

Yang–Mills theory for semidirect products $\mathfrak{G} \ltimes \mathfrak{g}^*$ and its instantons

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Abstract Yang–Mills theory with a symmetry algebra that is the semidirect product $\mathfrak{h} \ltimes \mathfrak{h}^*$ defined by the coadjoint action of a Lie algebra \mathfrak{h} on its dual \mathfrak{h}^* is studied. The gauge group is the semidirect product $G_{\mathfrak{h}} \ltimes \mathfrak{h}^*$, a noncompact group given by the coadjoint action on \mathfrak{h}^* of the Lie group $G_{\mathfrak{h}}$ of \mathfrak{h} . For \mathfrak{h} simple, a method to construct the self–antiself dual instantons of the theory and their gauge nonequivalent deformations is presented. Every $G_{\mathfrak{h}} \ltimes \mathfrak{h}^*$ instanton has an embedded $G_{\mathfrak{h}}$ instanton with the same instanton charge, in terms of which the construction is realized. As an example, $\mathfrak{h} = \mathfrak{su}(2)$ and instanton charge one is considered. The gauge group is in this case $SU(2) \ltimes \mathbf{R}^3$. Explicit expressions for the selfdual connection, the zero modes and the metric and complex structures of the moduli space are given.

1 Introduction

Motivated by an interest in finding new gauge configurations, we consider Yang–Mills theory with a symmetry algebra that is the classical double of a real Lie algebra and study its self–antiself dual solutions. By the classical double of a real Lie algebra \mathfrak{h} we understand in this paper the semidirect product $\mathfrak{h} \ltimes \mathfrak{h}^*$ defined by the action of \mathfrak{h} on its dual \mathfrak{h}^* via the coadjoint representation. Our concern here is Yang–Mills theory with gauge group the simply connected Lie group $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ obtained from $\mathfrak{h} \ltimes \mathfrak{h}^*$ by exponentiation.

The group $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ admits several descriptions. From a geometric point of view, it is the cotangent bundle of the Lie group $G_{\mathfrak{h}}$ of \mathfrak{h} . Algebraically, it can be regarded as the semidirect product $G_{\mathfrak{h}} \ltimes G_{\mathfrak{h}^*}$ of $G_{\mathfrak{h}}$ with the Lie group $G_{\mathfrak{h}^*}$ of \mathfrak{h}^* . The cotangent bundle construction is standard in symplectic mechanics. The semidirect product approach is not new either in the physics literature. The Chern–Simons formulation of three-dimensional gravity [1, 2] is probably the most cele-

brated example of a gauge theory with a gauge group of this type. In that case, \mathfrak{h} is the Lorentz algebra in three dimensions, \mathfrak{h}^* is the algebra of three-dimensional translations, $\mathfrak{h} \ltimes \mathfrak{h}^*$ is the algebra of isometries $\mathfrak{iso}(1, 2)$, and $G_{\mathfrak{h}} \ltimes G_{\mathfrak{h}^*}$ is the isometry group $ISO(1, 2)$. Other forms of semidirect products, some involving finite groups, have been employed in various scenarios, including quantization of monopoles with nonabelian magnetic charges [3], neutrino mixing [4, 5], and hypercharge quantization [6, 7].

