

## Research Article

# Characterizations of Ideals in Intermediate $C$ -Rings $A(X)$ via the $A$ -Compactifications of $X$

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Let  $X$  be a completely regular topological space. An intermediate ring is a ring  $A(X)$  of continuous functions satisfying  $C^*(X) \subseteq A(X) \subseteq C(X)$ . In Redlin and Watson (1987) and in Panman et al. (2012), correspondences  $\mathcal{Z}_A$  and  $\mathfrak{Z}_A$  are defined between ideals in  $A(X)$  and  $z$ -filters on  $X$ , and it is shown that these extend the well-known correspondences studied separately for  $C^*(X)$  and  $C(X)$ , respectively, to any intermediate ring. Moreover, the inverse map  $\mathcal{Z}_A^{-1}$  sets up a one-one correspondence between the maximal ideals of  $A(X)$  and the  $z$ -ultrafilters on  $X$ . In this paper, we define a function  $\mathfrak{R}_A$  that, in the case that  $A(X)$  is a  $C$ -ring, describes  $\mathfrak{Z}_A$  in terms of extensions of functions to realcompactifications of  $X$ . For such rings, we show that  $\mathfrak{Z}_A^{-1}$  maps  $z$ -filters to ideals. We also give a characterization of the maximal ideals in  $A(X)$  that generalize the Gelfand-Kolmogorov theorem from  $C(X)$  to  $A(X)$ .

## 1. Introduction

Let  $X$  be a completely regular space and  $A(X)$  an intermediate ring of continuous real-valued functions; that is,  $C^*(X) \subseteq A(X) \subseteq C(X)$ . It is well known that there is a natural correspondence  $\mathbf{Z}$  between ideals of  $C(X)$  and  $z$ -filters on  $X$  as described in [1, pages 26–27]. Such a correspondence  $\mathbf{E}$  also exists for  $C^*(X)$  [1, Problem 2L]. In [2], a correspondence  $\mathcal{Z}_A$  between the ideals of any  $A(X)$  and the  $z$ -filters on  $X$  was introduced, and its properties were further investigated in [3–5]. In [6], another correspondence  $\mathfrak{Z}_A$  between ideals of any  $A(X)$  and  $z$ -filters on  $X$  is introduced. It is shown in [6] that the correspondences  $\mathcal{Z}_A$  and  $\mathfrak{Z}_A$  extend the correspondences  $\mathbf{E}$  and  $\mathbf{Z}$  from  $C^*(X)$  and  $C(X)$ , respectively, to all intermediate rings, and an explicit formula is stated that relates the two correspondences. In this paper, we give a characterization (Definition 3 and Theorem 6) of the correspondence  $\mathfrak{Z}_A$  for intermediate  $C$ -rings  $A(X)$  in terms of the  $A$ -compactifications of  $X$  introduced in [7]. In this setting, we show (Theorem 14) that the inverse map  $\mathfrak{Z}_A^{-1}$  of the set map  $\mathfrak{Z}_A$  maps ideals in  $A(X)$  to  $z$ -filters on  $X$ . We also give a characterization of the maximal ideals in  $A(X)$ . This characterization generalizes from  $C(X)$  to  $A(X)$  the Gelfand-Kolmogorov theorem (Theorem 8). We follow the notation in [1, 6].

## 2. Preliminaries

For convenience we state some of the definitions and results needed in this paper.

Following the notation in [1], we set

$$\begin{aligned} \mathbf{Z}(X) &= \mathbf{Z}[C(X)] \\ &= \{\mathbf{Z}(f) \mid f \in C(X)\} \end{aligned} \quad (1)$$

to be the collection of the zero sets  $\mathbf{Z}(f) = f^{-1}(\{0\})$  of all functions  $f \in C(X)$ . In this paper, we generally work with functions  $f$  on a fixed set  $X$ , as well as some extensions  $g$  of  $f$  to larger domains. As expected,  $\mathbf{Z}(g)$  then denotes the zero set of  $g$  on the larger domain.

A  $z$ -filter ( $z$ -ultrafilter, resp.) on  $X$  is the intersection of  $\mathbf{Z}(X)$  with a filter (ultrafilter, resp.) on  $X$ . The kernel  $k$  of a set  $\mathcal{S}$  of  $z$ -ultrafilters is defined by

$$k(\mathcal{S}) = \bigcap \{\mathcal{U} \mid \mathcal{U} \in \mathcal{S}\}. \quad (2)$$

One can verify that the kernel of a set of  $z$ -ultrafilters is a  $z$ -filter. The hull  $h$  of a  $z$ -filter  $\mathcal{F}$  is defined by

$$h(\mathcal{F}) = \{\mathcal{U} \mid \mathcal{U} \text{ is a } z\text{-ultrafilter on } X, \mathcal{U} \supseteq \mathcal{F}\}. \quad (3)$$

Given a set  $E \subseteq X$ , let  $\langle E \rangle$  denote the set of all zero sets  $F \subseteq X$ , such that  $E \subseteq F$ .

Given any two functions  $f$  and  $g$  on a set  $X$  and  $H \subseteq X$ , we write  $f \equiv g$  on  $H$  if  $f(x) = g(x)$  for all  $x \in H$ . For each intermediate ring  $A(X)$  of continuous functions, each noninvertible  $f \in A(X)$ , and each  $H \subseteq X$ , we say that  $f$  is  $H$ -regular in  $A(X)$  if there exists  $g \in A(X)$  such that  $fg \equiv 1$  on  $H$ . We just say  $f$  is  $H$ -regular if  $A(X)$  is understood by context. For any set  $E \subseteq X$ , let  $E^c$  be the complement of  $E$  in  $X$ .

