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Research Article

Convergence Theorems for Fixed Points of Multivalued Strictly Pseudocontractive Mappings in Hilbert Spaces

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Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Suppose that $T : K \to 2^K$ is a multivalued strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. A Krasnoselskii-type iteration sequence $\{x_n\}$ is constructed and shown to be an approximate fixed point sequence of T; that is, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ holds. Convergence theorems are also proved under appropriate additional conditions.

1. Introduction

For several years, the study of fixed point theory of *multivalued nonlinear mappings* has attracted, and continues to attract, the interest of several well-known mathematicians (see, e.g., Brouwer [1], Kakutani [2], Nash [3, 4], Geanakoplos [5], Nadler [6], and Downing and Kirk [7]).

Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in *Game Theory and Market Economy,* and in other areas of mathematics, such as in *Nonsmooth Differential Equations*. We describe briefly the connection of fixed point theory of multivalued mappings and these applications.

Game Theory and Market Economy. In game theory and market economy, the existence of equilibrium was uniformly obtained by the application of a fixed point theorem. In fact, Nash [3, 4] showed the existence of equilibria for noncooperative static games as a direct consequence of Brouwer [1] or Kakutani [2] fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multivalued mapping* whose fixed points coincide with the equilibrium points of the game. A model example of such an application is the *Nash equilibrium theorem* (see, e.g., [3]).

Consider a game $G = (u_n, K_n)$ with N players denoted by n, $n = 1, \ldots, N$, where $K_n \subset \mathbb{R}^{m_n}$ is the set of possible strategies of the nth player and is assumed to be nonempty, compact, and convex, and $u_n : K := K_1 \times K_2 \cdots \times K_N \to \mathbb{R}$ is the payoff (or gain function) of the player n and is assumed to be continuous. The player n can take *individual actions*, represented by a vector $\sigma_n \in K_n$. All players together can take a *collective action*, which is a combined vector $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$. For each n, $\sigma \in K$ and $z_n \in K_n$, we will use the following standard notations:

$$K_{-n} := K_1 \times \cdots \times K_{n-1} \times K_{n+1} \times \cdots \times K_N,$$
$$\sigma_{-n} := (\sigma_1, \ldots, \sigma_{n-1}, \sigma_{n+1}, \ldots, \sigma_N), \tag{1}$$
$$(z_n, \sigma_{-n}) := (\sigma_1, \ldots, \sigma_{n-1}, z_n, \sigma_{n+1}, \ldots, \sigma_N).$$

A strategy $\overline{\sigma}_n \in K_n$ permits the n'th player to maximize his gain *under the condition* that the *remaining players* have chosen their strategies σ_{-n} if and only if

$$u_n(\overline{\sigma}_n, \sigma_{-n}) = \max_{z_n \in K_n} u_n(z_n, \sigma_{-n}). \tag{2}$$

Now, let $T_n : K_{-n} \rightarrow 2^{K_n}$ be the multivalued mapping defined by

$$T_n(\sigma_{-n}) := \operatorname*{Arg\,max}_{z_n \in K_n} u_n(z_n, \sigma_{-n}) \quad \forall \sigma_{-n} \in K_{-n}. \tag{3}$$

Definition 1. A collective action $\overline{\sigma} = (\overline{\sigma}_1, \ldots, \overline{\sigma}_N) \in K$ is called a *Nash equilibrium point* if, for each n, $\overline{\sigma}_n$ is the best response for the n'th player to the action $\overline{\sigma}_{-n}$ made by the remaining players. That is, for each n,

$$u_n(\overline{\sigma}) = \max_{z_n \in K_n} u_n(z_n, \overline{\sigma}_{-n}) \tag{4}$$

or, equivalently,

$$\overline{\sigma}_n \in T_n(\overline{\sigma}_{-n}). \tag{5}$$

This is equivalent to that $\overline{\sigma}$ is a fixed point of the multivalued mapping $T : K \rightarrow 2^K$ defined by

$$T(\sigma) := T_1(\sigma_{-1}) \times T_2(\sigma_{-2}) \times \cdots \times T_N(\sigma_{-N}). \tag{6}$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of fixed point theory of multivalued mappings. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a nonequilibrium point and convergent to equilibrium solution. This is part of the problem that is being addressed by iterative methods for fixed point of multivalued mappings.

Nonsmooth Differential Equations. The mainstream of applications of fixed point theory for multivalued mappings has been initially motivated by the problem of differential equations (DEs) with discontinuous right-hand sides which gave birth to the existence theory of differential inclusion (DIs). Here is a simple model for this type of application.

Consider the initial value problem

$$\frac{du}{dt} = f(t, u), \quad \text{a.e. } t \in I := [-a, a], \ u(0) = u_0. \tag{7}$$

If $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is discontinuous with bounded jumps, measurable in t, one looks for *solutions* in the sense of Filippov [8, 9] which are solutions of the differential inclusion

$$\frac{du}{dt} \in F(t, u), \quad \text{a.e. } t \in I, \ u(0) = u_0, \tag{8}$$

where

$$F(t, x) = \left[\liminf_{y \to x} f(t, y), \limsup_{y \to x} f(t, y)\right]. \tag{9}$$

Now set $H := L^2(I)$ and let $N_F : H \rightarrow 2^H$ be the *multivalued NemyTskii operator* defined by

$$N_F(u) := \{v \in H : v(t) \in F(t, u(t)) \ \text{a.e. } t \in I\}. \tag{10}$$

Finally, let $T : H \rightarrow 2^H$ be the multivalued mapping defined by $T := N_F o L^{-1}$, where L^{-1} is the inverse of the derivative operator $Lu = u'$ given by

$$L^{-1} v(t) := u_0 + \int_0^t v(s)\, ds. \tag{11}$$

One can see that problem (8) reduces to the fixed point problem: $u \in Tu$.

