AN ELEMENTARY INEQUALITY

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ABSTRACT. An elementary inequality is proved in this note.

1. INTRODUCTION AND RESULTS.

The following theorems present a series of closely related inequalities. The form that was found originally is given first.

THEOREM 1. If $a, b \geq 0$ and n is a positive integer, then

$$\frac{a^n + a^{n-1} b + a^{n-2} b^2 + \ldots + b^n}{n + 1} \geq \left(\frac{a + b}{2}\right)^n \quad , \qquad (1.1)$$

PROOF. Let $[x]$ be, as usual, the integer part of x , and let the symbol $\displaystyle\sum_{0}^{[n/2]}{}^{*}$ be defined by :

$$\sum_{0}^{[n/2]}{}_{*} \; x_i \; = \; \begin{cases} x_0 + x_1 + \ldots + x_{[n/2]} & \text{if } n \text{ is odd} \\[2ex] x_0 + x_1 + \ldots + x_{[n/2]-1} + \dfrac{1}{2} x_{[n/2]} & \text{if } n \text{ is even.} \end{cases}$$

Assuming, without loss of generality, that $a \geq b$, we divide through by a^n in (1.1), and set $x = b/a$, obtaining :

$$\frac{1 + x + x^2 + \ldots + x^n}{n + 1} \; \geq \; (\frac{1 + x}{2})^n \; ; \quad 0 \leq x \leq 1 \; , \qquad (1.2)$$

Now

$$(1 + x)^n \; = \; \sum_{0}^{[n/2]}{}_{*} \; \binom{n}{i} \; (x^i + x^{n-i})$$

and

$$1 + x + x^2 + \ldots + x^n \; = \; \sum_{0}^{[n/2]}{}_{*} \; (x^i + x^{n-i}) \; ;$$

so that we may rewrite (1.2) as :

$$\sum_{0}^{[n/2]}{}_{*} \; (\frac{1}{n+1} - \frac{1}{2^n} \binom{n}{i}) \; (x^i + x^{n-i}) \geq 0 \; ; \quad 0 \leq x \leq 1 \; , \qquad (1.3)$$

We now note that $1 + x^n \geq x + x^{n-1} \geq x^2 + x^{n-2} \geq \ldots \geq x^{[n/2]} + x^{n-[n/2]}$. This is an example of the rearrangements inequality ([1], p.261), since we can write

$$x^k + x^{n-k} \; = \; x^{k+1/2} x^{-1/2} + x^{n-k-1/2} x^{1/2}$$

and

$$x^{k+1} + x^{n-k-1} \; = \; x^{k+1/2} x^{1/2} + x^{n-k-1/2} x^{-1/2} \; ;$$

and for $0 \leq k \leq [n/2] - 1$, $x^{k+1/2}$ and $x^{n-k-1/2}$ are in the same order as $x^{-1/2}$ and $x^{1/2}$. It follows that if we set

$$a_i = \begin{cases} x^i + x^{n-i} & \text{if } i < n/2 \\[2mm] \dfrac{x^i + x^{n-i}}{2} & \text{if } n \text{ is even and } i = n/2 \ , \end{cases}$$

then $a_0 \geq a_1 \geq \ldots \geq a_{[n/2]}$. If we also set

$$b_i = \frac{1}{n+1} - \frac{1}{2^n} \binom{n}{i} \ , \quad i = 1, 2, \ldots, [n/2] \ ,$$

it is clear that $b_1 \geq b_2 \geq \ldots \geq b_{[n/2]}$; and since (1.1) - (1.3) are equalities when $a = b$, the sum of the b_i is zero. (1.3) is then an immediate consequence of the following obvious lemma, which is also related to the rearrangements inequality.

LEMMA. If k is a positive integer and $a_1 \geq a_2 \geq \ldots \geq a_k \geq 0$, and $b_1 \geq b_2 \geq \ldots \geq b_k$ and $b_1 + b_2 + \ldots + b_k = 0$, then $a_1 b_1 + a_2 b_2 + \ldots + a_k b_k \geq 0$.

COROLLARIES.

$$x^n - 1 \geq n(x-1) \left(\frac{x+1}{2}\right)^{n-1} ; \quad x \geq 1 \text{ and } n = 0,1,2, \ldots \quad . \quad (1.4)$$

$$x^n - 1 \leq n(x-1) \left(\frac{x+1}{2}\right)^{n-1} ; \quad 0 \leq x \leq 1 \text{ and } n = 0,1,2,\ldots \quad . \quad (1.5)$$

Inequality (1.4) is a sharpening of Hardy, Littlewood, and Polya's (2.15.3) for part of the latter's range, while (1.5) is complementary to (2.15.3) for another part of its range. These are immediate consequences of (1.2), which is valid for all $x \geq 0$ by Theorem 1. (The case $n = 0$ is not a consequence of Theorem 1. but is obvious; similarly (1.2) holds for $n = 0$ if we interpret $1 + x + x^2 + \ldots + x^n$ as 1 for that case.)

