

Phenomenological applications of non-perturbative heavy quark effective theory¹

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Abstract. We briefly review the strategy to perform non-perturbative heavy quark effective theory computations and we specialize to the case of the b quark mass which has recently been computed including the $1/m$ term.

1. Introduction

Lattice QCD allows a first-principle study of non-perturbative properties of QCD (e.g. computation of the hadron spectrum, decay constants and other matrix elements). However many systematics have to be controlled in order to reliably compute quantities of phenomenological interests:

- (i) numerical computations can be performed only at finite non-zero lattice spacing. Results thus have to be extrapolated to the continuum limit;
- (ii) the presence of an ultra-violet cut-off (the lattice spacing a) and of an infra-red cut-off (the volume V) constrains quark masses, and extrapolations are needed to reach the chiral and the heavy quark regimes;
- (iii) dynamical light quark effects are numerically expensive to simulate and are thus often neglected in the so-called quenched approximation. This approximation turned out empirically to be a good approximation in many cases and is useful in order to develop new methods and to pin down systematics. All the results presented in this contribution are obtained in this approximation.

We now discuss the second point above in more detail. Pions have a Compton wavelength which is too large compared to lattices that can be presently simulated and in order to keep finite volume effects under control one has to simulate light quarks (u, d) which are heavier than the physical ones and then extrapolate to the physical point by matching lattice results with the chiral effective theory.

On the contrary, the B meson is too heavy and its Compton wavelength is too small compared to the lattice spacings that can be simulated at present (for large enough volumes). In order to keep lattice artifacts under control one is forced to simulate quark masses in the region of the charm quark and then extrapolate to the b-quark mass. Another interesting possibility is to use heavy-quark effective theory (HQET) to describe the b quark. HQET can be formulated on the

¹ based on a work in collaboration with M. Della Morte, N. Garron, R. Sommer and part of the ALPHA Coll.

lattice as shown in [1]. However, in order to be able to perform the continuum limit, it has to be renormalised non-perturbatively. A strategy has been devised and applied to the determination of the b-quark mass in the static limit of HQET [2] and presented in a more general framework in [3]. We will briefly review this strategy and then describe its application to the case of the computation of the b-quark mass including $1/m_b$ corrections [4].

2. Non-perturbative HQET

The HQET action on a lattice (up to terms of order $1/m^2$) reads

$$S_{\text{HQET}} = a^4 \sum_x \left\{ \bar{\psi}_h D_0 \psi_h + \omega_{\text{spin}} \underbrace{\bar{\psi}_h (-\sigma \cdot \mathbf{B}) \psi_h}_{\mathcal{O}_{\text{spin}}} + \omega_{\text{kin}} \underbrace{\bar{\psi}_h \left(-\frac{1}{2} \mathbf{D}^2\right) \psi_h}_{\mathcal{O}_{\text{kin}}} \right\} + \mathcal{O}(1/m^2) \quad (1)$$

where the first term is $\mathcal{O}(1)$ [1] while the second and third terms are $\mathcal{O}(1/m)$ (ω_{spin} and ω_{kin} are formally of order $1/m$)². The heavy quark fields are subject to the constraints $P_+ \psi_h = \psi_h$, $\bar{\psi}_h P_+ = \bar{\psi}_h$ with $P_+ = \frac{1}{2}(1 + \gamma_0)$. In the following we will consider two different discretisations of the heavy quark action where the naïve parallel transporter in the covariant derivative D_0 is replaced by HYP1 and HYP2 gauge links (see [6] for details).

Also the composite operators have an $1/m$ expansion in the effective theory. For example, the time component of the axial current, including $1/m$ terms, reads:

$$A_0^{\text{HQET}}(x) = Z_A^{\text{HQET}} \bar{\psi}_1 \gamma_0 \gamma_5 \psi_h + c_A^{\text{HQET}} \bar{\psi}_1 \gamma_j \overleftarrow{D}_j \psi_h + \mathcal{O}(1/m^2) \quad (2)$$

where $c_A^{\text{HQET}} = \mathcal{O}(1/m)$.