For each  $f \in A(X)$ , set

$$\begin{aligned} \mathcal{Z}_A(f) &= \{E \in \mathbf{Z}(X) \mid f \text{ is } E^c\text{-regular}\}, \\ \mathfrak{Z}_A(f) &= \{E \in \mathbf{Z}(X) \mid f \text{ is } H\text{-regular for each} \\ &\quad \text{zero-set } H \subseteq E^c\}. \end{aligned} \tag{4}$$

For an ideal  $I \subset A(X)$ , we set  $\mathcal{Z}_A[I] = \bigcup\{\mathcal{Z}_A(f) \mid f \in I\}$  and  $\mathfrak{Z}_A[I] = \bigcup\{\mathfrak{Z}_A(f) \mid f \in I\}$ . Several properties of  $\mathcal{Z}_A$  and  $\mathfrak{Z}_A$  are proved in [3, 5, 6]. In particular, we have the following lemma, which we state here for convenience.

**Lemma 1.** *Let  $A(X)$  be an intermediate ring. Then the following hold.*

- (a) *For any noninvertible  $f \in A(X)$ , both  $\mathcal{Z}_A(f)$  and  $\mathfrak{Z}_A(f)$  are  $z$ -filters on  $X$ . If  $f$  is invertible, then  $\mathcal{Z}_A(f) = \mathfrak{Z}_A(f) = \mathbf{Z}[X]$ , the set of all zero sets in  $X$ .*
- (b) *For an ideal  $I \subset A(X)$ , both  $\mathcal{Z}_A[I]$  and  $\mathfrak{Z}_A[I]$  are  $z$ -filters on  $X$ .*
- (c) *For any  $f \in A(X)$ , we have  $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$ .*
- (d) *If  $\mathcal{F}$  is a  $z$ -filter on  $X$ , then  $\mathcal{Z}_A(f) \subseteq \mathcal{F}$  if and only if  $\lim_{\mathcal{F}} fh = 0$  for all  $h \in A(X)$ .*

Item (a) is from [2, 6], Item (b) is from [3, 6], Item (c) is from [6], and Item (d) is from [2].

The Stone-Čech compactification of  $X$ , denoted  $\beta X$ , is any topological space homeomorphic to the space of  $z$ -ultrafilters on  $X$  topologized with the hull-kernel closure operator as follows: the closure of any set  $U$  of  $z$ -ultrafilters is  $hk(U)$ . Throughout this paper, we will in particular take  $\beta X$  to consist of a superset of  $X$ , whose points, which we denote  $p, q, \dots$ , can be viewed as indices of  $z$ -ultrafilters on  $X$ . Let  $\mathcal{U}$  be the map which associates every  $p \in \beta X$  with a  $z$ -ultrafilter  $\mathcal{U}_p$ :

$$p \xrightarrow{\mathcal{U}} \mathcal{U}_p, \tag{5}$$

such that for each  $p \in X$ ,  $\mathcal{U}_p = \langle\{p\}\rangle$  is the fixed  $z$ -ultrafilter containing  $p$ , and for each  $p \in \beta X \setminus X$ ,  $\mathcal{U}_p$  is a unique free  $z$ -ultrafilter, such that  $\mathcal{U}$  is a one-to-one correspondence between  $\beta X$  and the set of  $z$ -ultrafilters on  $X$ . The topology on  $\beta X$  is defined in such a way that the map  $\mathcal{U}$  is a homeomorphism. Making use of the fact that the zero sets form a base for the collection of closed sets [1, page 38], one can check that  $\mathcal{U}$  maps  $X$  homeomorphically onto the subspace of fixed  $z$ -ultrafilters, and hence  $X$  is a subspace of  $\beta X$ .

A  $z$ -filter  $\mathcal{F}$  on  $X$  is called  $A$ -stable if for every  $f \in A(X)$  there exists a set in  $\mathcal{F}$  on which  $f$  is bounded (see [7, 8]).

Following [7], for each  $A(X)$  we define the  $A$ -compactification  $v_A X$  of  $X$  as the subspace of  $\beta X$  where

$$v_A X = \{p \in \beta X \mid \mathcal{U}_p \text{ is } A\text{-stable}\}. \tag{6}$$

From [5, Theorem 4.6], it holds that  $v_A X$  is a realcompactification of  $X$ . Note that if  $A(X) = C^*(X)$ , then  $v_A X = \beta X$ . If  $A(X) = C(X)$ , then  $v_A X = vX$  is the Hewitt realcompactification.