An important property of $\mathfrak{h} \ltimes \mathfrak{h}^*$ is that it is a metric Lie algebra. This means that it admits an invariant, nondegenerate, symmetric, bilinear form, called metric, that takes values in \mathbf{R} . The relevance of this property comes from the observation that if \mathfrak{g} is a metric Lie algebra and Ω is a metric on it, it is possible to formulate Yang–Mills theory with gauge group the Lie group $G_{\mathfrak{g}}$ of \mathfrak{g} . To do this on a d -dimensional spacetime manifold, introduce a one-form gauge field κ and its two-form field strength $K = d\kappa + \kappa \wedge \kappa$, both valued in \mathfrak{g} , and consider the Yang–Mills d -form $\mathcal{L}_{\text{YM}} = \Omega(K, \star K)$. Nondegeneracy of Ω ensures that \mathcal{L}_{YM} contains a kinetic term for the gauge field κ , while invariance of Ω guarantees that \mathcal{L}_{YM} is invariant under $G_{\mathfrak{g}}$ gauge transformations. By considering the classical double $\mathfrak{h} \ltimes \mathfrak{h}^*$, it is thus possible to define a gauge theory even if \mathfrak{h} is not metric. Similarly, four-dimensional topological field theory and three-dimensional Chern–Simons theory can be considered, with Lagrangians given by $\Omega(K, K)$ and $\Omega(\kappa, d\kappa + \frac{2}{3}\kappa \wedge \kappa)$.

In view of this, it seems natural to ask how many different real metric Lie algebras there are. The list of them is exhausted by (i) reductive algebras, (ii) classical doubles, and (iii) double extensions. Reductive algebras are direct sums of semisimple Lie algebras and the Abelian algebra. They are the Lie algebras of the compact Lie groups, and their gauge theories have been the subject of continuous study over the last 40 years. Less is known about the gauge theories for algebras of type (ii) and (iii). Yang–Mills theory for classical doubles is the object of this paper. As regards double extensions, they are obtained by a nontrivial generalization [8] due to

Dedicated to Ramón F. Alvarez-Estrada on occasion of his 70th birthday.

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Medina and Revoy of the semidirect product that defines the classical double. In fact, a classical double can be regarded as a double extension of the trivial algebra. These authors proved a structure theorem that states (a) that every real metric Lie algebra is an orthogonal sum of indecomposable real metric Lie algebras, and (b) that every indecomposable real metric Lie algebra is simple, one-dimensional or the double extension of a metric Lie algebra by either a simple or a one-dimensional Lie algebra. A discussion of the theorem can be found in Ref. [9]. Some Wess–Zumino–Witten models and gauge theories for double extensions have been considered in Refs. [9–12].

Let us center on the case of interest here, gauge theories with symmetry algebra $\mathfrak{h} \ltimes \mathfrak{h}^*$. In these theories, the gauge field κ takes values in $\mathfrak{h} \ltimes \mathfrak{h}^*$ and has nonzero projections onto \mathfrak{h} and \mathfrak{h}^* . New degrees of freedom are thus introduced when \mathfrak{h} is replaced with $\mathfrak{h} \ltimes \mathfrak{h}^*$. In Sect. 2, it is shown, however, that the homology and homotopy invariants for the group $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ are the same as for $G_{\mathfrak{h}}$. This has two implications. Homotopically nontrivial solutions for $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ gauge theory exist if they do for $G_{\mathfrak{h}}$ gauge theory, and the \mathfrak{h}^* -component of the gauge field κ does not contribute to the theory’s invariants. Here we study these questions. It will be shown that $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ instantons indeed have the same instanton charge as their embedded $G_{\mathfrak{h}}$ instantons, but larger moduli spaces. A method to construct $G_{\mathfrak{h} \ltimes \mathfrak{h}^*} \cong T^*G_{\mathfrak{h}} \cong G_{\mathfrak{h}} \ltimes G_{\mathfrak{h}^*}$ instantons and their moduli spaces from those of $G_{\mathfrak{h}}$ instantons will be presented.

This paper is organized as follows. Section 2 is dedicated to review the definition and basic properties of $\mathfrak{h} \ltimes \mathfrak{h}^*$ and its Lie group $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$. The Lagrangian and field content of $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ Yang–Mills theory are discussed in Sect. 3. The construction of self–antiself dual $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ instantons in terms of the embedded $G_{\mathfrak{h}}$ instantons is presented in Sect. 4. This construction is explicitly realized for $\mathfrak{h} = \mathfrak{su}(2)$ and instanton charge one in Sect. 5, where expressions for the gauge field, the zero modes and the metric and complex structures of the moduli space are presented. In Sect. 6 we collect our final comments.