Finally, a variety of fixed point theorems for multivalued mappings with nonempty and convex values is available to conclude the existence of solution. We used a first-order differential equation as a model for simplicity of presentation, but this approach is most commonly used with respect to second-order boundary value problems for ordinary differential equations or partial differential equations. For more about these topics, one can consult [10–13] and references therein as examples.

We have seen that a *Nash equilibrium point* is a fixed point $\overline{\sigma}$ of a *multivalued mapping* $T : K \rightarrow 2^K$, that is, a solution of the inclusion $x \in Tx$ for some nonlinear mapping T. This inclusion can be rewritten as $0 \in Ax$, where $A := I - T$ and I is the identity mapping on K.

Many problems in applications can be modeled in the form $0 \in Ax$, where, for example, $A : H \rightarrow 2^H$ is a *monotone* operator, that is, $\langle u - v, x - y \rangle \geq 0$ for all $u \in Ax, v \in Ay$, $x, y \in H$. Typical examples include the equilibrium state of *evolution equations* and critical points of some functionals defined on Hilbert spaces H. Let $f : H \rightarrow (-\infty, +\infty]$ be a proper, lower-semicontinuous convex function; then it is known (see, e.g., Rockafellar [14] or Minty [15]) that the multivalued mapping $T := \partial f$, the *subdifferential* of f, is *maximal monotone*, where for $w \in H$,

$$\begin{aligned} w \in \partial f(x) &\Longleftrightarrow f(y) - f(x) \geq \langle y - x, w \rangle \quad \forall y \in H \\ &\Longleftrightarrow x \in \operatorname{Arg\,min}(f - \langle \cdot, w \rangle). \end{aligned} \tag{12}$$

In this case, the solutions of the inclusion $0 \in \partial f(x)$, if any, correspond to the critical points of f, which are exactly its minimizer points.

Also, the *proximal point algorithm* of Martinet [16] and Rockafellar [17] studied also by a host of authors is connected with iterative algorithm for approximating a solution of $0 \in Ax$ where A is a maximal monotone operator on a Hilbert space.

In studying the equation $Au = 0$, Browder introduced an operator T defined by $T : I - A$ where I is the identity mapping on H. He called such an operator T *pseudocontractive*. It is clear that solutions of $Au = 0$ now correspond to fixed points of T. In general, pseudocontraactive mappings are not continuous. However, in studying fixed point theory for pseudocontractive mappings, some continuity condition (e.g., Lipschitz condition) is imposed on the operator. An important subclass of the class of Lipschitz pseudocontractive mappings is the class of *nonexpansive* mappings, that is, mappings $T : K \rightarrow K$ such that $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Apart from being an obvious generalization of the contraction mappings, nonexpansive mappings are important, as has been observed by Bruck [18], mainly for the following two reasons.

(i) Nonexpansive mappings are intimately connected with the monotonicity methods developed since the early 1960s and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.

(ii) Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in (du/dt) + T(t)u$, where the operators $\{T(t)\}$ are, in general, set-valued and are *accretive* or *dissipative* and *minimally continuous*.

The class of strictly pseudocontractive mappings defined in Hilbert spaces which was introduced in 1967 by Browder and Petryshyn [19] is a superclass of the class of nonexpansive mappings and a subclass of the class of Lipschitz pseudocontractions. While pseudocontractive mappings are generally not continuous, the strictly pseudocontractive mappings inherit *Lipschitz property* from their definitions. The study of fixed point theory for strictly pseudocontractive mappings may help in the study of fixed point theory for nonexpansive mappings and for Lipschitz pseudocontractive mappings. Consequently, the study by several authors of iterative methods for fixed point of *multivalued nonexpansive mappings* has motivated the study in this paper of iterative methods for approximating fixed points of the more general strictly pseudocontractive mappings. Part of the novelty of this paper is that, even in the special case of multivalued nonexpansive mappings, convergence theorems are proved here for the *Krasnoselskii-type sequence* which is known to be superior to the Mann-type and Ishikawa-type sequences so far studied. It is worth mentioning here that iterative methods for approximating fixed points of nonexpansive mappings constitute the *central tools* used in *signal processing* and *image restoration* (see, e.g., Byrne [20]).

Let K be a nonempty subset of a normed space E. The set K is called *proximinal* (see, e.g., [21–23]) if for each $x \in E$ there exists $u \in K$ such that

$$d(x,u) = \inf\{\|x-y\| : y \in K\} = d(x,K), \tag{13}$$

where $d(x,y) = \|x-y\|$ for all $x, y \in E$. Every nonempty, closed, and convex subset of a real Hilbert space is proximinal. Let $CB(K)$ and $P(K)$ denote the families of nonempty, closed, and bounded subsets and of nonempty, proximinal, and bounded subsets of K, respectively. The *Hausdorff metric* on $CB(K)$ is defined by

$$D(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \tag{14}$$

for all $A, B \in CB(K)$. Let $T : D(T) \subseteq E \to CB(E)$ be a *multivalued mapping* on E. A point $x \in D(T)$ is called a *fixed point of* T if $x \in Tx$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

A multivalued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$D(Tx,Ty) \le L\|x-y\| \quad \forall x, y \in D(T). \tag{15}$$

When $L \in (0,1)$ in (15), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

Several papers deal with the problem of approximating fixed points of *multivalued nonexpansive* mappings (see, e.g., [21–26] and the references therein) and their generalizations (see, e.g., [27, 28]).

Sastry and Babu [21] introduced the following iterative schemes. Let $T : E \to P(E)$ be a multivalued mapping, and let x^* be a fixed point of T. Define iteratively the sequence $\{x_n\}_{n\in\mathbb{N}}$ from $x_0 \in E$ by

$$\begin{gathered} x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \quad y_n \in Tx_n, \\ \|y_n - x^*\| = d(Tx_n, x^*), \end{gathered} \tag{16}$$

where α_n is a real sequence in (0,1) satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\lim \alpha_n = 0$.