In the path integral one keeps the $\mathcal{O}(1)$ term of the HQET action in the weight and expands the rest in powers of $1/m$ considering the higher order terms of the HQET action ($\mathcal{L}^{(\nu)}(x) = \mathcal{O}(1/m^\nu)$) as operator insertions:

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi e^{-S_{\text{light}} - a^4 \sum_x \bar{\psi}_h(x) D_0 \psi_h(x)} \mathcal{O} \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \dots \right\} \quad (3)$$

In this way the effective theory is renormalisable order by order in $1/m$ (this statement is equivalent to assert the existence of the continuum limit due to universality) and results in an asymptotic expansion in $1/m$. It is important to notice that these properties are not automatic for an effective field theory: e.g. ChPT shares these properties while NRQCD does not.

It is important to mention that for HQET on the lattice, as for any other theory with a dimensional (hard) cut-off, the possibility of mixing with operators of lower dimension implies that renormalisation has to be carried out non-perturbatively, in order to be able to perform the continuum limit.

3. Matching between QCD and HQET

The bare couplings of HQET ($m_{\text{bare}}, \omega_{\text{kin}}, \omega_{\text{spin}}, c_A^{\text{HQET}}, Z_A^{\text{HQET}}, \dots$) are unknown parameters and have to be determined by matching the effective theory with QCD, thus allowing a transfer of predictivity from QCD to HQET. A way of determining the bare couplings of HQET in a non-perturbative way (thus assuring the existence of the continuum limit) is to perform the matching non-perturbatively. In practice this consists in properly choosing N_{HQET} (the number of HQET bare couplings to be determined at a certain order in $1/m$) QCD observables and to equate them to their corresponding expansion in HQET:

$$\Phi_k^{\text{QCD}} = \Phi_k^{\text{HQET}}, \quad k = 1, 2, \dots, N_{\text{HQET}} \quad (4)$$

² for the unexplained notation we refer to [4]

The bare couplings of HQET can consequently be extracted by solving the resulting system of equations. It is important to notice that this procedure requires to be able to simulate the b-quark around its physical mass. How is this possible? The trick consists in performing the matching in a small physical volume and at small lattice spacing (using the Schrödinger Functional) in such a way that $m_b a \ll 1$. At the same time, in small volume the expansion in $1/m$ will have corrections in $1/m_b L$ and thus L can not be chosen too small. We choose $L = L_1 \approx 0.4 \text{ fm}$ which allows both conditions $m_b a \ll 1$ and $1/(m_b L) \ll 1$ to be satisfied.

However, physical observables need a large volume, such that the B -meson fits comfortably, say $L \approx 4L_1 \approx 1.6 \text{ fm}$. The connection between these different volumes is achieved by a recursive finite size method based on the concept of step scaling functions (SSFs) [5]:

$$\Phi_k^{\text{HQET}}(2L) = F_k \left(\left\{ \Phi_j^{\text{HQET}}(L), j = 1, \dots, N \right\} \right) \quad (5)$$

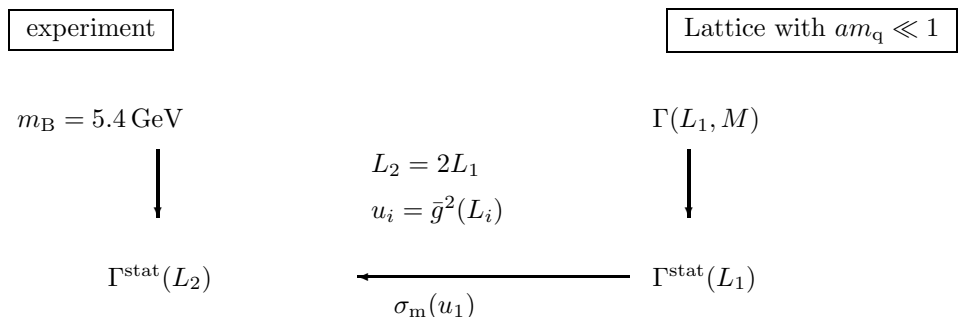
All these steps allow a fully non-perturbative formulation of HQET where the continuum limit can be taken in all steps [3].

