Research Article
Further Result on Finite-Time Stabilization of Stochastic Nonholonomic Systems

Fangzheng Gao,[1] Fushun Yuan,[1] Jian Zhang,[2] and Yuqiang Wu[3]

[1] *School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China*
[2] *Department of Mathematics, Zhengzhou University, Zhengzhou 450001, China*
[3] *Institute of Automation, Qufu Normal University, Qufu 273165, China*

Correspondence should be addressed to Fangzheng Gao; gaofz@126.com

This paper further investigates the problem of finite-time state feedback stabilization for a class of stochastic nonholonomic systems in chained form. Compared with the existing literature, the stochastic nonholonomic systems under investigation have more uncertainties, such as the x_0-subsystem contains stochastic disturbance. This renders the existing finite-time control methods highly difficult to the control problem of the systems or even inapplicable. In this paper, by extending adding a power integrator design method to a stochastic system and by skillfully constructing C^2 Lyapunov function, a novel switching control strategy is proposed, which renders that the states of closed-loop system are almost surely regulated to zero in a finite time. A simulation example is provided to demonstrate the effectiveness of the theoretical results.

1. Introduction

The nonholonomic systems, which can be used to model many frequently met mechanical systems, such as wheeled mobile robot, knife edge and rolling disk, have been an active research field over the past decades. From Brockett's necessary condition [1], it is well known that the nonholonomic systems cannot be stabilized to the origin by any static continuous state feedback, so the classical smooth control theory cannot be applied directly. In order to overcome this obstruction, several novel approaches have been developed for the problem, such as discontinuous time-invariant stabilization [2, 3], smooth time-varying stabilization [4–6], and hybrid stabilization [7]. Using these valid approaches, many fruitful results have been developed [8–15]. Particularly, the stochastic nonholonomic systems, which can be viewed as the extension of the classical nonholonomic systems, have been recently achieved investigations [16–18]. However, it should be noted that those aforementioned papers consider the feedback stabilizer that makes the trajectories of the systems converge to the equilibrium as the time goes to infinity.

Compared to asymptotic stabilization, the closed-loop system with finite-time convergence usually demonstrates faster convergence rates, higher accuracies, better disturbance rejection properties, and robustness against uncertainties [19]. Hence, it is more meaningful to investigate the finite-time stabilization problem than the classical asymptotical stability. For the deterministic case, in [20], a novel switching finite-time control strategy was proposed to nonholonomic systems in a chained form with uncertain parameters and perturbed terms by the use of time rescaling and Lyapunov based method. Later this result was essentially extended under weaker constraints on drift terms in [21]. As a natural extension, the finite-time stabilization for stochastic nonholonomic systems is an interesting and challenging subject of intensive study. However, it cannot be solved by simply extending the methods for deterministic systems because of the presence of stochastic disturbance. As pointed out by Yin et al. [22], "*the existence of a unique solution and the non-satisfaction of local Lipschitz condition are two prerequisites of discussing the finite-time stability for a stochastic nonlinear system. That is, as a common assumption condition*

to guarantee the existence of a unique solution for stochastic nonlinear system, local Lipschitz condition cannot be applied to study stochastic finite-time stability." Therefore, the finite-time controller design for stochastic nonholonomic systems should solve the following questions: *under what conditions, the stochastic nonholonomic systems exist possibly finite-time stabilizer? Under these conditions, how can one design a finite-time state feedback stabilizing controller?*

To the best of the authors' knowledge, only the authors in [23] considered the aforementioned problems when the x_0-subsystem is ordinary differential equation. This paper continues the investigation in [23] and addresses the finite-time stabilizing control design for stochastic nonholonomic systems with more uncertainties than those in [16–18, 23], which cannot be handled by general existing methods. A constructive method in designing finite-time stabilizing controller for such uncertain system is proposed. The contribution of this paper is highlighted as follows. First, inspired by the deterministic works in [24, 25], we generalize adding a power integrator design method [26] to a stochastic system, and a novel C^2 Lyapunov function is constructed to deal with stochastic disturbances. Second, based on a new switching method, a systematic control design procedure is proposed to solve the finite-time stabilization problem for all plants in the considered class.

The remainder of this paper is organized as follows. Section 2 presents some necessary notations, definitions, and preliminary results. Section 3 describes the systems to be studied and formulates the control problem. Section 4 presents the design scheme to the controller and the main result. Section 5 gives a simulation example to illustrate the theoretical finding of this paper. Finally, concluding remarks are proposed in Section 6.