2 The classical double of a Lie algebra and its Lie group

Let us start by reviewing the construction of the classical double as a semidirect product. Assume that \mathfrak{h} is a real Lie algebra of dimension n with basis $\{T_i\}$ satisfying $[T_i, T_j] = f_{ij}^k T_k$. Denote by \mathfrak{h}^* its dual vector space, and take for \mathfrak{h}^* the canonical dual basis $\{Z^i\}$, defined by $Z^i(T_j) = \delta^i_j$. Form the vector space $\mathfrak{h} \oplus \mathfrak{h}^*$. Its elements are pairs (T, Z) , with T in \mathfrak{h} and Z in \mathfrak{h}^* , and as a basis on it one may take $\{(0, T_i), (0, Z^j)\}$. Consider the semidirect product $\mathfrak{h} \ltimes \mathfrak{h}^*$ that results from acting with \mathfrak{h} on \mathfrak{h}^* via the coadjoint representation. For T in \mathfrak{h} , the coadjoint representation $\text{ad}_T^* : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$

associates $Z \mapsto \text{ad}_T^* Z$, with action on T' in \mathfrak{h} given by $\text{ad}_T^* Z(T') = Z(\text{ad}_T T') = Z([T, T'])$. This results in a Lie algebra of dimension $2n$ with Lie bracket

$$[(T, Z), (T', Z')] = ([T, T'], -\text{ad}_T^* Z' + \text{ad}_{T'}^* Z). \tag{2.1}$$

For the bases $\{T_i\}$ and $\{Z^j\}$, one has $\text{ad}_{T_i}^* Z^j(T_k) = f_{ik}^j$, so the Lie bracket becomes

$$[T_i, T_j] = f_{ij}^k T_k, \quad [T_i, Z^j] = -f_{ik}^j Z^k, \quad [Z^i, Z^j] = 0. \tag{2.2}$$

Here we have introduced the notation, which we will often use, $T_i + Z^j := (T_i, Z^j)$, so that $T_i := (T_i, 0)$ and $Z^j := (0, Z^j)$. The semidirect product $\mathfrak{h} \ltimes \mathfrak{h}^*$ is a particular type of Drinfeld double [13], namely the one specified by the trivial bialgebra structure on \mathfrak{h} .

Let us also recall that a bilinear symmetric form Ω on a Lie algebra is invariant if, for all A, B , and C in the algebra, it satisfies

$$\Omega(A, [B, C]) = \Omega([A, B], C). \tag{2.3}$$

This in turn implies invariance under the group adjoint action, or more precisely,

$$\Omega(e^{-C} A e^C, e^{-C} B e^C) = \Omega(A, B). \tag{2.4}$$

Coming back to $\mathfrak{h} \ltimes \mathfrak{h}^*$, it is very easy to see that

$$\Omega = \begin{matrix} & T_j & Z^j \\ \begin{matrix} T_i \\ Z^i \end{matrix} & \begin{pmatrix} \omega_{ij} & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix} \end{matrix} \tag{2.5}$$

is nondegenerate and solves condition (2.3) for the commutators (2.2), where $\omega_{ij} = \omega(T_i, T_j)$ are the components of an arbitrary symmetric, possibly degenerate, invariant, bilinear form ω on \mathfrak{h} . Hence $\mathfrak{h} \ltimes \mathfrak{h}^*$ is a real metric Lie algebra, even if \mathfrak{h} is not, and Ω is a metric on it.