To refine our understanding of the topology on  $v_A X$ , we define the  $A$ -stable hull  $h^A$  of a  $z$ -filter  $\mathcal{F}$  by

$$\begin{aligned} h^A(\mathcal{F}) &= \{\mathcal{U} \mid \mathcal{U} \text{ is an } A\text{-stable } z\text{-ultrafilter} \\ &\quad \text{on } X, \mathcal{U} \supseteq \mathcal{F}\}. \end{aligned} \tag{7}$$

It is immediate from the definition of the subspace topology that  $v_A X$  is homeomorphic (via  $\mathcal{U}|_{v_A X}$ ) to the hull-kernel topology restricted to  $A$ -stable  $z$ -ultrafilters, that is, the topology with the following closure operator: the closure of any set  $U$  is  $h^A k(U)$ . It follows that

$$p \in \text{cl}_{v_A X} E \quad \text{iff} \quad E \in \mathcal{U}_p. \tag{8}$$

From [7], we have that the space  $v_A X$  consists of the points of  $\beta X$  to which every function  $f \in A(X)$  can be continuously extended. We denote the extension of  $f$  to  $v_A X$  by  $f^{v_A}$ . From [7, Theorem 9], we have the value of  $f^{v_A}$  at a point  $p \in v_A X$  is given by

$$f^{v_A}(p) = \lim_{\mathcal{U}_p} f. \tag{9}$$

In [7], a ring  $A(X)$  of continuous functions is called a  $C$ -ring if there is a completely regular space  $Y$  such that  $A(X)$  is isomorphic to  $C(Y)$ . Clearly  $C(X)$  and  $C^*(X)$  are  $C$ -rings (with  $C^*(X)$  isomorphic to  $C(\beta X)$ ). We use the following result from [7, Theorem 7].

**Lemma 2.** *Let  $A(X)$  be an intermediate ring. Then the following hold.*

- (a)  *$A(X)$  is a  $C$ -ring if and only if  $A(X)$  is isomorphic to  $C(v_A X)$ .*
- (b) *If  $A(X)$  is a  $C$ -ring, then  $f \in A(X)$  is invertible in  $A(X)$  if and only if  $\mathbf{Z}(f^{v_A}) = \emptyset$ .*

In addition, it is shown in [5, Theorem 4.7] that there is a bijective correspondence between the realcompactifications of  $X$  and the  $C$ -rings on  $X$ .

### 3. Characterizations Using Realcompactifications

In this section, we utilize the realcompactifications of  $X$  to provide a new description of the function  $\mathfrak{Z}_A$  and of maximal ideals of  $A(X)$ , when  $A(X)$  is an intermediate  $C$ -ring. The new description of the maximal ideals of  $A(X)$  generalizes the Gelfand-Kolmogorov theorem [1, page 102] for  $C(X)$  to all intermediate  $C$ -rings.

**3.1. A New Description of  $\mathfrak{Z}_A$  for  $C$ -Rings.** While the definition of  $\mathfrak{Z}_A$  is essentially algebraic (using the property of local invertibility), we now define a function  $\mathfrak{R}_A$  that provides a “topological description” (using realcompactifications of  $X$ ) of a mapping from ideals of an intermediate  $C$ -ring  $A(X)$  to  $z$ -filters on  $X$ , and we will show (Theorem 6) that  $\mathfrak{R}_A$  coincides with  $\mathfrak{Z}_A$  when  $A(X)$  is an intermediate  $C$ -ring.

*Definition 3.* Let  $A(X)$  be an intermediate ring and  $f \in A(X)$ . We set

$$\mathfrak{R}_A(f) = \{E \in \mathbf{Z}(X) \mid \mathbf{Z}(f^{v_A}) \subseteq \text{cl}_{v_A X} E\}. \quad (10)$$

For an ideal  $I \subset A(X)$ , we set  $\mathfrak{R}_A[I] = \bigcup\{\mathfrak{R}_A(f) \mid f \in I\}$ .

That  $\mathfrak{R}_A$  is indeed a mapping from ideals of an intermediate  $C$ -ring  $A(X)$  to  $z$ -filters on  $X$  will follow from our main result of the section that establishes that  $\mathfrak{R}_A = \mathfrak{Z}_A$  when  $A(X)$  is an intermediate  $C$ -ring. We also show that  $\mathfrak{R}_A$  does not necessarily map ideals in  $A(X)$  to  $z$ -filters on  $X$  when  $A(X)$  is not an intermediate  $C$ -ring (Example 7).

First, to illustrate some connections that motivated the development of  $\mathfrak{R}_A$ , let us now observe a similarity between  $\mathfrak{R}_A$  and a simple function  $\widehat{\mathbf{Z}}$  that we define in terms of  $\mathbf{Z}$ . We define  $\widehat{\mathbf{Z}}_X$  on  $C(X)$  by

$$\widehat{\mathbf{Z}}_X : f \mapsto \langle \mathbf{Z}(f) \rangle = \{E \in \mathbf{Z}(X) \mid \mathbf{Z}(f) \subseteq E\}. \quad (11)$$

We drop the subscript  $X$  when it is understood by context. It is easy to see that given an ideal  $I$ ,  $\widehat{\mathbf{Z}}[I] = \mathbf{Z}[I]$ .