2. Notations and Preliminary Results

The following notations, definitions, and lemmas are to be used throughout the paper. R^+ denotes the set of all nonnegative real numbers, and R^n denotes the real n-dimensional space. $(a, b)_{\text{odd}}$ denotes the set $\{x \mid x = p/q \text{ and } x \in (a, b),$ where p and q are odd integers$\}$. $(a, b)_{\text{even}}$ denotes the set $\{x \mid x = p/q \text{ and } x \in (a, b),$ where p is an even integer and q is an odd integer$\}$. For a given vector or matrix X, X^T denotes its transpose, $\text{Tr}\{X\}$ denotes its trace when X is square, and $|X|$ is the Euclidean norm of a vector X. C^i denotes the set of all functions with continuous ith partial derivatives. K denotes the set of all functions: $R^+ \rightarrow R^+$, which are continuous, strictly increasing, and vanishing at zero; K_∞ denotes the set of all functions which are of class K and unbounded. The arguments of the functions (or the functionals) will be omitted or simplified, whenever no confusion can arise from the context. For instance, we sometimes denote a function $f(x(t))$ by simply $f(x)$ or f.

We begin with some basic concepts and terminologies related to the notion of stochastic finite-time stability. The reader is referred to [22, 27, 28] as well as the references therein for additional details.

Consider the stochastic nonlinear system

$$dx = f(t, x) \, dt + g(t, x) \, d\omega, \tag{1}$$

where $x \in R^n$ is the system state with the initial condition $x(0) = x_0$; ω is an m-dimensional independent standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t \geq 0}$ being a filtration, and P being a probability measure. The functions: $f : R^+ \times R^n \rightarrow R^n$ and $g : R^+ \times R^n \rightarrow R^{n \times m}$ are piecewise continuous and continuous with respect to the first and second arguments, respectively, and satisfy $f(t, 0) \equiv 0$ and $g(t, 0) \equiv 0$. Moreover, system (1) is assumed to have a pathwise unique strong solution, denoted by $x(t, x_0)$.

Definition 1. For system (1), define $T_0(x_0, w) = \inf\{T \geq 0 : x(t, x_0) = 0, \text{ for all } t \geq T\}$, which is called the stochastic settling time function. Especially, $T_0(x_0, w) := +\infty$ if $x(t, x_0) \neq 0$, for all $t \geq 0$.

Definition 2. The equilibrium $x \equiv 0$ of the system (1) is said to be a stochastic finite-time stable equilibrium if

(i) it is stable in probability: for every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists $\delta > 0$ such that $P\{|x(t, x_0)| < r, \text{ for all } t \geq 0\} \geq 1 - \varepsilon$, whenever $|x| < \delta$;

(ii) its stochastic settling-time function $T_0(x_0, w)$ exists finitely with probability and $E[T_0(x_0, w)] < \infty$.

Next, we give two lemmas where the first one provides sufficient conditions to ensure the existence of pathwise unique strong solution to system (1), and the other one has been used to determine the finite-time stability of stochastic nonlinear systems.

Lemma 3 (see [23, 29]). *Assume that $f(t, x)$ and $g(t, x)$ are continuous in x. Further, for any $0 < \delta < 1$, each $N = 1, 2, \ldots,$ and each $0 \leq T < \infty$, if the following conditions hold:*

$$|f(t, x)| \leq c(t)(1 + |x|),$$

$$|g(t, x)|^2 \leq c(t)(1 + |x|^2),$$

$$|f(t, x_1) - f(t, x_2)| \leq c_T^N(t)|x_1 - x_2|, \tag{2}$$

$$|g(t, x_1) - g(t, x_2)| \leq c_T^N(t)|x_1 - x_2|,$$

as $0 < \delta \leq |x_i| \leq N$, $i = 1, 2$, $t \in [0, T]$, where $c(t)$ and $c_T^N(t)$ are nonnegative functions such that $\int_0^T c(t)dt < \infty$ and $\int_0^T c_T^N(t)dt < \infty$. Then for any given $x_0 \in R^n$, system (1) has a pathwise unique strong solution.

Lemma 4 (see [30]). *Consider the stochastic nonlinear system described in (1). Suppose that there exist a C^2 function $V(x)$, class K_∞ functions μ_1 and μ_2, real numbers $c > 0$ and $0 < \alpha < 1$, such that*

$$\mu_1(|x|) \leq V(x) \leq \mu_2(|x|),$$

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x} f + \frac{1}{2} \text{Tr}\left\{ g^T \frac{\partial^2 V}{\partial x^2} g \right\} \leq -cV^\alpha(x). \tag{3}$$

Then it is globally finite-time stable in probability, and the stochastic settling time function $T_0(x_0, w)$ satisfies

$$E\left[T_0(x_0, w)\right] \leq \frac{V^{1-\alpha}(x_0)}{c(1-\alpha)}. \tag{4}$$

Then we list two lemmas that serve as the basis of the key tools for the adding a power integrator technique.