They also introduced the following scheme:

$$\begin{gathered} y_n = (1-\beta_n)x_n + \beta_n z_n, \quad z_n \in Tx_n, \\ \|z_n - x^*\| = d(x^*, Tx_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \quad u_n \in Ty_n, \\ \|u_n - x^*\| = d(Ty_n, x^*), \end{gathered} \tag{17}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $0 \le \alpha_n, \beta_n < 1$,

(ii) $\lim_{n\to\infty}\beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.

Sastry and Babu called a process defined by (16) a Mann iteration process and a process defined by (17) where the iteration parameters α_n and β_n satisfy conditions (i), (ii), and (iii) an Ishikawa iteration process. They proved in [21] that the Mann and Ishikawa iteration schemes for a multivalued mapping T with fixed point p converge to a fixed point of T under certain conditions. More precisely, they proved the following result for a multivalued nonexpansive mapping with compact domain.

Theorem SB (Sastry and Babu [21]). *Let H be real Hilbert space, let K be a nonempty, compact, and convex subset of H, and let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with a fixed point p. Assume that (i) $0 \le \alpha_n, \beta_n < 1$, (ii) $\beta_n \to 0$, and (iii) $\sum \alpha_n\beta_n = \infty$. Then, the sequence $\{x_n\}$ defined by* (17) *converges strongly to a fixed point of T.*

Panyanak [22] extended the above result of Sastry and Babu [21] to uniformly convex real Banach spaces. He proved the following result.

Theorem P1 (Panyanak [22]). *Let E be a uniformly convex real Banach space, and let K be a nonempty, compact, and convex subset of E and $T : K \to CB(K)$ a multivalued nonexpansive mapping with a fixed point p. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$, (ii) $\beta_n \to 0$, and (iii) $\sum \alpha_n \beta_n = \infty$. Then, the sequence $\{x_n\}$ defined by* (17) *converges strongly to a fixed point of T.*

Panyanak [22] also modified the iteration schemes of Sastry and Babu [21]. Let K be a nonempty, closed, and convex subset of a real Banach space, and let $T : K \to P(K)$ be a multivalued mapping such that $F(T)$ is a nonempty proximinal subset of K.

The sequence of Mann iterates is defined by $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n y_n, \quad \alpha_n \in [a, b], \ 0 < a < b < 1, \tag{18}$$

where $y_n \in Tx_n$ is such that $\|y_n - u_n\| = d(u_n, Tx_n)$ and $u_n \in F(T)$ is such that $\|x_n - u_n\| = d(x_n, F(T))$.

The sequence of Ishikawa iterates is defined by $x_0 \in K$,

$$y_n = (1 - \beta_n) x_n + \beta_n z_n, \quad \beta_n \in [a, b], \ 0 < a < b < 1, \tag{19}$$

where $z_n \in Tx_n$ is such that $\|z_n - u_n\| = d(u_n, Tx_n)$ and $u_n \in F(T)$ is such that $\|x_n - u_n\| = d(x_n, F(T))$. The sequence is defined iteratively by the following way

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n z'_n, \quad \alpha_n \in [a, b], \ 0 < a < b < 1, \tag{20}$$

where $z'_n \in Ty_n$ is such that $\|z'_n - v_n\| = d(v_n, Ty_n)$ and $v_n \in F(T)$ is such that $\|y_n - v_n\| = d(y_n, F(T))$. Before we state his theorem, we need the following definition.

Definition 2. A mapping $T : K \to CB(K)$ is said to satisfy *condition (I)* if there exists a strictly increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, T(x)) \geq f(d(x, F(T)) \quad \forall x \in D. \tag{21}$$

Theorem P2 (Panyanak [22]). *Let E be a uniformly convex real Banach space, let K be a nonempty, closed, bounded, and convex subset of E, and let $T : K \to P(K)$ be a multivalued nonexpansive mapping that satisfies condition (I). Assume that (i) $0 \leq \alpha_n < 1$ and (ii) $\sum \alpha_n = \infty$. Suppose that $F(T)$ is a nonempty proximinal subset of K. Then, the sequence $\{x_n\}$ defined by* (18) *converges strongly to a fixed point of T.*

Panyanak [22] then asked the following question.

Question (P). Is Theorem P2 true for the Ishikawa iteration defined by (19) and (20)?

For multivalued mappings, the following lemma is a consequence of the definition of Hausdorff metric, as remarked by Nadler [6].

Lemma 3. *Let $A, B \in CB(X)$ and $a \in A$. For every $\gamma > 0$, there exists $b \in B$ such that*

$$d(a, b) \leq D(A, B) + \gamma. \tag{22}$$

Recently, Song and Wang [23] modified the iteration process due to Panyanak [22] and improved the results therein. They gave their iteration scheme as follows.

Let K be a nonempty, closed, and convex subset of a real Banach space, and let $T : K \to CB(K)$ be a multivalued mapping. Let $\alpha_n, \beta_n \in [0, 1]$ and $\gamma_n \in (0, \infty)$ be such that $\lim_{n \to \infty} \gamma_n = 0$. Choose $x_0 \in K$,

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n z_n, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n u_n, \end{aligned} \tag{23}$$

where $z_n \in Tx_n$ and $u_n \in Ty_n$ are such that

$$\begin{aligned} \|z_n - u_n\| &\leq D(Tx_n, Ty_n) + \gamma_n, \\ \|z_{n+1} - u_n\| &\leq D(Tx_{n+1}, Ty_n) + \gamma_n. \end{aligned} \tag{24}$$

They then proved the following result.