We now turn our attention to  $C^*(X)$  and we note that  $C^*(X)$  is an intermediate  $C$ -ring, as  $C^*(X)$  is isomorphic to  $C(\beta X)$ . In light of this, we have for  $f \in C(X)$  that

$$\widehat{\mathbf{Z}}_{\beta X} : f^\beta \mapsto \langle \mathbf{Z}(f^\beta) \rangle = \{E_\beta \in \mathbf{Z}(\beta X) \mid \mathbf{Z}(f^\beta) \subseteq E_\beta\}. \quad (12)$$

Ultimately, we would like to find a map defined on  $C^*(X)$  itself rather than  $C(\beta X)$  and that maps to  $z$ -filters on  $X$  rather than  $\beta X$  (it is possible that a  $z$ -filter on  $\beta X$  may contain a set that does not meet  $X$ ; for example,  $\beta\mathbb{N} \setminus \mathbb{N}$  is the zero set of the function  $f^{\beta\mathbb{N}}$ , where  $f(n) = 1/n$ ). Observe that

$$\begin{aligned} \bigcap \{E_\beta \in \mathbf{Z}(\beta X) \mid \mathbf{Z}(f^\beta) \subseteq E_\beta\} &= \mathbf{Z}(f^\beta) \\ &= \bigcap \{\text{cl}_{\beta X} E \mid E \in \mathbf{Z}(X), \mathbf{Z}(f^\beta) \subseteq \text{cl}_{\beta X} E\}, \end{aligned} \quad (13)$$

where the first equality is immediate from the fact that  $\mathbf{Z}(f^\beta) \in \mathbf{Z}(\beta X)$ , and the second equality holds because the sets of the form  $\text{cl}_{\beta X} E$  are a base for the closed sets in  $\beta X$  (see [1, page 94]). This motivates the following definition that relates  $\mathbf{Z}(f^\beta)$  to  $X$ :

$$\mathfrak{R}_{C^*} : f \mapsto \{E \in \mathbf{Z}(X) \mid \mathbf{Z}(f^\beta) \subseteq \text{cl}_{\beta X} E\}. \quad (14)$$

We then generalize  $\mathfrak{R}_{C^*}$  to all intermediate rings  $A(X)$  to arrive at Definition 3.

In order to prove our main theorem that  $\mathfrak{R}_A = \mathfrak{Z}_A$  whenever  $A$  is an intermediate  $C$ -ring, we need some lemmas. We first show that the zero set of  $f^{v_A}$ , viewed as a set of  $z$ -ultrafilters, is the  $A$ -stable hull of  $\mathcal{Z}_A(f)$ .

**Lemma 4.** If  $A(X)$  is an intermediate ring of continuous functions,

$$\mathbf{Z}(f^{v_A}) \equiv h^A(\mathcal{Z}_A(f)), \quad (15)$$

where the symbol  $\equiv$  indicates that one set is the homeomorphic image of the other. In particular, if  $f \in A(X)$ , then  $p \in \mathbf{Z}(f^{v_A})$  if and only if  $\mathcal{U}_p \in h^A(\mathcal{Z}_A(f))$ .

*Proof.* We observe that the following are equivalent:

- (i)  $p \in \mathbf{Z}(f^{v_A})$ ,
- (ii)  $\mathcal{U}_p$  is  $A$ -stable and  $\lim_{\mathcal{U}_p} f = 0$ ,
- (iii)  $\mathcal{U}_p$  is  $A$ -stable and  $\lim_{\mathcal{U}_p} fh = 0$  for all  $h \in A(X)$ ,
- (iv)  $\mathcal{U}_p$  is  $A$ -stable and  $\mathcal{U}_p \supseteq \mathcal{Z}_A(f)$ , that is,  $\mathcal{U}_p \in h^A(\mathcal{Z}_A(f))$ .

The equivalence (i) $\Leftrightarrow$ (ii) follows from (6) and (9). The equivalence (ii) $\Leftrightarrow$ (iii) follows from the fact that  $\mathcal{U}_p$  is  $A$ -stable, and hence  $h$  is bounded on some set in  $\mathcal{U}_p$ . The equivalence (iii) $\Leftrightarrow$ (iv) follows from Lemma 1(d).  $\square$

We use the notation  $kh\mathcal{F}$  to denote the kernel of the hull of the  $z$ -filter  $\mathcal{F}$ , that is, the intersection of the set of  $z$ -ultrafilters containing  $\mathcal{F}$ .

**Lemma 5.** Let  $A(X)$  be an intermediate  $C$ -ring and  $f \in A(X)$ . Then

$$kh\mathcal{Z}_A(f) = kh^A\mathcal{Z}_A(f). \quad (16)$$

*Proof.* Without loss of generality, we may assume that  $f \geq 0$  (because  $\mathcal{Z}_A(f) = \mathcal{Z}_A(f^2)$ ). Of course  $kh\mathcal{Z}_A(f) \subseteq kh^A\mathcal{Z}_A(f)$ . For the other containment, suppose  $E \in kh^A\mathcal{Z}_A(f)$ , so  $E$  belongs to every  $A$ -stable  $z$ -ultrafilter containing  $\mathcal{Z}_A(f)$ . We first note that by Lemma 4,  $\mathcal{U}_p \in h^A\mathcal{Z}_A(f)$  if and only if  $p \in \mathbf{Z}(f^{v_A})$ . In other words, for any  $E \in \mathbf{Z}[X]$ , we have  $E \in kh^A\mathcal{Z}_A(f)$  if and only if  $E \in \mathcal{U}_p$  for every  $p \in \mathbf{Z}(f^{v_A})$  if and only if  $\text{cl}_{v_A X} E \supseteq \mathbf{Z}(f^{v_A})$ . Now suppose  $\mathcal{U}_q$  is any  $z$ -ultrafilter ( $A$ -stable or not) containing  $\mathcal{Z}_A(f)$ . We show that  $E$  belongs to  $\mathcal{U}_q$ . If not, then there exists  $F \in \mathcal{U}_q$  such that