Lemma 5 (see [31]). *For $x \in R$, $y \in R$, and $p \geq 1$ being a constant, the following inequalities hold:*

$$|x+y|^p \leq 2^{p-1}|x^p + y^p|,$$
$$(|x| + |y|)^{1/p} \leq |x|^{1/p} + |y|^{1/p} \tag{5}$$
$$\leq 2^{(p-1)/p}(|x| + |y|)^{1/p}.$$

If $p \geq 1$ is odd, then

$$|x-y|^p \leq 2^{p-1}|x^p - y^p|,$$
$$\left|x^{1/p} - y^{1/p}\right| \leq 2^{(p-1)/p}(|x-y|)^{1/p}. \tag{6}$$

Lemma 6 (see [32]). *Let x, y be real variables; then for any positive real numbers a, m, and n, one has*

$$a|x|^m|y|^n \leq b|x|^{m+n} + \frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-m/n}$$
$$\times a^{(m+n)/n}b^{-m/n}|y|^{m+n}, \tag{7}$$

where $b > 0$ is any real number.

3. Problem Formulation

Motivated by the statement in [18, 23] that many nonholonomic mechanical systems, such as wheeled mobile robot, subject to stochastic disturbances, can be transformed to a kind of stochastic nonholonomic systems in the so-called chained form, this paper considers the following class of stochastic nonholonomic systems in chained form:

$$dx_0 = d_0(t)u_0 dt + f_0(t, x_0)dt + g_0^T(t, x_0)dw,$$
$$dx_i = d_i(t)x_{i+1}u_0 dt + f_i(t, x_0, x)dt$$
$$+ g_i^T(t, x_0, x)dw, \quad i = 1, \ldots, n-1, \tag{8}$$
$$dx_n = d_n(t)u_1 dt + f_n(t, x_0, x)dt + g_n^T(t, x_0, x)dw,$$

where $x_0 \in R$ and $x = (x_1, \ldots, x_n)^T \in R^n$ are system states and $u_0 \in R$ and $u_1 \in R$ are control inputs, respectively; $d_i : R^+ \rightarrow R$, $i = 0, \ldots, n$ are uncertain and continuous, called the control coefficients. The nonlinear functions $f_0 : R^+ \times R \rightarrow R$ and $g_0 : R^+ \times R \rightarrow R^m$, $f_i : R^+ \times R^{i+1} \rightarrow R$ and $g_i : R^+ \times R^{i+1} \rightarrow R^m$, $i = 1, \ldots, n$ are assumed to be C^1 with their arguments with $f_0(t, 0) = 0$, $g_0(t, 0) = 0$, $f_i(t, x_0, 0) = 0$ and $g_i(t, x_0, 0) = 0$. and ω is an m-dimensional independent standard Wiener process defined on a complete probability space (Ω, F, P) with Ω being a sample space, F being a filtration, and P being a probability measure.

Remark 7. Obviously, the stochastic nonholonomic system (8) contains more uncertainties in drift and diffusion terms than those in the closely related papers [16–18, 23]. This, especially the existence of stochastic disturbance in the x_0-subsystem, renders the procedure of control design in [23] inapplicable to the finite-time control problem of the system (8). Up to now, how to design a stabilizing controller to achieve the finite-time stabilization of the system (8) is unsolved. It is precisely our intention of this paper.

The following assumptions regarding system (8) are imposed throughout the paper.

Assumption 8. For $i = 0, \ldots, n$, there are known positive constants c_{i1} and c_{i2} such that

$$c_{i1} \leq d_i(t) \leq c_{i2}. \tag{9}$$

Assumption 9. For f_0 and g_0, there are constants $a > 0$ and $r \in (0, 1)_{\text{odd}}$ such that

$$|f_0(t, x_0) - f_0(t, \hat{x}_0)| \leq a|x_0 - \hat{x}_0|^r,$$
$$|g_0(t, x_0) - g_0(t, \hat{x}_0)| \leq a|x_0 - \hat{x}_0|^{(r+1)/2}. \tag{10}$$

Assumption 10. For $i = 1, \ldots, n$, there is a constant $b > 0$ such that for any $\tau \in (-2/(4n+1), 0)$,

$$|f_i(t, x_0, x) - f_i(t, x_0, \hat{x})|$$
$$\leq b\left(|x_1 - \hat{x}_1|^{(m_i+\tau)/m_1} + \cdots + |x_i - \hat{x}_i|^{(m_i+\tau)/m_i}\right),$$
$$|g_i(t, x_0, x) - g_i(t, x_0, \hat{x})|$$
$$\leq b\left(|x_1 - \hat{x}_1|^{(2m_i+\tau)/2m_1} + \cdots + |x_i - \hat{x}_i|^{(2m_i+\tau)/2m_i}\right), \tag{11}$$

where $m_i = 1 + (i-1)\tau$.

For simplicity, in this paper we assume that $\tau = -p/q$ with p being any even integer and q being any odd integer, under which and the definition of m_i in Assumption 10, we know that m_i is an odd number.

