscr{B}^\mu}\leq 1} \sup_{a\in\mathbb{D}} &\int_{\{|\varphi(z)|>t\}} \left|(f\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &\lesssim C\varepsilon, \end{aligned} \tag{37}$$

which implies the desired estimate (29) by using Lemma 1 in a similar way as in the proof of Theorem 2.

Conversely, we assume that $\varphi \in \mathcal{Q}_p$ and (29) holds. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in the unit ball of $\mathcal{B}^\mu$, such that $f_n \to 0$ uniformly on the compact subsets of the unit disc as $n \to \infty$. We notice that, for $t \in (0,1)$,

$$\begin{aligned}\|f_n \circ \varphi\|^2_{\mathcal{Q}_p} &\lesssim |f_n(\varphi(0))|^2 \\ &+ \sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|\le t\}} \left|(f_n\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &+ \sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|> t\}} \left|(f_n\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &=: J_1 + J_2 + J_3.\end{aligned} \tag{38}$$

Since $f_n \to 0$ uniformly on the compact subsets of the unit disc, as $n \to \infty$, then $f_n' \to 0$ as $n \to \infty$ uniformly on the compact subsets of the unit disc. So for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that, for each $n > n_0$, $J_1 \le \varepsilon$, and $J_2 \le \varepsilon\|\varphi\|^2_{\mathcal{Q}_p}$. Also notice that

$$J_3 \le \sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} \frac{|\varphi'(z)|^2 \left(1-|\sigma_a(z)|^2\right)^p}{\mu(|\varphi(z)|)^2} dm(z). \tag{39}$$

By (29) there exists $t_0 \in (0,1)$ such that, for every $t > t_0$, $J_3 \le \varepsilon$. Thus $\|C_\varphi(f_n)\|_{\mathcal{Q}_p} \to 0$ as $n \to \infty$, which completes the proof by Lemma 4. □

The following corollary is an immediate result of Theorems 2 and 5.

Corollary 6. *Let $p \in (0,\infty)$. Then*

(1) *$\mathcal{B}^\mu$ is embedded boundedly into $\mathcal{Q}_p$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left(1-|\sigma_a(z)|^2\right)^p}{\mu(|z|)^2} dm(z) < \infty. \tag{40}$$

(2) *$\mathcal{B}^\mu$ is embedded compactly into $\mathcal{Q}_p$ if and only if*

$$\lim_{t\to 1}\sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>t\}} \frac{\left(1-|\sigma_a(z)|^2\right)^p}{\mu(|z|)^2} dm(z) < 0. \tag{41}$$

3. Composition Operators from $\mathcal{B}^\mu$ to $\mathcal{Q}_{p,0}$

In this section, we investigate composition operators from $\mathcal{B}^\mu$ to $\mathcal{Q}_{p,0}$. Contrast with the case $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_p$, here the boundedness and compactness of $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ are equivalent. Last, we also characterize the Fredholmness of composition operators on $\mathcal{Q}_{p,0}$. We first begin with the following.

Lemma 7. *Suppose that $0 < p < \infty$ and φ is an analytic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ is compact if and only if*

$$\lim_{|a|\to 1} \sup_{\|f\|_{\mathcal{B}^\mu}\le 1} \int_{\mathbb{D}} \left|(f\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) = 0. \tag{42}$$

Proof. Suppose that $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ is compact; then $C_\varphi(\mathbb{B}^\mu)$ is relatively compact in $\mathcal{Q}_{p,0}$, where $\mathbb{B}^\mu$ is the unit ball of $\mathcal{B}^\mu$. Let $\varepsilon > 0$; then there is an $(\varepsilon/4)$-net $f_1, f_2, \dots, f_N$ of $C_\varphi(\mathbb{B}^\mu)$. Then for any fixed $f \in \mathbb{B}^\mu$, there exists $i_0 \in \{1,2,\dots,N\}$ such that

$$\left\|\left(f-f_{i_0}\right)\circ(\varphi)\right\|_{\mathcal{Q}_p} < \frac{\varepsilon}{4}. \tag{43}$$

Clearly, there is $\delta > 0$ such that

$$\int_{\mathbb{D}} \left|\left(f_{i_0}\circ\varphi\right)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) < \frac{\varepsilon}{4} \tag{44}$$

for $|a| > \delta$. So

$$\begin{aligned}&\int_{\mathbb{D}} \left|(f\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &\le 2\int_{\mathbb{D}} \left|(f\circ\varphi)'(z) - \left(f_{i_0}\circ\varphi\right)'(z)\right|^2 \\ &\qquad \times \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &\quad + 2\int_{\mathbb{D}} \left|\left(f_{i_0}\circ\varphi\right)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) \\ &< 2\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon\end{aligned} \tag{45}$$

for $|a| > \delta$. So (42) is proved.