$$E \cap F = \emptyset. \quad (17)$$

Since  $E$  and  $F$  are disjoint zero sets in  $X$ , they are completely separated (see [1, page 17]). So there is a continuous function  $h$  that takes the value 1 on  $E$  and 0 on  $F$  and  $0 \leq h \leq 1$ , so  $h \in A(X)$ . Consider the function  $k = f + h \in A(X)$ . Note that  $k(x) = f(x)$  for  $x \in F$ . Also, since  $k \geq 1$  on  $E$  and since  $\text{cl}_{v_A X} E \supseteq \mathbf{Z}(f^{v_A})$ , it follows that  $\mathbf{Z}(k^{v_A}) = \emptyset$ . By Lemma 2(b),  $k$  is invertible, and as  $k \equiv f$  on  $F$ , it follows that  $k^{-1}f \equiv 1$  on  $F$ . Since  $F \in \mathcal{U}_q$ , it also follows that  $\lim_{\mathcal{U}_q} k^{-1}f \neq 0$ , but this contradicts the fact that  $\mathcal{U}_q \supseteq \mathcal{Z}_A(f)$  because of Lemma 1(d). This contradiction stems from the assumption that  $E$  does not belong to  $\mathcal{U}_q$ . So  $E \in \mathcal{U}_q$  for every  $\mathcal{U}_q \supseteq \mathcal{Z}_A(f)$ , that is,  $E \in kh\mathcal{Z}_A(f)$ .  $\square$

Recall that for any intermediate ring  $A(X)$ , Lemma 1(c) gives the relationship  $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$ . For intermediate  $C$ -rings, we can characterize  $\mathfrak{Z}_A$  topologically as  $\mathfrak{R}_A$ .

**Theorem 6.** *Let  $A(X)$  be an intermediate  $C$ -ring and  $f \in A(X)$ . Then  $\mathfrak{Z}_A(f) = \mathfrak{R}_A(f)$ .*

*Proof.* Suppose  $E \in \mathfrak{Z}_A(f)$ . By Lemma 1(c),  $E \in kh\mathcal{Z}_A(f)$ . We show that in this case  $p \in \mathbf{Z}(f^{v_A})$  implies  $p \in cl_{v_A X} E$ . Now if  $p \in \mathbf{Z}(f^{v_A})$ , then by Lemma 4,  $\mathcal{U}_p \in h^A(\mathcal{Z}_A(f))$ . Since  $E \in kh\mathcal{Z}_A(f)$ , it follows that  $E \in \mathcal{U}_p$ , and hence by (8), we have that  $p \in cl_{v_A X} E$ .

For the other containment, suppose  $E \in \mathfrak{R}_A(f)$ , that is,  $\mathbf{Z}(f^{v_A}) \subseteq cl_{v_A X} E$ . We show that  $E$  belongs to every  $A$ -stable  $z$ -ultrafilter containing  $\mathcal{Z}_A(f)$ . First, let  $\mathcal{U}_p$  be an  $A$ -stable  $z$ -ultrafilter containing  $\mathcal{Z}_A(f)$ . Then by Lemma 4,  $p \in \mathbf{Z}(f^{v_A})$ , so by hypothesis,  $p \in cl_{v_A X} E$ , and hence  $E \in \mathcal{U}_p$ . Thus  $E$  belongs to every  $A$ -stable  $z$ -ultrafilter containing  $\mathcal{Z}_A(f)$ ; in other words,  $E \in kh^A\mathcal{Z}_A(f)$ . Since  $A(X)$  is an intermediate  $C$ -ring, we can apply Lemma 5 so that  $E \in kh\mathcal{Z}_A(f)$ , and hence by Lemma 1(c),  $E \in \mathfrak{Z}_A(f)$ .  $\square$

The theorem does not hold if the assumption that  $A(X)$  being an intermediate  $C$ -ring is removed. In fact,  $\mathfrak{R}_A$  need not map ideals in  $A(X)$  to  $z$ -filters on  $X$  when  $A(X)$  is not a  $C$ -ring, as the following example shows.

*Example 7.* Let  $X = [1, \infty)$ . Let  $A(X)$  be the smallest ring of continuous functions containing both  $C^*(X)$  and  $f(x) = x$ . Then each function  $h \in A(X)$  has the form  $h(x) = \sum_{k=0}^n f_k(x)x^k$  for  $f_k \in C^*(X)$ . Note that any function  $h \in A(X)$  is therefore bounded by some function of the form  $Mx^n$  for  $M \in \mathbb{R}$  and  $n \in \mathbb{N}$ , where  $M$  is  $n + 1$  times a common bound for all of  $f_k$ . We now observe that  $e^x \notin A(X)$ , since  $e^x$  cannot be bounded by any function of the form  $Mx^n$  for  $M \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Let  $g = e^{-x}$ . Then  $g$  is in  $A(X)$  but is not invertible in  $A(X)$ . Thus  $A(X)$  and  $C(X)$  are not isomorphic.

Since every set of a free  $z$ -ultrafilter on  $X = [1, \infty)$  must be unbounded, the identity function  $f$  is not bounded on any such set either. Hence any free  $z$ -ultrafilter on  $X$  is not  $A$ -stable, and  $v_A(X) = X$ . Then by Lemma 2(a),  $A(X)$  is not a  $C$ -ring, since  $C(v_A(X)) = C(X)$  and  $C(X) \neq A(X)$ .