Remark 11. Noting that $f_0(t, 0) = 0$, $g_0(t, 0) = 0$, $f_i(t, x_0, 0) = 0$ and $g_i(t, x_0, 0) = 0$ are assumed, Assumptions 9 and 10 imply that

$$|f_0(t, x_0)| \leq a|x_0|^r,$$
$$|g_0(t, x_0)| \leq a|x_0|^{(r+1)/2},$$
$$|f_i(t, x_0, x)| \leq b\left(|x_1|^{(m_i+\tau)/m_1} + \cdots + |x_i|^{(m_i+\tau)/m_i}\right),$$
$$|g_i(t, x_0, x)| \leq b\left(|x_1|^{(2m_i+\tau)/2m_1} + \cdots + |x_i|^{(2m_i+\tau)/2m_i}\right). \tag{12}$$

It is noteworthy that Assumption 10 is a generalization of the homogeneous growth condition introduced in [33] where $\hat{x} = 0$ and $\tau \geq 0$. Assumptions 9 and 10 are necessary, which play an essential role in ensuring the existence of finite-time stabilizer for stochastic nonholonomic system (8).

4. Finite-Time Control Design

In this section, we give a constructive procedure for the finite-time stabilizer of system (8). The design of finite-time state feedback controller is divided into two steps.

Step A. We first stabilize the x-subsystem in a finite time almost surely.

For the x_0-subsystem, we choose the control u_0 as

$$u_0 \equiv u_0^*, \tag{13}$$

where u_0^* is a positive constant. In this case, the x_0-subsystem becomes

$$dx_0 = (d_0 u_0^* + f_0)\, dt + g_0^T\, dw. \tag{14}$$

With the help of Assumptions 8 and 9 and Lemma 3, it easy to obtain that system (14) has a unique solution on $[0, T]$, for any given finite time $T > 0$. Hence, x_0 is well defined on $[0, T]$. Next, we stabilize the x-subsystem

$$dx_i = d_i(t)\, u_0^* x_{i+1} dt + f_i(t, x_0, x)\, dt$$
$$+ g_i^T(t, x_0, x)\, dw, \quad i = 1, \ldots, n-1, \tag{15}$$

$$dx_n = d_n(t)\, u_1 dt + f_n(t, x_0, x)\, dt + g_n^T(t, x_0, x)\, dw,$$

within a finite time. The control law u_1 can be recursively constructed by applying the method of adding a power integrator.

Step 1. Let $\xi_1 = x_1^\sigma$, where $\sigma \in (2, \infty)_{odd}$ is a constant. Choosing the Lyapunov function $V_1 = x_1^{(4l\sigma-\tau)}/(4l\sigma - \tau)$, where l is a constant satisfying $4l \in (0, \infty)_{even}$ and $(4l-2)\sigma \geq 1$. With the help of Remark 11 and Lemma 4, it can be verified that

$$\mathscr{L}V_1 \leq d_1 u_0^* x_1^{4l\sigma-\tau-1} (x_2 - x_2^*) + d_1 u_0^* x_1^{4l\sigma-\tau-1} x_2^*$$
$$+ b x_1^{4l\sigma} + \frac{1}{2}(4\sigma - \tau - 1)\, b^2 x_1^{4l\sigma}. \tag{16}$$

Obviously, the first virtual controller

$$x_2^* = -\frac{1}{c_{11} u_0^*}\left[M + n - 1 + b + \frac{1}{2}(4\sigma - \tau - 1)\, b^2\right] x_1^{1+\tau}$$
$$:= -\beta_1^{m_2/\sigma} \xi_1^{m_2/\sigma}, \tag{17}$$

with design constant $M > 0$, results in

$$\mathscr{L}V_1 \leq -(M + n - 1)\, \xi_1^{4l} + d_1 u_0^* \xi_1^{(4l\sigma-\tau-1)/\sigma} (x_2 - x_2^*). \tag{18}$$

Inductive Step. Suppose at step $k-1$, there are a C^2, proper and positive definite Lyapunov function V_{k-1}, and a set of virtual controllers $x_1^*, \ldots, x_k^*$ defined by

$$
\begin{aligned}
x_1^* &= 0, & \xi_1 &= x_1^{\sigma/m_1} - x_1^{*\sigma/m_1}, \\
x_2^* &= -\beta_1^{m_2/\sigma} \xi_1^{m_2/\sigma}, & \xi_2 &= x_2^{\sigma/m_2} - x_2^{*\sigma/m_2}, \\
&\vdots & &\vdots \\
x_k^* &= -\beta_{k-1}^{m_k/\sigma} \xi_{k-1}^{m_k/\sigma}, & \xi_k &= x_k^{\sigma/m_k} - x_k^{*\sigma/m_k},
\end{aligned}
\tag{19}
$$

with constants $\beta_1 > 0, \ldots, \beta_{k-1} > 0$, such that

$$\mathscr{L}V_{k-1} \leq -(M + n - k + 1)\left(\xi_1^{4l} + \cdots + \xi_{k-1}^{4l}\right)$$
$$+ d_{k-1} u_0^* \xi_{k-1}^{(4l\sigma-\tau-m_{k-1})/\sigma} (x_k - x_k^*). \tag{20}$$