Theorem SW (Song and Wang [23]). *Let K be a nonempty, compact and convex subset of a uniformly convex real Banach space E. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with $F(T) \neq \emptyset$ satisfying $T(p) = \{p\}$ for all $p \in F(T)$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$, (ii) $\beta_n \to 0$, and (iii) $\sum \alpha_n \beta_n = \infty$. Then, the Ishikawa sequence defined by* (23) *converges strongly to a fixed point of T.*

More recently, Shahzad and Zegeye [29] extended and improved the results of Sastry and Babu [21], Panyanak [22], and Son and Wang [23] to multivalued quasi-nonexpansive mappings. Also, in an attempt to remove the restriction $Tp = \{p\}$ for all $p \in F(T)$ in Theorem SW, they introduced a new iteration scheme as follows.

Let K be a nonempty, closed, and convex subset of a real Banach space, and let $T : K \to P(K)$ be a multivalued mapping and $P_T x := \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Let $\alpha_n, \beta_n \in [0, 1]$. Choose $x_0 \in K$, and define $\{x_n\}$ as follows:

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n z_n, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n u_n, \end{aligned} \tag{25}$$

where $z_n \in P_T x_n$ and $u_n \in P_T y_n$. They then proved the following result.

Theorem SZ (Shahzad and Zegeye [29]). *Let X be a uniformly convex real Banach space, let K be a nonempty, closed, and convex subset of X, and let $T : K \to P(K)$ be a multivalued mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the Ishikawa iterates defined by* (25). *Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then, $\{x_n\}$ converges strongly to a fixed point of T.*

Remark 4. In recursion formula (16), the authors take $y_n \in T(x_n)$ such that $\|y_n - x^*\| = d(x^*, Tx_n)$. The existence of y_n satisfying this condition is guaranteed by the assumption that

Tx_n is proximinal. In general such a y_n is extremely difficult to pick. If Tx_n is proximinal, it is not difficult to prove that it is closed. If, in addition, it is a convex subset of a real Hilbert space, then y_n is *unique* and is characterized by

$$\langle x^* - y_n, y_n - u_n \rangle \geq 0 \quad \forall u_n \in Tx_n. \tag{26}$$

One can see from this inequality that it is not easy to pick $y_n \in Tx_n$ satisfying

$$\| y_n - x^* \| = d(x^*, Tx_n) \tag{27}$$

at every step of the iteration process. So, recursion formula (16) is not convenient to use in any possible application. Also, the recursion formula defined in (23) is not convenient to use in any possible application. The sequences $\{u_n\}$ and $\{z_n\}$ are not known precisely. Only their *existence* is guaranteed by Lemma 3. Unlike as in the case of formula (16), characterizations of $\{u_n\}$ and $\{z_n\}$ guaranteed by Lemma 3 are not even known. So, recursion formulas (23) are not really useable.

It is our purpose in this paper to first introduce the important class of *multivalued strictly pseudocontractive mappings* which is more general than the class of *multivalued nonexpansive mappings*. Then, we prove strong convergence theorems for this class of mappings. The recursion formula used in our more general setting is of the *Krasnoselskii type* [30] which is known to be superior (see, e.g., Remark 20) to the recursion formula of Mann [31] or Ishikawa [32]. We achieve these results by means of an incisive result similar to the result of Nadler [6] which we prove in Lemma 7.

2. Preliminaries

In the sequel, we will need the following definitions and results.

Definition 5. Let H be a real Hilbert space and let T be a multivalued mapping. The multivalued mapping $(I - T)$ is said to be *strongly demiclosed* at 0 (see, e.g., [27]) if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges strongly to x^* and $d(x_n, Tx_n)$ converges strongly to 0, then $d(x^*, Tx^*) = 0$.

Definition 6. Let H be a real Hilbert space. A multivalued mapping $T : D(T) \subseteq H \to CB(H)$ is said to be *k-strictly pseudocontractive* if there exist $k \in (0, 1)$ such that for all $x, y \in D(T)$ one has

$$(D(Tx, Ty))^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2 \quad \forall u \in Tx,\ v \in Ty. \tag{28}$$

If $k = 1$ in (28), the mapping T is said to be *pseudocontractive*.

We now prove the following lemma which will play a central role in the sequel.

Lemma 7. *Let E be a reflexive real Banach space and let $A, B \in CB(X)$. Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that*

$$\|a - b\| \leq D(A, B). \tag{29}$$

Proof. Let $a \in A$ and let $\{\lambda_n\}$ be a sequence of positive real numbers such that $\lambda_n \to 0$ as $n \to \infty$. From Lemma 3, for each $n \geq 1$, there exists $b_n \in B$ such that

$$\|a - b_n\| \leq D(A, B) + \lambda_n. \tag{30}$$

It then follows that the sequence $\{b_n\}$ is bounded. Since E is reflexive and B is weakly closed, there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ that converges weakly to some $b \in B$. Now, using inequality (30), the fact that $\{a - b_{n_k}\}$ converges weakly to $a - b$ and $\lambda_{n_k} \to 0$, as $k \to \infty$, it follows that

$$\|a - b\| \leq \liminf \|a - b_{n_k}\| \leq D(A, B). \tag{31}$$

This proves the lemma. □

Proposition 8. *Let K be a nonempty subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multivalued k-strictly pseudocontractive mapping. Assume that for every $x \in K$, the set Tx is weakly closed. Then, T is Lipschitzian.*

Proof. Let $x, y \in D(T)$ and $u \in Tx$. From Lemma 7, there exists $v \in Ty$ such that

$$\|u - v\| \leq D(Tx, Ty). \tag{32}$$

Using the fact that T is k-strictly pseudocontractive, and inequality (32), we obtain the following estimates:

$$\begin{aligned}(D(Tx, Ty))^2 &\leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2 \\ &\leq \left(\|x - y\| + \sqrt{k}\,\|x - u - (y - v)\|\right)^2,\end{aligned} \tag{33}$$

so that

$$\begin{aligned}D(Tx, Ty) &\leq \|x - y\| + \sqrt{k}\left(\|x - y\| + \|u - v\|\right) \\ &\leq \|x - y\| + \sqrt{k}\left(\|x - y\| + D(Tx, Ty)\right).\end{aligned} \tag{34}$$

Hence,

$$D(Tx, Ty) \leq \left(\frac{1 + \sqrt{k}}{1 - \sqrt{k}}\right)\|x - y\|. \tag{35}$$

Therefore, T is L-Lipschitzian with $L =: (1 + \sqrt{k})/(1 - \sqrt{k})$. □

Remark 9. We note that for a single-valued mapping T, for each $x \in D(T)$, the set Tx is always weakly closed.