4. M_b static (at order $1/m^0$)

We present the strategy described above in the simple case of M_b in the static approximation (i.e. at order $1/m^0$). We start from the definition of the finite volume B -meson “mass” $\Gamma = -\partial_0 \log[f_A(x_0)]_{x_0=L/2, T=L}$ which is defined in term of the Schrödinger functional correlator $f_A(x_0)$ of the temporal component of the axial current (inserted at time x_0) with a pseudoscalar source on one boundary (the temporal extension T of the Schrödinger functional is here chosen to be equal to the spatial size L). From Γ we construct the observable $\Phi_2(L, M)^{\text{QCD}} = L\Gamma(L, M)$. In the effective theory this observable has the expansion $\Phi_2(L, M)^{\text{HQET}} = L[\Gamma^{\text{stat}}(L) + m_{\text{bare}}]$, where m_{bare} is the overall energy shift between the effective theory and QCD. The matching condition $\Phi_2(L, M)^{\text{QCD}} = \Phi_2(L, M)^{\text{HQET}}$ in infinite volume becomes $m_B = E_{\text{stat}} + m_{\text{bare}}$ where $E_{\text{stat}} = \lim_{L \rightarrow \infty} \Gamma^{\text{stat}}(L)$. Multiplying by $L_2 = 2L_1$ and eliminating m_{bare} one obtains:

$$\begin{aligned} m_B &= L_2 E_{\text{stat}} - L_2 \Gamma^{\text{stat}}(L_1) + \frac{L_2}{L_1} \Phi_2^{\text{HQET}}(L_1, M_b) \\ &= L_2 E_{\text{stat}} - L_2 \Gamma^{\text{stat}}(L_1) + \frac{L_2}{L_1} \Phi_2^{\text{QCD}}(L_1, M_b) \\ &= L_2 E_{\text{stat}} - L_2 \Gamma^{\text{stat}}(L_2) + \underbrace{L_2 \Gamma^{\text{stat}}(L_2) - L_2 \Gamma^{\text{stat}}(L_1)}_{=\sigma_m(\bar{g}^2(L_1))} + \frac{L_2}{L_1} \Phi_2^{\text{QCD}}(L_1, M_b) \\ &= \underbrace{L_2 [E_{\text{stat}} - \Gamma^{\text{stat}}(L_2)]}_{a \rightarrow 0 \text{ in HQET}} + \underbrace{\sigma_m(u_1)}_{a \rightarrow 0 \text{ in HQET}} + 2 \underbrace{\frac{L_2}{L_1} \Gamma(L_1, M_b)}_{a \rightarrow 0 \text{ for } M_b L_1 \gg 1} \end{aligned} \quad (6)$$

where $\sigma_m(\bar{g}^2(L_1))$ is the SSF for the static effective mass Γ^{stat} . The whole procedure is represented by the following diagram:



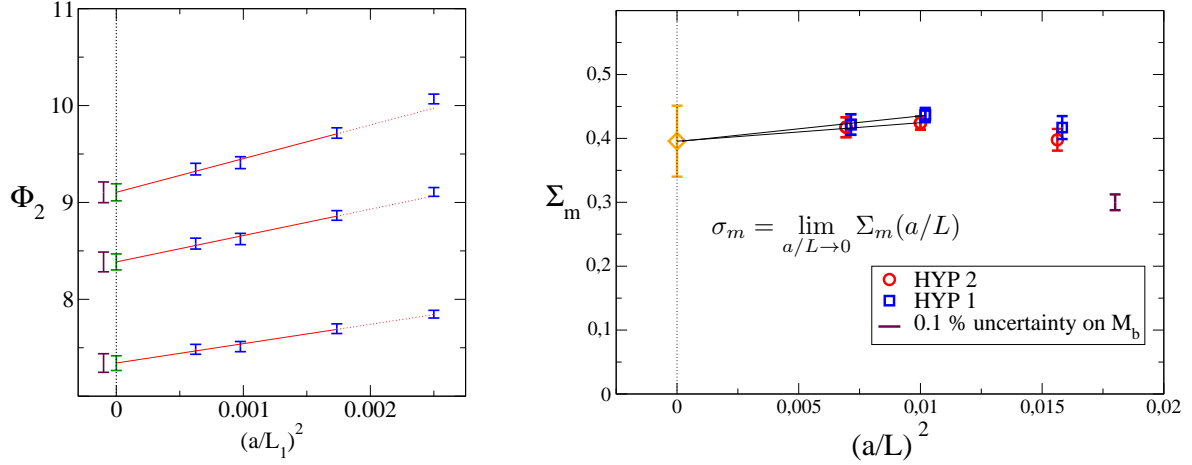


Figure 1. example of continuum limit extrapolation for the QCD observable Φ_2^{QCD} (for three values of the mass around M_b) and for the static SSF σ_m .