To complete the induction, at the kth step, we choose the following Lyapunov function:

$$V_k(x_1, \ldots, x_k) = V_{k-1}(x_1, \ldots, x_{k-1}) + W_k(x_1, \ldots, x_k), \tag{21}$$

where

$$W_k(x_1, \ldots, x_k) = \int_{x_k^*}^{x_k} (s^{\sigma/m_k} - x_k^{*\sigma/m_k})^{(4l\sigma-\tau-m_k)/\sigma}\, ds. \tag{22}$$

Noting that

$$x_k^{*\sigma/m_k} = -\beta_{k-1}\xi_{k-1},$$

$$\xi_k = x_k^{\sigma/m_k} - \beta_{k-1} x_{k-1}^{\sigma/m_{k-1}} + \cdots + (-1)^{k-1}$$
$$\times \beta_{k-1} \cdots \beta_1 x_1^{\sigma/m_1}, \tag{23}$$

with the help of the fact $\sigma/m_i > 2$, the function V_k can be shown to be C^2, proper and positive definite using a similar method as in [34]. Moreover, we can obtain

$$\frac{\partial W_k}{\partial x_k} = \xi_k^{(4l\sigma-\tau-m_k)/\sigma},$$

$$\frac{\partial^2 W_k}{\partial x_k^2} = \frac{4l\sigma - \tau - m_k}{m_k} \xi_k^{((4l-1)\sigma-\tau-m_k)/\sigma} x_k^{(\sigma-m_k)/m_k},$$

$$\frac{\partial^2 W_k}{\partial x_k \partial x_i} = \frac{\partial^2 W_k}{\partial x_i \partial x_k} = \frac{4l\sigma - \tau - m_k}{\sigma}$$
$$\times \xi_k^{((4l-1)\sigma-\tau-m_k)/\sigma} \frac{\partial x_k^{*\sigma/m_k}}{\partial x_i},$$

$$\frac{\partial W_k}{\partial x_i} = -\frac{4l\sigma - \tau - m_k}{\sigma} \frac{\partial x_k^{*\sigma/m_k}}{\partial x_i}$$
$$\times \int_{x_k^*}^{x_k} (s^{\sigma/m_k} - x_k^{*\sigma/m_k})^{((4l-1)\sigma-\tau-m_k)/\sigma}\, ds,$$

$$\frac{\partial^2 W_k}{\partial x_i^2} = \frac{4l\sigma - \tau - m_k}{\sigma} \frac{(4l-1)\sigma - \tau - m_k}{\sigma}\left(\frac{\partial x_k^{*\sigma/m_k}}{\partial x_i}\right)^2$$
$$\times \int_{x_k^*}^{x_k} (s^{\sigma/m_k} - x_k^{*\sigma/m_k})^{((4l-2)\sigma-\tau-m_k)/\sigma}\, ds$$
$$- \frac{4\sigma - \tau - m_k}{\sigma} \frac{\partial^2 x_k^{*\sigma/m_k}}{\partial x_i^2}$$
$$\times \int_{x_k^*}^{x_k} (s^{\sigma/m_k} - x_k^{*\sigma/m_k})^{((4l-1)\sigma-\tau-m_k)/\sigma}\, ds,$$

$$\frac{\partial^2 W_k}{\partial x_i \partial x_j} = \frac{4l\sigma - \tau - m_k}{\sigma}$$

$$\cdot \frac{(4l-1)\sigma - \tau - m_k}{\sigma} \frac{\partial x_k^{*\sigma/m_k}}{\partial x_i} \frac{\partial x_k^{*\sigma/m_k}}{\partial x_j}$$

$$\times \int_{x_k^*}^{x_k} \left(s^{\sigma/m_k} - x_k^{*\sigma/m_k} \right)^{((4l-2)\sigma - \tau - m_k)/\sigma} ds,$$

$$(24)$$

where $i, j = 1, \ldots, k-1, \ i \neq j$.

Using (20)–(22) and (4), it follows that

$$\mathscr{L}V_k \leq -(M+n-k+1)\left(\xi_1^{4l} + \cdots + \xi_{k-1}^{4l}\right)$$

$$+ d_k u_0^* \xi_k^{(4l\sigma - \tau - m_k)/\sigma} x_{k+1}$$

$$+ d_{k-1} u_0^* \xi_{k-1}^{(4l\sigma - \tau - m_{k-1})/\sigma} (x_k - x_k^*)$$

$$+ \xi_k^{(4l\sigma - \tau - m_k)/\sigma} f_k + \sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i} (d_i u_0^* x_{i+1} + f_i)$$

$$+ \frac{1}{2} \sum_{i,j=1,\ i \neq j}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_j} \right| \left| g_i^T \right| \left| g_j^T \right|$$

$$+ \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i^2} \right| \left| g_i^T \right|^2$$

$$+ \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_k \partial x_i} \right| \left| g_k^T \right| \left| g_i^T \right| + \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_k^2} \right| \left| g_k^T \right|^2.$$

$$(25)$$

Based on Lemmas 5 and 6, the following proposition is given to estimate the last seven terms on the right-hand side of (25).