The algebras \mathfrak{h} , \mathfrak{h}^* and $\mathfrak{h} \ltimes \mathfrak{h}^*$ define through exponentiation simply connected Lie groups that we denote by $G_{\mathfrak{h}}$, $G_{\mathfrak{h}^*}$ and $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$. From a geometric point of view, $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ is the cotangent bundle $T^*G_{\mathfrak{h}}$ of $G_{\mathfrak{h}}$, a standard construction in geometry. $T^*G_{\mathfrak{h}}$ is in turn isomorphic to the semidirect product $G_{\mathfrak{h}} \ltimes \mathfrak{h}^*$, where $G_{\mathfrak{h}}$ acts on \mathfrak{h}^* by the coadjoint action. For h in $G_{\mathfrak{h}}$, the coadjoint representation $\text{Ad}_h^* : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ maps Z to $\text{Ad}_h^* Z$, whose action on T' in \mathfrak{h} is given by $\text{Ad}_h^* Z(T') = Z(\text{Ad}_h T') = Z(h^{-1} T' h)$. The elements of $G_{\mathfrak{h}} \ltimes \mathfrak{h}^*$ are pairs (h, Z) with product law $(h_1, Z_1)(h_2, Z_2) = (h_1 h_2, \text{Ad}_{h_2}^* Z_1 + Z_2)$. Since h in $G_{\mathfrak{h}}$ can be uniquely written as $h = e^T$, with T in \mathfrak{h} , the derivative of Ad_h^* is the coadjoint action ad_T^* used to construct the semidirect product $\mathfrak{h} \ltimes \mathfrak{h}^*$. As a group, \mathfrak{h}^* is Abelian, noncompact and homeomorphic to \mathbf{R}^n , and $\{0\} \times \mathfrak{h}^*$ is a normal subgroup. For example, for $\mathfrak{h} = \mathfrak{su}(2)$, this gives $G_{\mathfrak{h} \ltimes \mathfrak{h}^*} \cong SU(2) \times \mathbf{R}^3$.

One may also adopt the following approach to $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$. Consider the Cartesian product $G_{\mathfrak{h}} \times G_{\mathfrak{h}^*}$, whose elements

are pairs (h, n) that can be uniquely written as (e^T, e^Z) , for some T in \mathfrak{h} and some Z in \mathfrak{h}^* . The homomorphism $\varphi: G_{\mathfrak{h}} \rightarrow \text{Aut}(G_{\mathfrak{h}^*})$, where $\varphi(h) = \varphi_h$ acts on $G_{\mathfrak{h}^*}$ by conjugation, $\varphi_h(n) = h^{-1}nh$, defines a group structure on $G_{\mathfrak{h}} \times G_{\mathfrak{h}^*}$. This results in the semidirect product $G_{\mathfrak{h}} \ltimes G_{\mathfrak{h}^*}$, with group law $(h_1, n_1)(h_2, n_2) = (h_1h_2, (h_2^{-1}n_1h_2)n_2)$ and Lie algebra $\mathfrak{h} \ltimes \mathfrak{h}^*$. As a group, $G_{\mathfrak{h}^*}$ is Abelian, noncompact, and homeomorphic to \mathbf{R}_+^n . The map $[0, 1] \times (G_{\mathfrak{h}} \ltimes G_{\mathfrak{h}^*}) \rightarrow G_{\mathfrak{h}} \times \{0\}$, given by $(t, (h, n)) \mapsto (h, tn)$, is then a homotopy. This means that $G_{\mathfrak{h}} \ltimes G_{\mathfrak{h}^*}$ and $G_{\mathfrak{h}} \times \{0\}$ are homotopically equivalent, hence have the same homology and homotopy invariants. In particular, they have the same third homotopy group. For the elements of $G_{\mathfrak{h}} \times G_{\mathfrak{h}^*}$ we will use the notation $g = hn = (h, n)$. It is clear that $G_{\mathfrak{h}} \ltimes G_{\mathfrak{h}^*}$ and $G_{\mathfrak{h}} \ltimes \mathfrak{h}^*$ are isomorphic.

We finish this section with two comments, one on representations and one on deformations.