We now prove the following lemma which will also be crucial in what follows.

Lemma 10. *Let K be a nonempty and closed subset of a real Hilbert space H and let $T : K \to P(K)$ be a k-strictly pseudocontractive mapping. Assume that for every $x \in K$, the set Tx is weakly closed. Then, $(I - T)$ is strongly demiclosed at zero.*

Proof. Let $\{x_n\} \subseteq K$ be such that $x_n \to x$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Since K is closed, we have that $x \in K$. Since, for every n, Tx_n is proximinal, let $y_n \in Tx_n$ such that $\|x_n - y_n\| = d(x_n, Tx_n)$. Using Lemma 7, for each n, there exists $z_n \in Tx$ such that

$$\|y_n - z_n\| \leq D(Tx_n, Tx). \quad (36)$$

We then have

$$\begin{aligned} \|x - z_n\| &\leq \|x - x_n\| + \|x_n - y_n\| + \|y_n - z_n\| \\ &\leq \|x - x_n\| + \|x_n - y_n\| + D(Tx_n, Tx) \\ &\leq \|x - x_n\| + \|x_n - y_n\| + \frac{(1 + \sqrt{k})}{(1 - \sqrt{k})} \|x_n - x\|. \end{aligned} \quad (37)$$

Observing that $d(x, Tx) \leq \|x - z_n\|$, it then follows that

$$d(x, Tx) \leq \|x - x_n\| + \|x_n - y_n\| + \left(\frac{1 + \sqrt{k}}{1 - \sqrt{k}}\right) \|x_n - x\|. \quad (38)$$

Taking the limit as $n \to \infty$, we have that $d(x, Tx) = 0$. Therefore, $x \in Tx$, completing the proof. □

3. Main Results

We prove the following theorem.

Theorem 11. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Suppose that $T : K \to CB(K)$ is a multivalued k-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1 - \lambda) x_n + \lambda y_n, \quad (39)$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1 - k)$. Then, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Let $p \in F(T)$. We have the following well-known identity:

$$\begin{aligned} &\|tx + (1 - t) y\|^2 \\ &\quad = t\|x\|^2 + (1 - t) \|y\|^2 - t (1 - t) \|x - y\|^2, \end{aligned} \quad (40)$$

which holds for all $x, y \in H$ and for all $t \in [0, 1]$. Using inequality (28) and the assumption that $Tp = \{p\}$ for all $p \in F(T)$, we obtain the following estimates:

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \lambda)(x_n - p) + \lambda (y_n - p)\|^2 \\ &= (1 - \lambda) \|x_n - p\|^2 + \lambda\|y_n - p\|^2 \\ &\quad - \lambda (1 - \lambda) \|x_n - y_n\|^2 \\ &\leq (1 - \lambda) \|x_n - p\|^2 + \lambda(D(Tx_n, Tp))^2 \\ &\quad - \lambda (1 - \lambda) \|x_n - y_n\|^2 \\ &\leq (1 - \lambda) \|x_n - p\|^2 \\ &\quad + \lambda \left(\|x_n - p\|^2 + k\|x_n - y_n\|^2\right) \\ &\quad - \lambda (1 - \lambda) \|x_n - y_n\|^2 \\ &= \|x_n - p\|^2 + \lambda k\|x_n - y_n\|^2 \\ &\quad - \lambda (1 - \lambda) \|x_n - y_n\|^2 \end{aligned} \quad (41)$$

$$= \|x_n - p\|^2 - \lambda (1 - k - \lambda) \|x_n - y_n\|^2. \quad (42)$$

It then follows that

$$\lambda (1 - k - \lambda) \sum_{n=1}^{\infty} \|x_n - y_n\|^2 \leq \|x_0 - p\|^2 \quad (43)$$

which implies that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < \infty. \quad (44)$$

Hence, $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since $y_n \in Tx_n$, we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

A mapping $T : K \to CB(K)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$. We note that if K is compact, then every multivalued mapping $T : K \to CB(K)$ is hemicompact.

We now prove the following corollaries of Theorem 11.

Corollary 12. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a multivalued k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact and continuous. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1 - \lambda) x_n + \lambda y_n, \quad (45)$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1 - k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From Theorem 11, we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is hemicompact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ as $k \to \infty$ for some $q \in K$. Since T is continuous, we also have $d(x_{n_k}, Tx_{n_k}) \to d(q, Tq)$ as $k \to \infty$. Therefore, $d(q, Tq) = 0$ and so $q \in F(T)$. Setting $p = q$ in the proof of Theorem 11, it follows from inequality (42) that $\lim_{n\to\infty} \| x_n - q \|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. □

Corollary 13. *Let K be a nonempty, compact, and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a multivalued k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1 - \lambda) x_n + \lambda y_n, \quad (46)$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1 - k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Observing that if K is compact, every mapping $T : K \to CB(K)$ is hemicompact, the proof follows from Corollary 12. □

Corollary 14. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a multivalued nonexpansive mapping such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)\, x_n + \lambda y_n, \tag{47}$$

where $y_n \in Tx_n$ and $\lambda \in (0,1)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since T is nonexpansive and hemicompact, then it is strictly pseudocontractive, hemicompact, and continuous. So, the proof follows from Corollary 12. □

Remark 15. In Corollary 12, the continuity assumption on T can be dispensed with if we assume that for every $x \in K$, Tx is proximinal and weakly closed. In fact, we have the following result.