Finally, since  $g^{v_A} = g$  and  $\mathbf{Z}(g) = \emptyset$ , the set  $\mathfrak{R}_A(g) = \{E \in \mathbf{Z}(X) \mid \mathbf{Z}(g^{v_A}) \subseteq cl_{v_A X} E\}$  consists of all zero sets of  $X$  (hence is not a  $z$ -filter), while  $\mathfrak{Z}_A(g)$  is the  $z$ -filter consisting of all zero sets in  $X$  whose complement in  $X$  has an upper bound. Thus  $\mathfrak{R}_A(g) \neq \mathfrak{Z}_A(g)$ , which is in contrast to the conclusion of Theorem 6.

We leave open the question as to precisely what rings  $A(X)$  are such that  $\mathfrak{R}_A = \mathfrak{Z}_A$ . We also leave open the question as to whether there exists a ring  $A(X)$  such that  $\mathfrak{R}_A$  does map ideals in  $A(X)$  to  $z$ -filters on  $X$ , but  $\mathfrak{R}_A \neq \mathfrak{Z}_A$ .

**3.2. Characterizing Maximal Ideals in  $C$ -Rings.** The following characterization for maximal ideals in  $A(X)$  is proved in [6]. Every maximal ideal in  $A(X)$  is of the form

$$M_A^p = \{f \in A(X) \mid \mathfrak{Z}_A(f) \subseteq \mathcal{U}_p\}, \tag{18}$$

for  $p \in \beta X$ . By Lemma 1(c),  $\mathfrak{Z}_A(f) \subseteq \mathcal{U}_p$  whenever  $\mathcal{Z}_A(f) \subseteq \mathcal{U}_p$ , and hence by (9) and Lemma 1(d), we have

$$M_{C^*}^p = \{f \in C^*(X) \mid f^\beta(p) = 0\}. \tag{19}$$

Since  $\mathfrak{Z}_C(f) = \langle \mathbf{Z}(f) \rangle$ , we have by (8) that

$$\begin{aligned} M_C^p &= \{f \in C(X) \mid \langle \mathbf{Z}(f) \rangle \subseteq \mathcal{U}_p\} \\ &= \{f \in C(X) \mid p \in cl_{\beta X} \mathbf{Z}(f)\}. \end{aligned} \tag{20}$$

The characterizations in (19) and (20) that we obtained from (18) agree with those given in [1, pages 101-102]; the latter characterization of  $M_C^p$  is called the Gelfand-Kolmogorov theorem [1, page 102]. The following theorem provides a characterization of  $M_A^p$  that we see extends both (20) and (19) to all intermediate  $C$ -rings  $A(X)$ .

**Theorem 8.** *Let  $A(X)$  be an intermediate  $C$ -ring. Then each maximal ideal in  $A(X)$  is of the form*

$$M_A^p = \{f \in A(X) : p \in cl_{\beta X} \mathbf{Z}(f^{v_A})\}, \tag{21}$$

where  $p \in \beta X$ .

*Proof.*  $f \in M_A^p$  if and only if  $\mathcal{Z}_A(f) \subseteq \mathcal{U}_p$  if and only if  $\mathcal{U}_p \in h(\mathcal{Z}_A(f))$ . By identifying a set in  $\beta X$  with its image under  $\mathcal{U}$ , it follows from the definition of closure in  $\beta X$ , Lemmas 4 and 5, that

$$\begin{aligned} cl_{\beta X} \mathbf{Z}(f^{v_A}) &= hk\mathbf{Z}(f^{v_A}) = hkh^A\mathcal{Z}_A(f) \\ &= hkh\mathcal{Z}_A(f) = h\mathcal{Z}_A(f). \end{aligned} \tag{22}$$

The result now follows from the fact that  $h(\mathcal{Z}_A(f)) = cl_{\beta X} \mathbf{Z}(f^{v_A})$ .  $\square$

We now verify that this theorem generalizes the Gelfand-Kolmogorov Theorem [1, page 102] to intermediate  $C$ -rings. If  $A(X) = C(X)$ , then  $v_A X = vX$ , the Hewitt realcompactification of  $X$ . Now using the fact that  $cl_{vX} \mathbf{Z}(f) = \mathbf{Z}(f^v)$  [1, page 118] and Theorem 8, we have

$$\begin{aligned} M_C^p &= \{f \in C(X) \mid p \in cl_{\beta X} \mathbf{Z}(f^v)\} \\ &= \{f \in C(X) \mid p \in cl_{\beta X} cl_{vX} \mathbf{Z}(f)\} \\ &= \{f \in C(X) \mid p \in cl_{\beta X} \mathbf{Z}(f)\}. \end{aligned} \tag{23}$$

For the case where  $A(X) = C^*(X)$ , we have

$$\begin{aligned} M_{C^*}^p &= \{f \in C^*(X) \mid p \in cl_{\beta X} \mathbf{Z}(f^\beta)\} \\ &= \{f \in C^*(X) \mid f^\beta(p) = 0\}. \end{aligned} \tag{24}$$

So Theorem 8 simultaneously generalizes the results of (20) for  $C(X)$  and (19) for  $C^*(X)$  to all intermediate  $C$ -rings.