Proposition 12. *There exists a positive constant l_k such that*

$$d_{k-1} u_0^* \xi_{k-1}^{(4l\sigma - \tau - m_k)/\sigma} (x_k - x_k^*)$$

$$+ \xi_k^{(4l\sigma - \tau - m_k)/\sigma} f_k + \sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i} (d_i u_0^* x_{i+1} + f_i)$$

$$+ \frac{1}{2} \sum_{i,j=1,\ i \neq j}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_j} \right| \left| g_i^T \right| \left| g_j^T \right|$$

$$+ \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i^2} \right| \left| g_i^T \right|^2$$

$$+ \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_k \partial x_i} \right| \left| g_k^T \right| \left| g_i^T \right| + \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_k^2} \right| \left| g_k^T \right|^2$$

$$\leq \xi_1^{4l} + \cdots + \xi_{k-1}^{4l} + l_k \xi_k^{4l}.$$

$$(26)$$

Proof. The similar proof can be found in [23] and thus it is omitted here.

Substituting (26) into (25), we obtain

$$\mathscr{L}V_k \leq -(M+n-k)\left(\xi_1^{4l} + \cdots + \xi_{k-1}^{4l}\right)$$

$$+ d_k u_0^* \xi_k^{(4l\sigma - \tau - m_k)/\sigma} x_{k+1} + l_k \xi_k^{4l}.$$

$$(27)$$

Clearly, the C^0 virtual controller

$$x_{k+1}^* = -\frac{1}{c_{k1} u_0^*} \xi_k^{(m_k + \tau)/\sigma} (M+n-k+l_k) := -\beta_k^{m_{k+1}/\sigma} \xi_k^{m_{k+1}/\sigma},$$

$$(28)$$

with $\beta_k > 0$ being constant, results in

$$\mathscr{L}V_k \leq -(M+n-k)\left(\xi_1^{4l} + \cdots + \xi_k^{4l}\right)$$

$$+ d_k u_0^* \xi_k^{(4l\sigma - \tau - m_k)/\sigma} \left(x_{k+1} - x_{k+1}^*\right).$$

$$(29)$$

This completes the proof of the inductive step. □

Using the previous inductive argument, one concludes that at the nth step, there exists a non-Lipschitz continuous state feedback control law of the form

$$u = x_{n+1}^* = -\beta_n^{m_{n+1}/\sigma} \xi_n^{m_{n+1}/\sigma},$$

$$(30)$$

with $\beta_n > 0$ being constant and a C^2 positive definite and proper Lyapunov function V_n of the form (21)-(22), such that

$$\mathscr{L}V_n \leq -M\left(\xi_1^{4l} + \cdots + \xi_n^{4l}\right).$$

$$(31)$$

So the following result is obtained.

Lemma 13. *If Assumptions 8 and 10 hold for the stochastic nonlinear system (15), then the solution of the closed-loop system consisting of (15) and (30) is finite-time stable in probability.*

Proof. From (23) and (30), it is not hard to verify that all conditions in Lemma 3 are satisfied, which means that the closed-loop system admits a unique solution. Next we prove that the closed-loop system is globally finite-time stable in probability.

First of all, by using Lemma 5, it is easy to see that

$$W_k = \int_{x_k^*}^{x_k} \left(s^{\sigma/m_k} - x_k^{*\sigma/m_k} \right)^{(4l\sigma - \tau - m_k)/\sigma} ds$$

$$\leq \left| \xi_k \right|^{(4l\sigma - \tau - m_k)/\sigma} \left| x_k - x_k^* \right|$$

$$\leq 2^{(\sigma - m_k)/\sigma} \left| \xi_k \right|^{(4l\sigma - \tau)/\sigma}.$$

$$(32)$$

So we have the following estimate

$$V_n = \frac{1}{(4l\sigma - \tau)} x_1^{4l\sigma - \tau} + \sum_{k=2}^{n} W_k$$

$$\leq \frac{1}{(4l\sigma - \tau)} \xi_1^{(4l\sigma - \tau)/\sigma} + \sum_{k=2}^{n} 2^{(\sigma - m_k)/\sigma} \left| \xi_k \right|^{(4l\sigma - \tau)/\sigma}$$