Comment 1. Given any p -dimensional matrix representation of \mathfrak{h} that associates to its basis $\{T_i\}$ matrices $\{\mathbf{M}_i\}$ with $[\mathbf{M}_i, \mathbf{M}_j] = f_{ij}^k \mathbf{M}_k$, it is very easy to see that

$$\rho(T_i, 0) = \begin{pmatrix} \mathbf{M}_i & | & 0 \\ \hline 0 & | & \mathbf{M}_i \end{pmatrix}, \quad \rho(0, Z_i) = \begin{pmatrix} 0 & | & 0 \\ \hline \mathbf{M}_i & | & 0 \end{pmatrix} \quad (2.6)$$

is a $2p$ -dimensional matrix representation of $\mathfrak{h} \ltimes \mathfrak{h}^*$. In the adjoint representation of \mathfrak{h} , the matrices $\{\mathbf{M}_i\}$ are $n \times n$ and have entries $(\mathbf{M}_i^{\text{ad}})_{j^k} = -f_{ij}^k$. It is straightforward to check that ρ above is then the adjoint representation of $\mathfrak{h} \ltimes \mathfrak{h}^*$. Representations other than (2.6) are possible. An example is the following. Let \mathbf{e}_i be the unit column vector in \mathbf{R}^n , with components $(\mathbf{e}_i)_j = \delta_{ij}$. Some simple algebra shows that the matrices

$$\rho'(T_i, 0) = \begin{pmatrix} \mathbf{M}_i^{\text{ad}} & | & 0 \\ \hline 0 & | & 0 \end{pmatrix}, \quad \rho'(0, Z_i) = \begin{pmatrix} 0 & | & \mathbf{e}_i \\ \hline 0 & | & 0 \end{pmatrix} \quad (2.7)$$

form a $(n + 1)$ -dimensional representation of $\mathfrak{h} \ltimes \mathfrak{h}^*$. Note finally that every matrix representation of $\mathfrak{h} \ltimes \mathfrak{h}^*$ induces a matrix representation of $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ via matrix exponentiation.

Comment 2. Assume that the algebra \mathfrak{h} is metric, so that ω_{ij} in Eq. (2.5) can be taken as the components of a metric. One may use ω_{ij} and its inverse ω^{ij} , given by $\omega^{ik}\omega_{kj} = \delta^i_j$, to lower and raise indices in the structure constants f_{ij}^k . This yields completely antisymmetric structure constants

$$f_{ijk} = f_{ij}^l \omega_{lk} = \omega([T_i, T_j], T_k), \quad f_{ijk} = -f_{jik} = f_{kij}. \quad (2.8)$$

Perform in \mathfrak{h}^* the change of generators $\{Z^i\} \rightarrow \{Z_i\}$, with $Z_i = \omega_{ik} Z^k$. This gives

$$[T_i, T_j] = f_{ij}^k T_k, \quad [T_i, Z_j] = f_{ij}^k Z_k, \quad [Z_i, Z_j] = 0. \quad (2.9)$$

Consider the commutators

$$[T_i, T_j] = f_{ij}^k T_k, \quad [T_i, Z_j] = f_{ij}^k Z_k, \quad [Z_i, Z_j] = s^2 f_{ij}^k T_k, \quad (2.10)$$

where s in $[Z_i, Z_j]$ is an arbitrary real parameter. These commutators satisfy the Jacobi identity for all s and reduce to the Lie bracket (2.9) of the classical double when $s \rightarrow 0$. The vector space $\mathfrak{h} \oplus \mathfrak{h}^*$ with the Lie bracket (2.10) is thus a Lie algebra, call it $\mathfrak{h} \ltimes_s \mathfrak{h}^*$, and a deformation of $\mathfrak{h} \ltimes \mathfrak{h}^*$ with deformation parameter s . The algebra $\mathfrak{h} \ltimes_s \mathfrak{h}^*$ is metric since it admits the metric

$$\Omega_s = \begin{matrix} & T_j & Z_j \\ \begin{matrix} T_i \\ Z_i \end{matrix} & \begin{pmatrix} \omega_{ij} & \omega_{ij} \\ \omega_{ij} & s^2 \omega_{ij} \end{pmatrix} \end{matrix}. \quad (2.11)$$