Conversely, suppose that (42) holds and $(f_n) \subseteq \mathcal{B}^\mu$ with $\|f_n\|_{\mathcal{B}^\mu} \le 1$, converging uniformly to 0 on compact subsets of $\mathbb{D}$; we now prove

$$\lim_{n\to\infty} \left\|C_\varphi(f_n)\right\|_{\mathcal{Q}_p} = 0. \tag{46}$$

For any given $\varepsilon > 0$, by (42), there is $\delta > 0$ such that, for all f_n,

$$\sup_{\delta<|a|<1} \int_{\mathbb{D}} \left|(f_n\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z) < \varepsilon; \tag{47}$$

that is, $f_n \circ \varphi \in \mathcal{Q}_{p,0}$. For $a \in \mathbb{D}$, $r \in (0,1)$ and $\mathbb{D}_r = \{z \in \mathbb{D} : |\varphi(z)| > r\}$, set

$$T_r(a) = \int_{\mathbb{D}_r} \left|(f_n\circ\varphi)'(z)\right|^2 \left(1-|\sigma_a(z)|^2\right)^p dm(z), \tag{48}$$

and then $\lim_{r\to 1} T_r(a) = 0$, which means that, for each $a \in \mathbb{D}$, there exists r_a such that $T_r(a) < \varepsilon$ for all $r > r_a$. The same as in the proof of Lemma 1.3 in [14], $T_r(a)$ is a continuous function of a, so there is a neighbourhood $N(a) \subseteq \mathbb{D}$ of a such that $T_{r_a}(z) < \varepsilon$ for all $z \in N(a)$. Since $\{a : |a| \le \delta\} \subseteq \bigcup_{a\in\{a:|a|\le\delta\}} N(a)$ and $\{a : |a| \le \delta\}$ is compact, there exist

$N(a_1), \ldots, N(a_M)$ such that $\{a : |a| \le \delta\} \subseteq \bigcup_{i=1}^{M} N(a_i)$. For a_i, $i = 1, \ldots, M$, there exists r_{a_i} such that $T_{r_{a_i}}(z) < \varepsilon$, $z \in N(a_i)$, $i = 1, \ldots, M$. Setting $r_0 = \max\{r_{a_1}, \ldots, r_{a_M}\}$, $T_{r_0}(a) < \varepsilon$ for all $|a| \le \delta$. That is,

$$\sup_{|a|\le\delta} \int_{\mathbb{D}_{r_0}} \left|(f_n \circ \varphi)'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) < \varepsilon. \quad (49)$$

On the other hand, since f_n converge to 0 uniformly on compact subsets of $\mathbb{D}$, there exists n_0, such that, for all $n \ge n_0$, $|f_n'(z)|^2 \le \varepsilon$ for $|z| \le r_0$. It follows from (42) that $\varphi \in \mathcal{Q}_{p,0}$. So

$$\begin{aligned} &\sup_{|a|\le\delta} \int_{\mathbb{D}\setminus\mathbb{D}_{r_0}} \left|(f \circ \varphi)'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ &\quad\le \sup_{|a|\le\delta} \varepsilon \int_{\mathbb{D}} \left|\varphi'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ &\quad\le \varepsilon \|\varphi\|_{\mathcal{Q}_p}^2. \end{aligned} \quad (50)$$

It follows from (49) and (50) that for $n \ge n_0$

$$\begin{aligned} &\sup_{|a|\le\delta} \int_{\mathbb{D}} \left|(f_n \circ \varphi)'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ &\quad\le \left(1 + \|\varphi\|_{\mathcal{Q}_p}^2\right) \varepsilon. \end{aligned} \quad (51)$$

Combining (47) and (51) implies that $\|C_\varphi(f_n)\|_{\mathcal{Q}_p} \to 0$ as $n \to \infty$, which completes the proof. □

The following theorem characterizes the equivalence of boundedness and compactness of composition operators from $\mathcal{B}^\mu$ to $\mathcal{Q}_{p,0}$.