After having taken the continuum limit in the various parts of the last line of Eq. 6 one has to solve it for M_b (the RGI b-quark mass). In the simulations we fix $L_1 \approx 0.4$ fm (from $\bar{g}^2(L_1) \approx 3.48$), we set the light quark mass to zero and we fix the RGI quark masses of the heavy quark around M_b . In infinite volume ($L_\infty = 4L_1$) we set the light quark mass to the strange quark mass because as physical input we use the mass of the B_s meson.

Two examples of continuum limit extrapolation are shown in Fig. 1. The solution of Eq. 6 is represented graphically in Fig. 2, where the red square represent the value of $L_2 m_B - L_2 [E_{\text{stat}} - \Gamma^{\text{stat}}(L_1)] - \sigma_m(\bar{g}^2(L_1))$ (with m_B set to the experimental value) while the green circles correspond to the values of $2\Phi_2^{\text{QCD}}(L_1, M)$ for tree values of $z = L_1 M$ around $L_1 M_b$. From the solution of Eq. 6 one obtains also the slope

$$S = \frac{1}{L_1} \frac{\partial}{\partial M} \Phi_2^{\text{QCD}}(L_1, M) = 0.61(5) \quad (7)$$

which will be needed in the computation of the $1/m$ corrections. The results for M_b in the static approximation are

$$M_b^{\text{stat}} = 6806 \pm 79 \text{ MeV} \quad (8)$$

and the error is dominated by the error on the renormalisation constant of the quark mass [4].

5. M_b at order $1/m$

At order $1/m$, beyond the parameter m_{bare} one has to determine at least (depending on the strategy) the bare couplings ω_{kin} and ω_{spin} which are of order $1/m$. In our case we consider the spin averaged B -meson mass and in this quantity ω_{spin} cancels out. We rewrite Eq. 6 in the case $1/m$:

$$\infty \text{ volume} \quad m_B^{\text{av}} = E^{\text{stat}} + m_{\text{bare}} + \omega_{\text{kin}} E^{\text{kin}}$$

$$\text{Matching 1} \quad \Phi_1^{\text{QCD}}(L, M) = \omega_{\text{kin}} R_1^{\text{kin}}(L) = \Phi_1^{\text{HQET}}$$

$$\text{Matching 2} \quad \Gamma_1(L, M) = \Gamma_1^{\text{stat}}(L) + m_{\text{bare}} + \omega_{\text{kin}} \Gamma_1^{\text{kin}}(L) = \frac{\Phi_2^{\text{HQET}}}{L}$$

where now $\Phi_2^{\text{QCD}} \equiv L\Gamma_1$ and Γ_1 is defined similarly to Γ but in terms of the Schrödinger

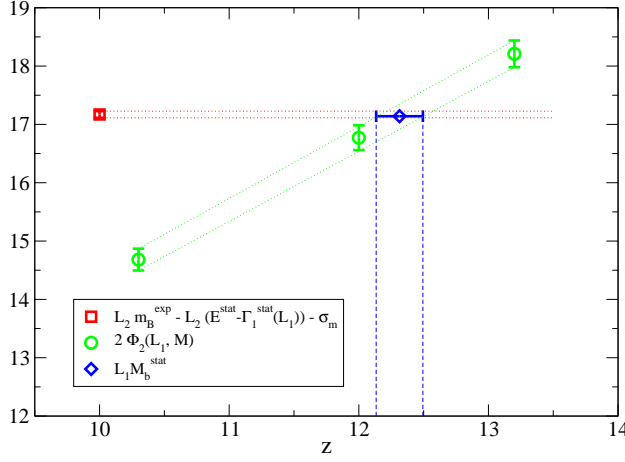


Figure 2. Graphical solution of Eq. 6 to determine M_b^{stat} .