Corollary 16. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H, and let $T : K \to P(K)$ be a multivalued k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)\, x_n + \lambda y_n, \tag{48}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Following the same arguments as in the proof of Corollary 12, we have $x_{n_k} \to q$ and $\lim_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$. Furthermore, from Lemma 10, $(I-T)$ is strongly demiclosed at zero. It then follows that $q \in Tq$. Setting $p = q$ and following the same computations as in the proof of Theorem 11, we have from inequality (42) that $\lim \|x_n - q\|$ exists. Since $\{x_{n_k}\}$ converges strongly to q, it follows that $\{x_n\}$ converges strongly to $q \in F(T)$, completing the proof. □

Corollary 17. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H, and let $T : K \to P(K)$ be a multivalued k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T satisfies condition (I). Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)\, x_n + \lambda y_n, \tag{49}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From Theorem 11, we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Using the fact that T satisfies condition (I), it follows that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Thus there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F(T)$ such that

$$\left\| x_{n_k} - p_k \right\| < \frac{1}{2^k} \quad \forall k. \tag{50}$$

By setting $p = p_k$ and following the same arguments as in the proof of Theorem 11, we obtain from inequality (42) that

$$\left\| x_{n_{k+1}} - p_k \right\| \le \left\| x_{n_k} - p_k \right\| < \frac{1}{2^k}. \tag{51}$$

We now show that $\{p_k\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned} \|p_{k+1} - p_k\| &\le \left\| p_{k+1} - x_{n_{k+1}} \right\| + \left\| x_{n_{k+1}} - p_k \right\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned} \tag{52}$$

This shows that $\{p_k\}$ is a Cauchy sequence in K and thus converges strongly to some $q \in K$. Using the fact that T is L-Lipschitzian and $p_k \to q$, we have

$$\begin{aligned} d(p_k, Tq) &\le D(Tp_k, Tq) \\ &\le L \|p_k - q\|, \end{aligned} \tag{53}$$

so that $d(q, Tq) = 0$ and thus $q \in Tq$. Therefore, $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Setting $p = q$ in the proof of Theorem 11, it follows from inequality (42) that $\lim_{n\to\infty}\|x_n - q\|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. □

Corollary 18. *Let K be a nonempty compact convex subset of a real Hilbert space H, and let $T : K \to P(K)$ be a multivalued k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)\, x_n + \lambda y_n, \tag{54}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From Theorem 11, we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since $\{x_n\} \subseteq K$ and K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges strongly to some $q \in K$. Furthermore, from Lemma 10, $(I-T)$ is strongly demiclosed at zero. It then follows that $q \in Tq$. Setting $p = q$ and following the same arguments as in the proof of Theorem 11, we have from inequality (42) that $\lim \|x_n - q\|$ exists. Since $\{x_{n_k}\}$ converges strongly to q, it follows that $\{x_n\}$ converges strongly to $q \in F(T)$. This completes the proof. □

Corollary 19. *Let K be a nonempty, compact, and convex subset of a real Hilbert space E, and let $T : K \to P(K)$ be a multivalued nonexpansive mapping. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)\, x_n + \lambda y_n, \tag{55}$$

where $y_n \in Tx_n$ and $\lambda \in (0,1)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 20. Recursion formula (39) of Theorem 11 is the Krasnoselskii type (see, e.g., [30]) and is known to be superior than the recursion formula of the Mann algorithm (see, e.g., Mann [31]) in the following sense.

(i) Recursion formula (39) requires less computation time than the Mann algorithm because the parameter λ in formula (39) is fixed in $(0, 1-k)$, whereas in the algorithm of Mann, λ is replaced by a sequence $\{c_n\}$ in $(0, 1)$ satisfying the following conditions: $\sum_{n=1}^{\infty} c_n = \infty$ and $\lim c_n = 0$. The c_n must be computed at each step of the iteration process.

(ii) The Krasnoselskii-type algorithm usually yields rate of convergence as fast as that of a geometric progression, whereas the Mann algorithm usually has order of convergence of the form $o(1/n)$.

Remark 21. Any consideration of the Ishikawa iterative algorithm (see, e.g., [32]) involving *two* parameters (two sequences in $(0, 1)$) for the above problem is completely undesirable. Moreover, the rate of convergence of the Ishikawa-type algorithm is generally of the form $o(1/\sqrt{n})$ and the algorithm requires a lot more computation than even the Mann process. Consequently, the question asked in [22], *Question (P)* above, whether an Ishikawa-type algorithm will converge (when it was already known that a Mann-type process converges) has no merit.

Remark 22. Our theorem and corollaries improve convergence theorems for multivalued nonexpansive mappings in [21–23, 25, 26, 28] in the following sense.

(i) In our algorithm, $y_n \in \mathrm{T}x_n$ is arbitrary and does not have to satisfy the very restrictive condition $\|y_n - x^*\| = d(x^*, Tx_n)$ in recursion formula (16), and similar restrictions in recursion formula (17). These restrictions on y_n depend on x^*, a fixed point that is being approximated.

(ii) The algorithms used in our theorem and corollaries which are proved for the much larger class of multivalued strict pseudocontractions are of the Krasnoselskii type.

Remark 23. In [29], the authors replace the condition $Tp = \{p\}$ for all $p \in F(T)$ with the following two restrictions: (i) on the sequence $\{y_n\}$:$y_n \in P_T x_n$, for example, $y_n \in Tx_n$ and $\|y_n - x_n\| = d(x_n, Tx_n)$. We observe that if Tx_n is a closed convex subset of a real Hilbert space, then y_n is unique and is characterized by

$$\langle x_n - y_n, y_n - u_n \rangle \geq 0 \quad \forall u_n \in Tx_n; \tag{56}$$

(ii) on P_T: the authors demand that P_T be nonexpansive. So, the first restriction makes the recursion formula difficult to use in any possible application, while the second restriction reduces the class of mappings to which the results are applicable. This is the price to pay for removing the condition $Tp = \{p\}$ for all $p \in F(T)$.