Theorem 8. *Let $0 < p < \infty$ and φ is an analytic self-map of $\mathbb{D}$. Then the following are equivalent.*

(1) *$C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ is bound.*

(2) *$C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ is compact.*

(3)

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p}{\mu\left(|\varphi(z)|\right)^2} dm(z) = 0. \quad (52)$$

Proof. (1) ⇔ (3). Assume that $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ is bounded; then $\varphi(z) \in \mathcal{Q}_{p,0}$ by taking $f(z) = z$. By Lemma 1, there exist $f, g \in \mathcal{B}^\mu$ such that $|f'(z)| + |g'(z)| \gtrsim 1/\mu(z)$. So

$$\begin{aligned} 0 \longleftarrow &\lim_{a\to 1^-} \int_{\mathbb{D}} \left[\left|(f \circ \varphi)'(z)\right|^2 + \left|(g \circ \varphi)'(z)\right|^2\right] \\ &\times \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ \gtrsim &\lim_{a\to 1^-} \int_{\mathbb{D}} \left[\left|(f \circ \varphi)'(z)\right| + \left|(g \circ \varphi)'(z)\right|\right]^2 \\ &\times \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ = &\lim_{a\to 1^-} \int_{\mathbb{D}} \left[\left|f'(\varphi(z))\right| + \left|g'(\varphi(z))\right|\right]^2 \\ &\times \left|\varphi'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ \gtrsim &\lim_{a\to 1^-} \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p}{\mu\left(|\varphi(z)|\right)^2} dm(z), \end{aligned} \quad (53)$$

which implies (52).

Conversely, for any $f \in \mathcal{B}^\mu$, it is clear that

$$\begin{aligned} &\lim_{a\to 1^-} \int_{\mathbb{D}} \left|(f \circ \varphi)'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ &\quad= \lim_{a\to 1^-} \int_{\mathbb{D}} \left|f'(\varphi(z))\right|^2 \left|\varphi'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ &\quad\le \lim_{a\to 1^-} \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p}{\mu\left(|\varphi(z)|\right)^2} dm(z) \cdot \|f\|_{\mathcal{B}^\mu}^2. \end{aligned} \quad (54)$$

By (52), $C_\varphi f \in \mathcal{Q}_{p,0}$. Then $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ is bounded by the closed graph theorem.

(2) ⇔ (3). Let $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ be compact. By Lemma 7, for any given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\sup_{\|f\|_{\mathcal{B}^\mu} < 1} \int_{\mathbb{D}} \left|(f \circ \varphi)'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) < \varepsilon \quad (55)$$

for $|a| > \delta$, which implies (52) by Lemma 1.

Conversely, suppose that (52) holds; then for any function $f \in \mathcal{B}^\mu$,

$$\begin{aligned} &\int_{\mathbb{D}} \left|f'(\varphi(z))\right|^2 \left|\varphi'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z) \\ &\quad\le \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^2 \left(1 - |\sigma_a(z)|^2\right)^p}{\mu\left(|\varphi(z)|\right)^2} dm(z) \cdot \|f\|_{\mathcal{B}^\mu} \longrightarrow 0 \end{aligned} \quad (56)$$

as $|a| \to 1^-$. Hence, $C_\varphi : \mathcal{B}^\mu \to \mathcal{Q}_{p,0}$ is compact by Lemma 7, which completes the proof. □

Finally, we consider the Fredholmness of composition operators on $\mathcal{Q}_{p,0}$ spaces. For a Banach space X, recall that a bounded linear operator T on X is said to be Fredholm if both the dimension of its kernel and the codimension of its image are finite. This occurs if and only if T is invertible modulo the compact operators; that is, there is a bounded linear operator S such that both $TS - I$ and $ST - I$ are compact. We also notice that an operator is Fredholm if and only if its dual is Fredholm (see, e.g., [15]).

Before giving our result on Fredholmness, we need a useful result due to Wirths and Xiao [16].

Lemma 9. *Let $p \in (0,\infty)$ and $f \in \mathcal{Q}_p$ with $f_r(z) := f(rz)$ for $r \in (0,1)$. Then the following are equivalent.*

(1) $f \in \mathcal{Q}_{p,0}$.

(2) $\lim_{r\to 1}\|f_r - f\|_{\mathcal{Q}_p} = 0$.