functional boundary to boundary correlators (this allows a determination of M_b at order $1/m$ without need of determining c_A^{HQET}) while $\Gamma_1^{\text{stat}}(L)$ and $\Gamma_1^{\text{kin}}(L)$ arise naturally from the expansion of Γ_1 in the effective theory. $\Phi_1^{\text{QCD}}(L)$ is a more complicate quantity whose definition (together that of R_1^{kin}) can be found in [4]. The important thing here is that its expansion in the effective theory is proportional to ω_{kin} , thus allowing to eliminate this parameter from the above system of equations. By eliminating also m_{bare} one obtains the following equation

$$m_B^{\text{av}} = \left[E^{\text{stat}} - \Gamma_1^{\text{stat}}(L) \right] + \Gamma_1(L, M) + \left[\frac{\Phi_1^{\text{QCD}}(L, M)}{R_1^{\text{kin}}(L)} (E^{\text{kin}} - \Gamma_1^{\text{kin}}(L)) \right] \quad (9)$$

In this equation we set $L = L_2 = 2L_1$ and we use suitable SSFs to relate $\Phi_i(L_2)$ with $\Phi_i(L_1)$:

$$\Phi_1(2L, M) = \sigma_1^{\text{kin}}(u) \Phi_1(L, M), \quad \Phi_2(2L, M) = 2\Phi_2(L, M) + \sigma_m(u) + \sigma_2^{\text{kin}}(u) \Phi_1(L, M) \quad (10)$$

The continuum SSFs are defined in terms of those at finite lattice spacing as $\sigma(u) = \lim_{a/L \rightarrow 0} \Sigma(u, a/L)$ where the definition of the Σ 's (neglecting their θ dependence, see [4] for further details) is:

$$\Sigma_1^{\text{kin}}(u, a/L) = \left. \frac{R_1^{\text{kin}}(2L)}{R_1^{\text{kin}}(L)} \right|_{u=\bar{g}^2(L)}, \quad (11)$$

$$\Sigma_2^{\text{kin}}(u, a/L) = \left. \frac{2L [\Gamma_1^{\text{kin}}(2L) - \Gamma_1^{\text{kin}}(L)]}{R_1^{\text{kin}}(L)} \right|_{u=\bar{g}^2(L)}, \quad (12)$$

$$\Sigma_m(u, a/L) = 2L \left[\Gamma_1^{\text{stat}}(2L) - \Gamma_1^{\text{stat}}(L) \right]_{u=\bar{g}^2(L)}. \quad (13)$$

The r.h.s. of Eq. 9 can be split in three parts $m_B = m_B^{\text{stat}} + m_B^{(1a)} + m_B^{(1b)}$. M_b^{stat} is the solution of $m_B^{\text{stat}}(M_b^{\text{stat}}) = m_B^{\text{av}}$ while the $1/m$ correction can be found by solving the equation $m_B(M_b^{(1a)} + M_b^{(1b)}) = m_B^{\text{av}}$, which translates into $M_b^{(1a)} + M_b^{(1b)} = -\frac{1}{S} [m_B^{(1a)}(M_b^{\text{stat}}) + m_B^{(1b)}(M_b^{\text{stat}})]$. From Eqs. 9 and 10 one obtains the formulae for $m_B^{(1a)}$, $m_B^{(1b)}$:

$$m_B^{(1a)}(M) = \frac{1}{L_2} \sigma_2^{\text{kin}}(u_1) \Phi_1(L_1, M) \quad m_B^{(1b)}(M) = \frac{(E^{\text{kin}} - \Gamma_1^{\text{kin}}(L_2))}{R_1^{\text{kin}}} \Phi_1(L_2, M) \quad (14)$$

Some example of continuum extrapolations at order $1/m$ are shown in Fig. 3. These are the