#### 4. The Map $\mathfrak{Z}_A^{\leftarrow}$ and Ideals in $A(X)$

Recall that  $\mathfrak{Z}_A$  maps ideals to  $z$ -filters (Lemma 1(a)). Here we show that for an intermediate  $C$ -ring  $A(X)$ , the inverse map  $\mathfrak{Z}_A^{\leftarrow}$ , defined by

$$\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}] = \{f \in A(X) : \mathfrak{Z}_A(f) \subseteq \mathcal{F}\} \tag{25}$$

maps  $z$ -filters on  $X$  to ideals in  $A(X)$  (Theorem 14). The corresponding result for the maps  $\mathbf{Z}$  (for  $C(X)$ ) and  $\mathcal{Z}_A$  (for any intermediate ring  $A(X)$ ) is proved, respectively, in [1] and [3]. Our proof for  $\mathfrak{Z}_A$  makes use of Theorem 6.

We need some lemmas concerning meets and joins on the lattice of  $z$ -filters. Recall that  $\mathcal{F} \vee \mathcal{G}$  is the smallest  $z$ -filter containing both the  $z$ -filters  $\mathcal{F}$  and  $\mathcal{G}$ . Similarly,  $\mathcal{F} \wedge \mathcal{G}$  is the largest  $z$ -filter contained in both the  $z$ -filters  $\mathcal{F}$  and  $\mathcal{G}$ .

The following lemma is from [6].

**Lemma 9.** *Let  $A(X)$  be an intermediate ring and  $f, g \in A(X)$ .*

- (a)  $\mathcal{Z}_A(fg) = \mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)$ .
- (b)  $\mathcal{Z}_A(f + g) \subseteq \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$ .
- (c) *If  $f, g \geq 0$ , then  $\mathcal{Z}_A(f + g) = \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$ .*

The following is a special case of [6, Lemma 4.2(a)].

**Lemma 10.** *For all  $f, g \in A(X)$ ,  $kh(\mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)) = kh(\mathcal{Z}_A(f)) \wedge kh(\mathcal{Z}_A(g))$ .*

To obtain the analog of Lemma 10 for joins, we need the following lemma.

**Lemma 11.** *If  $E$  is a zero set in  $X$  and if  $\mathbf{Z}(f^{v_A}) \cap \mathbf{Z}(g^{v_A}) \subseteq cl_{v_A X} E$  then there exist zero sets  $Z_1$  and  $Z_2$  in  $X$  such that  $\mathbf{Z}(f^{v_A}) \subseteq cl_{v_A X} Z_1$ ,  $\mathbf{Z}(g^{v_A}) \subseteq cl_{v_A X} Z_2$ , and  $Z_1 \cap Z_2 = E$ .*

*Proof.* Let  $H = cl_{v_A X} E$  and let  $Y = v_A X \setminus H$ . Now, the sets  $\mathbf{Z}(f^{v_A}) \cap Y$  and  $\mathbf{Z}(g^{v_A}) \cap Y$  are disjoint zero sets in  $Y$ , so are contained in disjoint zero set neighborhoods  $W_1$  and  $W_2$  in  $Y$ . Moreover, since  $W_1$  is a neighborhood of  $\mathbf{Z}(f^{v_A}) \cap Y$ , it follows that

$$\mathbf{Z}(f^{v_A}) \cap Y \subseteq cl_{v_A X}(W_1) = cl_{v_A X}(W_1 \cap X). \tag{26}$$

Similarly  $\mathbf{Z}(g^{v_A}) \cap Y \subseteq cl_{v_A X}(W_2 \cap X)$ . Let

$$Z_1 = (W_1 \cap X) \cup E, \quad Z_2 = (W_2 \cap X) \cup E. \tag{27}$$

Since  $W_1 \cap X$  is a zero set in  $X \setminus E$ , it follows that  $Z_1$  is a zero set in  $X$  by the fact that the cozero set of a cozero set is a cozero set [9, Proposition 1.1]. Similarly,  $Z_2$  is a zero set in  $X$ . Also,  $\mathbf{Z}(f^{v_A}) \subseteq cl_{v_A X} Z_1$ , because

$$\begin{aligned} \mathbf{Z}(f^{v_A}) &= (\mathbf{Z}(f^{v_A}) \cap Y) \cup (\mathbf{Z}(f^{v_A}) \cap H) \\ &\subseteq cl_{v_A X}(W_1 \cap X) \cup cl_{v_A X} E \\ &= cl_{v_A X}((W_1 \cap X) \cup E) \\ &= cl_{v_A X} Z_1. \end{aligned} \tag{28}$$

Similarly,  $\mathbf{Z}(g^{v_A}) \subseteq cl_{v_A X} Z_2$ . Finally,  $Z_1 \cap Z_2 = E$  because  $W_1 \cap X$  and  $W_2 \cap X$  are disjoint.  $\square$

The next lemma shows that the kernel-hull operation distributes over the join operation on the lattice of  $z$ -filters.

**Theorem 12.** *If  $A(X)$  is an intermediate  $C$ -ring, then*

$$kh(\mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)) = kh(\mathcal{Z}_A(f)) \vee kh(\mathcal{Z}_A(g)). \tag{29}$$

*Proof.* The containment  $kh(\mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)) \supseteq kh(\mathcal{Z}_A(f)) \vee kh(\mathcal{Z}_A(g))$  is a special case of [6, Lemma 4.2(b)].

We show that the other containment is equivalent to Lemma 11 and hence must hold. First, we show the equivalence of the premises by showing that the following are equivalent. Recall that  $\mathcal{Z}_A(f) = \mathcal{Z}_A(f^2)$ , and hence we can assume without loss of generality that  $f, g \geq 0$ .