(3) *f belongs to the closure of the class of the polynomials in the norm $\|\cdot\|_{\mathcal{Q}_p}$.*

(4) *For any $\epsilon > 0$ there is a $g \in \mathcal{Q}_{p,0}$ such that $\|g - f\|_{\mathcal{Q}_p} < \epsilon$.*

Theorem 10. *Let φ be an analytic self-map of the unit disc $\mathbb{D}$. Then the following are equivalent.*

(1) *φ is a Möbius transformation of $\mathbb{D}$.*

(2) *$C_\varphi : \mathcal{Q}_{p,0} \to \mathcal{Q}_{p,0}$ is invertible.*

(3) *$C_\varphi : \mathcal{Q}_{p,0} \to \mathcal{Q}_{p,0}$ is Fredholm.*

Proof. (1) $\Rightarrow$ (2). If $\varphi(z) = \varphi_a(z) = (a - z)/(1 - \overline{a}z)$, $a \in \mathbb{D}$, then $\varphi_a \circ \varphi_a(z) = z$; that is, $\varphi_a = \varphi_a^{-1}$. Since $\mathcal{Q}_{p,0}$ is Möbius invariant by [15], we get $C_\varphi^{-1} = C_{\varphi^{-1}}$. If $\varphi(z) = \lambda z$ with $|\lambda| = 1$, we also have $C_\varphi^{-1} = C_{\varphi^{-1}}$. Since any Möbius transformation φ can be expressed that $\varphi(z) = \lambda((a-z)/(1-\overline{a}z))$ ($|\lambda| = 1, a \in \mathbb{D}$), C_φ is invertible.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Suppose $C_\varphi : \mathcal{Q}_{p,0} \to \mathcal{Q}_{p,0}$ is Fredholm. Note that φ cannot be a constant mapping. Otherwise, if $\varphi(z) \equiv a$, we have $(z - a)^n \in \ker C_\varphi$ and $\dim \ker C_\varphi = \infty$, which contradicts the Fredholmness of C_φ.

Assume φ is not one to one. So there exist $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$ with $\varphi(z_1) = \varphi(z_2)$. Select the neighborhoods U, V of z_1, z_2, respectively, such that $U \cap V = \emptyset$. $\varphi(U) \cap \varphi(V)$ is a nonempty and open set due to φ being open by the Open Mapping Theorem, so there exist infinite sequences $\{z_n^1\} \subseteq U$, $\{z_n^2\} \subseteq V$ such that $\varphi(z_n^1) = \varphi(z_n^2) = \omega_n$ which are distinct. Hence $C_\varphi^*\delta_{z_n^1} = \delta_{\varphi(z_n^1)} = \delta_{\varphi(z_n^2)} = C_\varphi^*\delta_{z_n^2}$; namely, $C_\varphi^*(\delta_{z_n^1} - \delta_{z_n^2}) = 0$, where $\delta_z : f \to f(z)$ is evaluation function, which is a bounded linear functional on $\mathcal{Q}_{p,0}$. Since $\mathcal{Q}_{p,0}$ contains all polynomials by Lemma 9, we have that each evaluation function is not a linear combination of other evaluation functions, so the sequence $\{\delta_{z_n^1} - \delta_{z_n^2}\}$ is linearly independent in the kernel of the adjoint operator C_φ^*. It is worth pointing out that C_φ^* is also Fredholm. It is a contradiction, so φ is injective.

We now show φ is surjective. Assume that φ is not surjective. Then we can find $z_0 \in \mathbb{D} \cap \partial\varphi(\mathbb{D})$ and $\{z_n\} \subseteq \mathbb{D}$ such that $\varphi(z_n) \to z_0$ as $n \to \infty$. Further, we get, by the Open Mapping Theorem, that $|z_n| \to 1$ as $n \to \infty$. For arbitrary $f \in \mathcal{Q}_{p,0}$,

$$C_\varphi^*\delta_{z_n} f = \delta_{\varphi(z_n)} f = f \circ \varphi(z_n) \longrightarrow f(z_0) = \delta_{z_0} f; \qquad (57)$$