Remark 24. Corollary 12 is an extension of Theorem 12 of Browder and Petryshyn [19] from single-valued to multivalued strictly pseudocontractive mappings.

Remark 25. A careful examination of our proofs in this paper reveals that all our results have carried over to the class of multivalued quasinonexpansive mappings.

Remark 26. The addition of *bounded* error terms to the recursion formula (39) leads to no generalization.

We conclude this paper with examples where for each $x \in K$, Tx is proximinal and weakly closed.

Example 27. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Define $T : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$Tx = [f(x-), f(x+)] \quad \forall x \in \mathbb{R}, \tag{57}$$

where $f(x-) := \lim_{y \to x^-} f(y)$ and $f(x+) := \lim_{y \to x^+} f(y)$. For every $x \in \mathbb{R}$, Tx is either a singleton or a closed and bounded interval. Therefore, Tx is always weakly closed and convex. Hence, for every $x \in \mathbb{R}$, the set Tx is proximinal and weakly closed.

Example 28. Let H be a real Hilbert space, and let $f : H \to \mathbb{R}$ be a convex continuous function. Let $T : H \to 2^H$ be the multivalued mapping defined by

$$Tx = \partial f(x) \quad \forall x \in H, \tag{58}$$

where $\partial f(x)$ is the subdifferential of f at x which is defined by

$$\partial f(x) = \{z \in H : \langle z, y - x \rangle \leq f(y) - f(x) \ \forall y \in H\}. \tag{59}$$

It is well known that for every $x \in H$, $\partial f(x)$ is nonempty, weakly closed, and convex. Therefore, since H is a real Hilbert space, it then follows that for every $x \in H$, the set Tx is proximinal and weakly closed.

The condition $Tp = \{p\}$ for all $p \in F(T)$ which is imposed in all our theorems of this paper is not crucial. Our emphasis in this paper is to show that a Krasnoselskii-type sequence converges. It is easy to construct trivial examples for which this condition is satisfied. We do not do this. Instead, we show how this condition can be replaced with another condition which does not assume that the multivalued mapping is single-valued on the nonempty fixed point set. This can be found in the paper by Shahzad and Zegeye [29].

Let K be a nonempty, closed, and convex subset of a real Hilbert space, let $T : K \to P(K)$ be a multivalued mapping, and let $P_T : K \to CB(K)$ be defined by

$$P_T(x) := \{y \in Tx : \|x - y\| = d(x, Tx)\}. \tag{60}$$

We will need the following result.

Lemma 29 (Song and Cho [33]). *Let K be a nonempty subset of a real Banach space, and let $T : K \to P(K)$ be a multivalued mapping. Then, the following are equivalent:*

(i) $x^* \in F(T)$;

(ii) $P_T(x^*) = \{x^*\}$;

(iii) $x^* \in F(P_T)$.

Moreover, $F(T) = F(P_T)$.

Remark 30. We observe from Lemma 29 that if $T : K \rightarrow P(K)$ *is any multivalued mapping* with $F(T) \neq \emptyset$, then the corresponding multivalued mapping P_T satisfies $P_T(p) = \{p\}$ for all $p \in F(P_T)$, condition imposed in all our theorems and corollaries. Consequently, examples of multivalued mappings $T : K \rightarrow CB(K)$ satisfying the condition $Tp = \{p\}$ for all $p \in F(T)$ abound.

Furthermore, we now prove the following theorem where we dispense with the condition $Tp = \{p\}$ for all $p \in F(T)$.

Theorem 31. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H, and let $T : K \rightarrow P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$. Assume that P_T is k-strictly pseudocontractive. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary point $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda) x_n + \lambda y_n, \tag{61}$$

where $y_n \in P_T(x_n)$ and $\lambda \in (0, 1 - k)$. Then, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Let $p \in F(T)$. We have the following well-known identity:

$$\begin{aligned} &\|tx + (1 - t) y\|^2 \\ &\quad = t\|x\|^2 + (1 - t) \|y\|^2 - t (1 - t) \|x - y\|^2, \end{aligned} \tag{62}$$

which holds for all $x, y \in H$ and for all $t \in [0, 1]$. Using recursion formula (61), the identity (62), the fact that P_T is k-strictly pseudocontractive, and Lemma 29, we obtain the following estimates:

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \lambda) (x_n - p) + \lambda (y_n - p)\|^2 \\ &= (1 - \lambda) \|x_n - p\|^2 + \lambda\|y_n - p\|^2 \\ &\quad - \lambda (1 - \lambda) \|x_n - y_n\|^2 \\ &\le (1 - \lambda) \|x_n - p\|^2 + \lambda[D (P_T (x_n), P_T (p))]^2 \\ &\quad - \lambda (1 - \lambda) \|x_n - y_n\|^2 \\ &\le (1 - \lambda) \|x_n - p\|^2 \\ &\quad + \lambda \left(\|x_n - p\|^2 + k\|x_n - y_n\|^2\right) \\ &\quad - \lambda (1 - \lambda) \|x_n - y_n\|^2 \\ &= \|x_n - p\|^2 + \lambda k\|x_n - y_n\|^2 \\ &\quad - \lambda (1 - \lambda) \|x_n - y_n\|^2 \\ &= \|x_n - p\|^2 - \lambda (1 - k - \lambda) \|x_n - y_n\|^2. \end{aligned} \tag{63}$$

It then follows that

$$\lambda (1 - k - \lambda) \sum_{n=1}^{\infty} \|x_n - y_n\|^2 \le \|x_0 - p\|^2, \tag{64}$$

which implies that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < \infty. \tag{65}$$

Hence, $\lim_{n\to\infty} \| x_n - y_n \| = 0$. Since $y_n \in P_T(x_n)$ (and hence, $y_n \in Tx_n$), we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, completing the proof. □

We conclude this paper with examples of multivalued mappings T for which P_T is strictly pseudocontractive, a condition assumed in Theorem 31. Trivially, every nonexpansive mapping is strictly pseudocontractive.