Setting $x = 1+t$ in (1.4) and (1.5) gives us the inequalities

$$(1+t)^n \geq 1+nt\left(1+\frac{t}{2}\right)^{n-1} ; \quad t \geq 0 \text{ and } n = 0,1,2,\ldots \quad . \quad (1.6)$$

and

$$(1+t)^n \leq 1+nt\left(1+\frac{t}{2}\right)^{n-1} ; \quad -1 \leq t \leq 0 \text{ and } n = 0,1,2\ldots \quad . \quad (1.7)$$

Putting $-t$ in place of t gives an alternative form:

$$(1-t)^n \leq 1-nt(1-\frac{t}{2})^{n-1} \quad ; \quad 0 \leq t \leq 1 \quad \text{and} \quad n = 0,1,2,\ldots \quad . \quad (1.8)$$

Replacing t by 1/t in (1.6) gives us

$$(t+1)^n \geq t^n + n(t+\frac{1}{2})^{n-1} \quad ; \quad t \geq 0 \quad \text{and} \quad n = 0,1,2,\ldots \quad . \quad (1.9)$$

This is better than (1.6) for t > 1   though not as good for t < 1 (for n > 2).
Similarly, from (1.8) we obtain :

$$(t-1)^n \leq t^n - n(t-\frac{1}{2})^{n-1} \quad ; \quad t \geq 1 \quad \text{and} \quad n = 0,1,2,\ldots \quad . \quad (1.10)$$

(1.6) is a sharpening of the observation that compound interest beats simple
interest: a period rate t, compounded for n periods, is better than simple
interest at the rate $t(1+t/2)^{n-1}$.

(1.6) – (1.10) were obtained from (1.4) or (1.5) by invertible bilinear changes
of variable; similarly each of (1.4) and (1.5) may be obtained from the other by
replacing x by 1/x. Thus (1.4)-(1.10) are equivalent, and equivalent to Theorem 1.
Another bilinear transformation – setting x = (1+t)/(1-t) in (1.4) – gives us

$$(1+t)^n - (1-t)^n \geq 2nt \quad ; \quad 0 \leq t \leq 1 \quad \text{and} \quad n = 0,1,2,\ldots \quad . \quad (1.11)$$

which is obvious! This provides a quick alternate proof of Theorem 1., but one
which does not show the connection with the rearrangements inequality.

The form of inequalities (1.4) – (1.11) suggests consideration of non-integral
n. The inequalities are equivalent in this precise sense; for any given n, one of
these inequalities holds for its stated range of values of the other variable
(x or t) if and only if all the other inequalities hold, for that n, for their
stated ranges of the other variable. Thus for each n we may choose which of (1.4)
– (1.11) to study, at our convenience.

For 0 < n < 1 (and 0 < t < 1) the binomial expansion of $(1+t)^n - (1-t)^n$ is

$$2n + 2 \left\{ \binom{n}{3} t^3 + \binom{n}{5} t^5 + \ldots \right\},$$

and each of the coefficients $\binom{n}{3}$, $\binom{n}{5}$,... is positive. It follows that (1.11)

holds for n between 0 and 1.

For n between 1 and 2, all of $\binom{n}{3}$, $\binom{n}{5}$, ... are negative. So for

$1 < n < 2$ (and in fact $1 \le n \le 2$), (1.11) holds with the inequality <u>reversed</u>.

For non-integral n > 2 we look at (1.6). Set

$$f(t,n) = \frac{1 + nt(1+\frac{t}{2})^{n-1}}{(1+t)^n} \quad ;$$

$f(0,n) \equiv 1$. To show that (1.6) holds, it is sufficient to show that

$$\frac{\partial f}{\partial t}(t,n) = \frac{n}{(1+t)^{n+1}} \left\{ (1 + \frac{2-n}{2}t)(1 + \frac{t}{2})^{n-2} - 1 \right\}$$

is ≤ 0 for $t \ge 0$, $n \ge 2$. Setting $s = n-2$, the quantity in brackets is

$$(1 - \frac{st}{2})(1 + \frac{t}{2})^s - 1.$$

This is certainly negative when $1 - st/2$ is negative; when the latter is non-

negative, the whole quantity is

$$\le (1 - \frac{st}{2})(e^{\frac{st}{2}}) - 1 \le 0 .$$

Summing up we have :

THEOREM 2. Each of the inequalities (1.4) - (1.11) holds also for non-integral

n, when $0 < n < 1$ and when n > 2: for $1 < n < 2$ each of (1.4) - (1.11) holds with

the direction of the inequality reversed.

REFERENCES

1. Hardy, G.H., J.E. Littlewood and G. Polya, Inequalities, Cambridge University Press (1959).