$$(33)$$

$$\leq 2 \sum_{k=1}^{n} \xi_k^{(4l\sigma - \tau)/\sigma}.$$
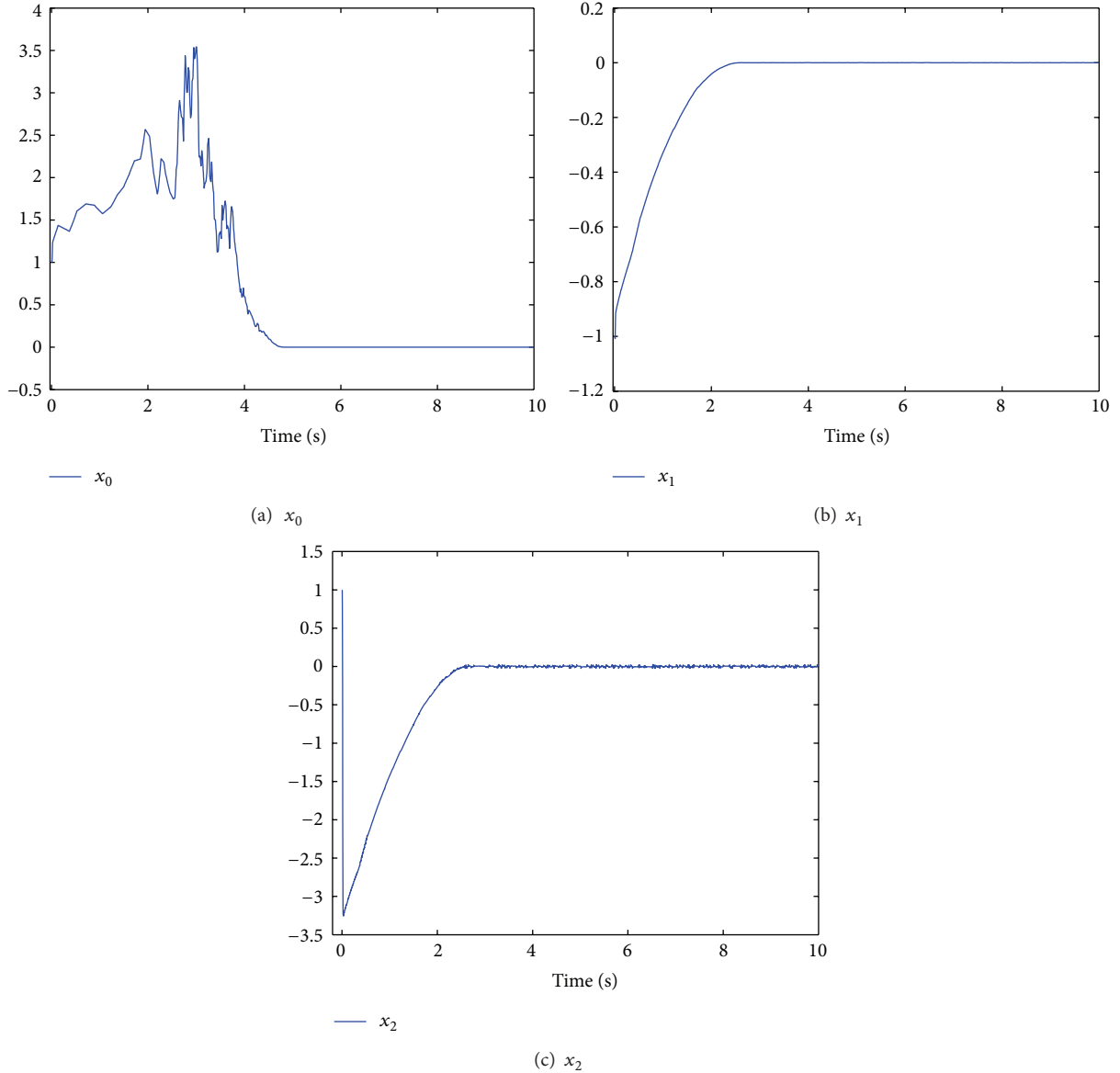
(a) x_0

(b) x_1

(c) x_2

FIGURE 1: States of the closed-loop system.

Let $\alpha = 4l\sigma/(4l\sigma - \tau)$. With (33) and (31) in mind, by Lemma 5, it is not difficult to obtain that

$$\mathscr{L}V_n \leq -\frac{MV_n^\alpha}{2^\alpha}. \tag{34}$$

Therefore, by Lemma 4, the uncertain system (15) with the controller (30) is globally finite-time stable in probability and its stochastic settling time $T_1(x(0), w)$ satisfies

$$E\left[T_1(x(0), w)\right] \leq \frac{2^\alpha V_n^{1-\alpha}(x(0))}{M(1-\alpha)}. \tag{35}$$

□

Step B. Then we design a state feedback controller such that the x_0-subsystem is finite-time stable in probability.

From Step A, we know that $x(t) \equiv 0$ when $t \geq T_1(x(0), w)$. Therefore, we just need to stabilize the x_0-subsystem in

a finite time almost surely. When $t \geq T_1(x(0), w)$, for the x_0-subsystem, we can take the following control law:

$$u_0(x_0) = -\frac{1}{c_{01}}\left(k_0 + a + \frac{3}{2}a^2\right)x_0^r, \tag{36}$$

where k_0 is a positive constant.