we get $\delta_{\varphi(z_n)} \underset{\longrightarrow}{w^*} \delta_{z_0}$ and $\{\delta_{\varphi(z_n)}\}$ is bounded uniformly. Again, it is obvious that $\|\delta_{z_n}\| \to \infty$ as $n \to \infty$. Therefore, $\|\delta_{\varphi(z_n)}/\|\delta_{z_n}\|\| = \|C_\varphi^*\delta_{z_n}/\|\delta_{z_n}\|\| \to 0$. On the other hand, since C_φ^* is also Fredholm, there are operators K and S on $\mathcal{Q}_{p,0}^*$, with K compact and S bounded, such that $SC_\varphi^* = I + K$. Thus, $\delta_{z_n}/\|\delta_{z_n}\| + K\delta_{z_n}/\|\delta_{z_n}\| \to 0$. Because K is compact and $\{\delta_{z_n}/\|\delta_{z_n}\|\}$ is bounded, there exists subsequence $\{\delta_{z_{n_k}}/\|\delta_{z_{n_k}}\|\}$ such that $K\delta_{z_{n_k}}/\|\delta_{z_{n_k}}\| \to h$, $\delta_{z_{n_k}}/\|\delta_{z_{n_k}}\| \to -h$, which means $\|h\| = 1$. Moreover, $\mathcal{Q}_{p,0}$ is the closure of all polynomials with respect to the norm $\|\cdot\|_{\mathcal{Q}_p}$ by Lemma 9, which gets $(\delta_{z_{n_k}}/\|\delta_{z_{n_k}}\|) \underset{\longrightarrow}{w^*} 0$. This implies that $(\delta_{z_{n_k}}/\|\delta_{z_{n_k}}\|) \underset{\longrightarrow}{w^*} -h = 0$. This is a contradiction. So φ is surjective. Thus φ is a Möbius transformation, which completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This research is partially supported by NSFC (11271293).

References

[1] J. Anderson, J. Claunie, and C. Pommerenke, "On the Bloch functions and normal functions," *Journal für die Reine und Angewandte Mathematik*, vol. 270, pp. 12–37, 1974.

[2] K. Zhu, *Operator Theory in Function Spaces*, vol. 139 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1990.

[3] K. Zhu, "Bloch type spaces of analytic functions," *Rocky Mountain Journal of Mathematics*, vol. 23, no. 3, pp. 1143–1177, 1993.

[4] H. Chen and P. Gauthier, "Composition operators on μ-Bloch spaces," *Canadian Journal of Mathematics*, vol. 61, no. 1, pp. 50–75, 2009.

[5] J. Xiao, *Holomorphic Q Classes*, vol. 1767 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 2001.

[6] S. Li, "Composition operators on Q^p spaces," *Georgian Mathematical Journal*, vol. 12, no. 3, pp. 505–514, 2005.

[7] K. Madigan and A. Matheson, "Compact composition operators on the Bloch space," *Transactions of the American Mathematical Society*, vol. 372, pp. 2679–2687, 1995.

[8] Z. Lou, "Composition operators on Q^p spaces," *Journal of the Australian Mathematical Society*, vol. 70, no. 2, pp. 161–188, 2001.

[9] H. Li and P. Liu, "Composition operators between generally weighted Bloch space and Q_{log}^q space," *Banach Journal of Mathematical Analysis*, vol. 3, no. 1, pp. 99–110, 2009.

[10] H. Li, P. Liu, and M. Wang, "Composition operators between generally weighted Bloch spaces of polydisk," *JIPAM: Journal of Inequalities in Pure and Applied Mathematics*, vol. 8, no. 3, article 85, 2007.

[11] S. Stević, "On new Bloch-type spaces," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 841–849, 2009.

[12] S. Stević, "Bloch-type functions with Hadamard gaps," *Applied Mathematics and Computation*, vol. 208, no. 2, pp. 416–422, 2009.

[13] M. Tjani, *Compact composition operators on some Möbius invariant Banach space [Ph.D. dissertation]*, Machigan State University, 1996.

[14] W. Smith and R. Zhao, "Composition operators mapping into the Q_p spaces," *Analysis*, vol. 17, no. 2-3, pp. 239–263, 1997.

[15] V. Müller, *Spectral Theory of Linear Operators*, Birkhäuser, Boston, Mass, USA, 2003.

[16] K. J. Wirths and J. Xiao, "Recognizing $Q_{p,0}$ functions per Dirichlet space structure," *Bulletin of the Belgian Mathematical Society*, vol. 8, no. 1, pp. 47–59, 2001.