Example 32. Let $H = \mathbb{R}$, with the usual metric and $T : \mathbb{R} \rightarrow CB(\mathbb{R})$ be the multivalued mapping defined by

$$Tx = \begin{cases} \left[0, \dfrac{x}{2}\right], & x \in (0, \infty), \\ \left[\dfrac{x}{2}, 0\right], & x \in (-\infty, 0]. \end{cases} \tag{66}$$

Then P_T is strictly pseudocontractive. In fact, $P_T x = \{x/2\}$ for all $x \in \mathbb{R}$.

Example 33. The following example is given in Shahzad and Zegeye [29]. Let K be nonempty subset of a normed space E. A multivalued mapping $T : K \rightarrow CB(E)$ is called $*$*-nonexpansive* (see, e.g., [34]) if for all $x, y \in K$ and $u_x \in Tx$ with $\|x - u_x\| = d(x, Tx)$, there exists $u_y \in Ty$ with $\|y - u_y\| = d(y, Ty)$ such that

$$\|u_x - u_y\| \le \|x - y\|. \tag{67}$$

It is clear that if T is $*$-nonexpansive, then P_T is nonexpansive and hence, strictly pseudocontractive. We also note that $*$-nonexpansiveness is different from nonexpansiveness for multivalued mappings. Let K = $[0, +\infty)$, and let T be defined by $Tx = [x, 2x]$ for $x \in K$. Then, $P_T(x) = \{x\}$ for $x \in K$ and thus it is nonexpansive and hence strictly pseudocontractive. Note also that T is $*$-nonexpansive but is not nonexpansive (see [35]).

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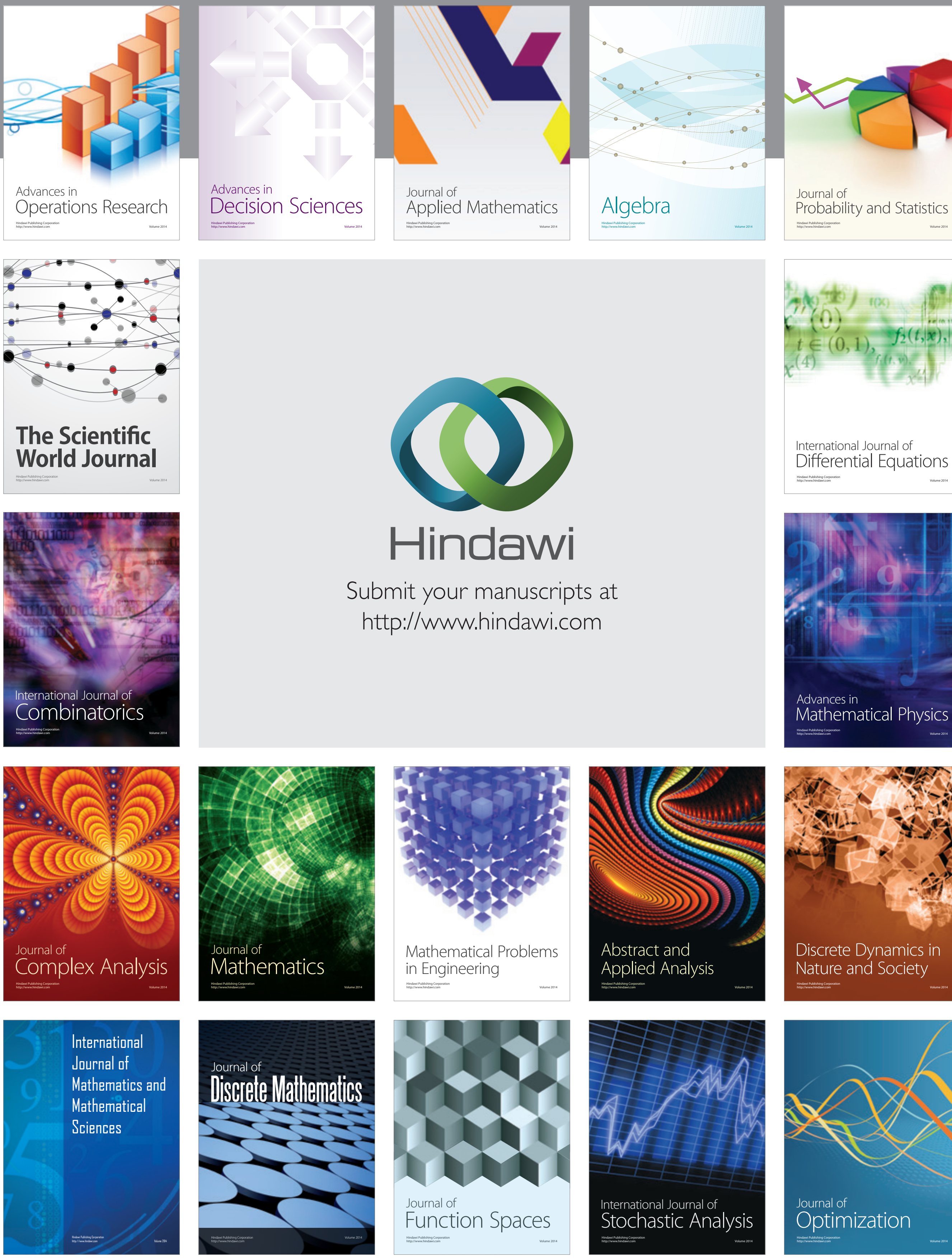

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