Taking the Lyapunov function $V_0 = x_0^4/4$, a simple computation gives

$$\mathscr{L}V_0 \leq -k_0 x_0^{3+r}. \tag{37}$$

Thus by Lemma 4, x_0 stochastically tends to 0 within a settling time denoted by $T_2(x_0(0), w)$ and

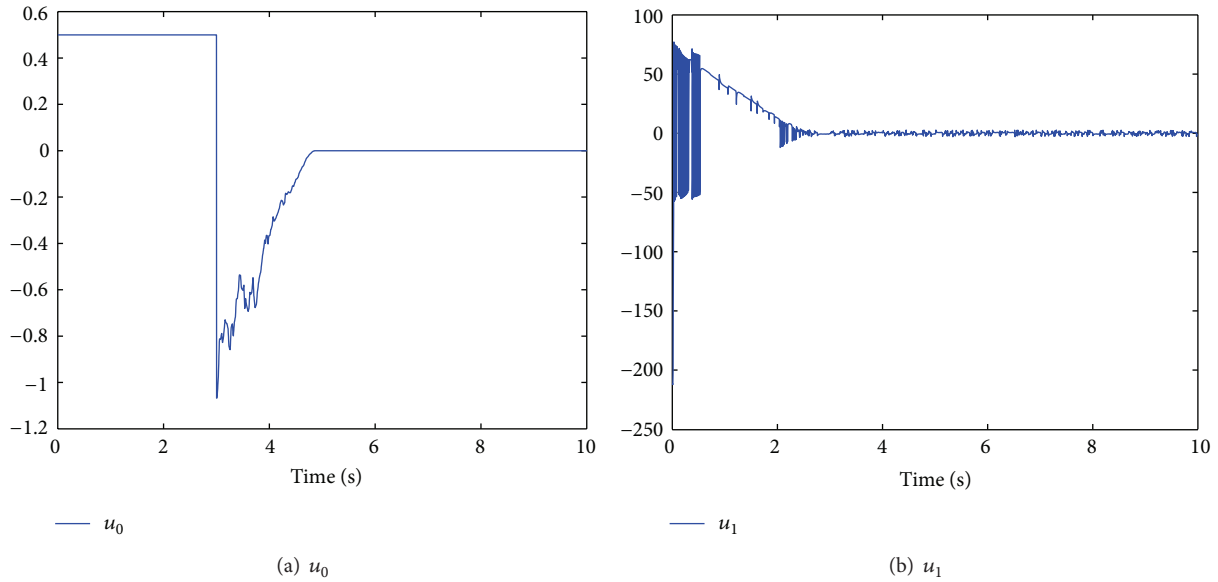$$E\left[T_2(x_0(0), w)\right] \leq \frac{2V_0^{(1-r)/4}(x_0(0))}{k_0(1-r)}. \tag{38}$$

FIGURE 2: Control inputs of the closed-loop system.

Up to now, we have finished the finite-time stabilizing controller design of the system (8). Consequently, the following theorem can be obtained to summarize the main result of the paper.

Theorem 14. *Under Assumptions 8–10, if the proposed control design procedure together with the previous switching control strategy is applied to system* (8), *then, for any initial conditions in the state space* $(x_0, x) \in R^{n+1}$, *the closed-loop system is globally finite time regulated at origin in probability.*

Remark 15. The settling time of the closed-loop system depends on design parameters k_0, M and initial conditions, that is, when the initial conditions are known, the settling-time can be arbitrarily adjusted by the choice of appropriate design parameters k_0 and M.

5. Simulation Example

To verify our proposed controller, we consider the following low-dimensional system:

$$dx_0 = (1.5 + 0.5 \cos t) u_0 dt + \frac{1}{4} x_0^{4/5} dw,$$

$$dx_1 = x_2 dt + \frac{1}{4} x_1^{9/11} dt + \frac{1}{8} \sin x_1 dw, \quad (39)$$

$$dx_2 = u dt + \frac{1}{4} x_1^{7/11} dt + \frac{1}{8} \cos 2x_2 dw.$$

We choose $\tau = -2/11$, which together with $m_1 = 1$ implies that $m_2 = 9/11$. By Lemma 5, it is verified that Assumptions 9-10 are satisfied with $a = b = 1/2$. Assumption 8 holds with $1 \leq 1.5 + 0.5 \cos t \leq 2$; hence the controller proposed in this paper is applicable.

When $(x_0(0), x_1(0), x_2(0)) = (1, -1, 1)$, by choosing $\sigma = 3$ and $l = 2/3$, according to the design procedure proposed

in Section 4, a state feedback stabilizing controller can be explicitly given. Using MATLAB, Figures 1 and 2 are obtained to exhibit the trajectories of the closed-loop system. From these figures, we can see that the states of the closed-loop system are finite time regulated to zero almost surely.

6. Conclusion

In this paper, the finite-time state feedback stabilization problem has been investigated for a class of nonholonomic systems with more general stochastic disturbances. With the help of adding a power integrator technique, a systematic control design procedure is developed in the stochastic setting. It should be noted that the proposed controller can only work well when the whole state vector is measurable. Therefore, a natural and more interesting problem is how to design output feedback stabilizing controller for the system studied in the paper if only partial state vector is measurable, which is now under our further investigation.

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