- (i)  $E \in kh(\mathcal{Z}_A(f) \vee \mathcal{Z}_A(g))$ .
- (ii)  $E \in kh(\mathcal{Z}_A(f + g))$ .
- (iii)  $\mathbf{Z}((f + g)^{v_A}) \subseteq cl_{v_A X} E$ .
- (iv)  $\mathbf{Z}(f^{v_A}) \cap \mathbf{Z}(g^{v_A}) \subseteq cl_{v_A X} E$ .

The equivalence (i) $\Leftrightarrow$ (ii) follows from Lemma 9(c). The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 6. The equivalence (iii) $\Leftrightarrow$ (iv) follows from the assumption that  $f, g \geq 0$ . This establishes the equivalence between the premises of left-to-right containment of this lemma and Lemma 11.

For the equivalence of the conclusions, note that  $E \in kh(\mathcal{Z}_A(f)) \vee kh(\mathcal{Z}_A(g))$  if and only if  $E$  is an intersection of a set in  $kh(\mathcal{Z}_A(f))$  with a set in  $kh(\mathcal{Z}_A(g))$ . In other words, there must exist  $Z_1 \in kh(\mathcal{Z}_A(f))$  and  $Z_2 \in kh(\mathcal{Z}_A(g))$  such that  $Z_1 \cap Z_2 = E$ . By Lemmas 4 and 5, this is equivalent to the statement that there exist zero sets  $Z_1, Z_2$  such that  $\mathbf{Z}(f^{v_A}) \subseteq cl_{v_A X} Z_1$ ,  $\mathbf{Z}(g^{v_A}) \subseteq cl_{v_A X} Z_2$ , and  $Z_1 \cap Z_2 = E$ , which is the conclusion of Lemma 11.  $\square$

**Corollary 13.** *Let  $A(X)$  be an intermediate ring of continuous functions and  $f, g \in A(X)$ . Then*

- (a)  $\mathfrak{Z}_A(fg) = \mathfrak{Z}_A(f) \wedge \mathfrak{Z}_A(g)$ ,
- (b)  $\mathfrak{Z}_A(f + g) \subseteq \mathfrak{Z}_A(f) \vee \mathfrak{Z}_A(g)$ .

*Proof.* Item (a) immediately follows from Lemmas 10 and 9(a).

Item (b) immediately follows from Theorem 12 and Lemma 9(b).  $\square$

We are now ready to prove the main result of this section.

**Theorem 14.** *Let  $A(X)$  be an intermediate  $C$ -ring. If  $\mathcal{F}$  is a  $z$ -filter on  $X$ , then  $\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]$  is an ideal in  $A(X)$ .*

*Proof.* Let  $\mathcal{F}$  be a  $z$ -filter and  $I = \mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]$ . If  $f \in I$  and  $g \in A(X)$ , then  $\mathfrak{Z}_A(f) \subseteq \mathcal{F}$ . By Corollary 13(a),  $\mathfrak{Z}_A(fg) = \mathfrak{Z}_A(f) \wedge \mathfrak{Z}_A(g) \subseteq \mathcal{F}$ , so  $fg \in I$ . If  $f, g \in I$ , then both  $\mathfrak{Z}_A(f) \subseteq \mathcal{F}$  and  $\mathfrak{Z}_A(g) \subseteq \mathcal{F}$ . By Corollary 13(b),  $\mathfrak{Z}_A(f + g) \subseteq \mathfrak{Z}_A(f) \vee \mathfrak{Z}_A(g) \subseteq \mathcal{F}$ , so  $f + g \in I$ . Thus  $I$  is an ideal.  $\square$

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## References

- [1] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer, New York, NY, USA, 1976.
- [2] L. Redlin and S. Watson, “Maximal ideals in subalgebras of  $C(X)$ ,” *Proceedings of the American Mathematical Society*, vol. 100, no. 4, pp. 763–766, 1987.
- [3] H. L. Byun and S. Watson, “Prime and maximal ideals in sub-rings of  $C(X)$ ,” *Topology and its Applications*, vol. 40, no. 1, pp. 45–62, 1991.
- [4] L. H. Byun, L. Redlin, and S. Watson, “Local bounded inversion in rings of continuous functions,” *Commentationes Mathematicae*, vol. 37, pp. 55–63, 1997.
- [5] L. Redlin and S. Watson, “Structure spaces for rings of continuous functions with applications to realcompactifications,” *Fundamenta Mathematicae*, vol. 152, no. 2, pp. 151–163, 1997.
- [6] P. Panman, J. Sack, and S. Watson, “Correspondences between ideals and  $z$ -filters for rings of continuous functions between  $C^*$  and  $C$ ,” *Commentationes Mathematicae*, vol. 52, no. 1, pp. 11–20, 2012.
- [7] H. L. Byun, L. Redlin, S. Watson, and “ $A$ -compactifications and rings of continuous functions between  $C^*$  and  $C$ ,” *Topological Vector Spaces, Algebras, and Related Areas*, vol. 316, pp. 130–139, 1995.
- [8] M. Mandelker, “Supports of continuous functions,” *Transactions of the American Mathematical Society*, vol. 156, pp. 73–83, 1971.
- [9] R. L. Blair and A. W. Hager, “Extensions of zero-sets and of real-valued functions,” *Mathematische Zeitschrift*, vol. 136, pp. 41–52, 1974.



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