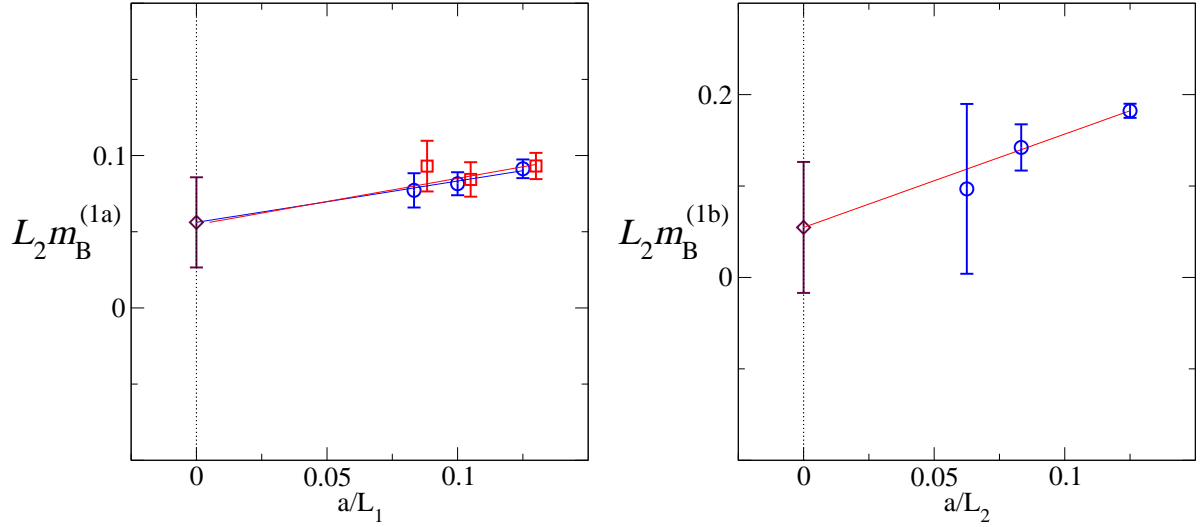


Figure 3. Continuum limit extrapolations for $m_B^{(1a)}$ (combining HYP1 and HYP2 actions) and for $m_B^{(1b)}$ (only HYP2 shown).

most difficult steps of the computation because in both Σ_2^{kin} and $(E^{\text{kin}} - \Gamma_1^{\text{kin}}(L_2))$ there are $1/a^2$ power divergences that have to cancel out and the extrapolation to the continuum limit is only linear in a (in the case $(E^{\text{stat}} - \Gamma_1^{\text{stat}}(L_2))$ the divergence goes only as $1/a$ and it extrapolates - as is the case for Φ_2^{QCD} and Σ_m - quadratically in a). The $1/m$ correction to M_b are then

$$\begin{aligned}
 M_b^{(1)} &= M_b^{(1a)} + M_b^{(1b)} \\
 M_b^{(1a)} &= -\frac{\sigma_2^{\text{kin}}(\bar{g}^2(L_1))\Phi_1(L_1, M_b^{\text{stat}})}{S L_2} = -25(13) \text{ MeV} \\
 M_b^{(1b)} &= -\frac{(E^{\text{kin}} - \Gamma_1^{\text{kin}}(L_2))\Phi_1(L_2, M)}{S R_1^{\text{kin}}} = -24(32) \text{ MeV}
 \end{aligned} \tag{15}$$

and in the $\overline{\text{MS}}$ scheme:

$$\begin{aligned}
 m_b(m_b) &= m_b^{\text{stat}} + m_b^{(1)} \\
 m_b^{\text{stat}} &= 4.347(48) \text{ GeV}, \quad m_b^{(1)} = -0.027(22) \text{ GeV}.
 \end{aligned} \tag{16}$$

which agrees with the value given by the Particle Data Group, despite the quenched approximation.

An alternative computation has been performed by using observables defined in terms of $f_A(x_0)$. In this case there is one additional combination of bare couplings of order $1/m$ that has to be determined and thus more step scaling functions are required. The final result is found to agree up to (small) $O(1/m^2)$ corrections.

We thank Rainer Sommer for a critical reading of the manuscript. Financial support by an EIF Marie Curie fellowship of the European Community's Sixth Framework Programme under contract number MEIF-CT-2006-040458 is acknowledged.

References

- [1] E. Eichten, *Nucl. Phys. Proc. Suppl.* **4** (1998) 170; E. Eichten and B. Hill, *Phys. Lett. B* **234** (1990) 511.
- [2] J. Heitger and R. Sommer [ALPHA collaboration], *Nucl. Phys. Proc. Suppl.* **106** (2002) 358
- [3] J. Heitger and R. Sommer [ALPHA Collaboration], *JHEP* **0402** (2004) 022
- [4] M. Della Morte, N. Garron, M. Papinutto and R. Sommer [ALPHA Collaboration], *JHEP* **0701** (2007) 007
- [5] M. Lüscher, P. Weisz and U. Wolff, *Nucl. Phys. B* **359** (1991) 221.
- [6] M. Della Morte, A. Shindler and R. Sommer, *JHEP* **0508** (2005) 051.