In $\mathfrak{h} \ltimes_s \mathfrak{h}^*$ introduce generators $\{X_i, Y_j\}$ given by

$$X_i = \frac{1}{2} \left(T_i + \frac{1}{s} Z_i \right), \quad Y_j = \frac{1}{2} \left(T_j - \frac{1}{s} Z_j \right). \quad (2.12)$$

In the new basis, the Lie bracket (2.10) becomes

$$[X_i, X_j] = f_{ij}^k X_k, \quad [X_i, Y_j] = 0, \quad [Y_i, Y_j] = f_{ij}^k Y_k, \quad (2.13)$$

and the metric Ω_s takes the diagonal form

$$\Omega_s = \begin{matrix} & X_j & Y_j \\ \begin{matrix} X_i \\ Y_i \end{matrix} & \begin{pmatrix} \frac{1}{2} \left(1 + \frac{1}{s}\right) \omega_{ij} & 0 \\ 0 & \frac{1}{2} \left(1 - \frac{1}{s}\right) \omega_{ij} \end{pmatrix} \end{matrix}. \quad (2.14)$$

The deformed algebra $\mathfrak{h} \ltimes_s \mathfrak{h}^*$ is thus the direct sum $\mathfrak{h} \oplus \mathfrak{h}$ and its simply connected Lie group $G_{\mathfrak{h} \ltimes_s \mathfrak{h}^*}$ becomes the direct product $G_{\mathfrak{h}} \times G_{\mathfrak{h}}$.

3 The gauge theory and its field content

Our interest here is Yang–Mills theory with gauge group $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$. Consider a spacetime manifold M_d of dimension d equipped with a metric γ . Greek letters μ, ν, \dots will label coordinate indices $1, 2, \dots, d$ in a local chart $\{x^\mu\}$. In such a chart, $\gamma_{\mu\nu}$ will denote the metric components and $\gamma^{\mu\nu}$ the components of the inverse metric. For an r -form ζ we will adopt the normalization $\zeta = \frac{1}{r!} \zeta_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$. Indices will be raised and lowered using $\gamma^{\mu\nu}$ and $\gamma_{\mu\nu}$ metric. For the commutator of an r -form ζ with an s -form ξ , both taking values in $\mathfrak{h} \ltimes \mathfrak{h}^*$, we will use $[\zeta, \xi] = \zeta \wedge \xi - (-)^{rs} \xi \wedge \zeta$.

The gauge field is a connection one-form κ on M_d that takes values in $\mathfrak{h} \ltimes \mathfrak{h}^*$. The connection defines a covariant derivative d_κ , whose action on an $(\mathfrak{h} \ltimes \mathfrak{h}^*)$ -valued r -form ζ is given by $d_\kappa \zeta = d\zeta + [\kappa, \zeta]$, and a curvature two-form or field strength

$$K = d\kappa + \frac{1}{2} [\kappa, \kappa]. \quad (3.1)$$

The curvature takes values in $\mathfrak{h} \ltimes \mathfrak{h}^*$ and satisfies the Bianchi identity $d_\kappa K = 0$. Gauge transformations

$$\kappa \rightarrow \kappa' = g^{-1} dg + g^{-1} \kappa g \tag{3.2}$$

are implemented by $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ valued functions $g(x)$. Under such transformations, the curvature changes as

$$K \rightarrow K' = g^{-1} K g. \tag{3.3}$$

As usual, infinitesimal gauge transformation are obtained by expanding $g = e^T e^Z$ in powers of T and Z and keeping terms up to order one. With $\Lambda := T + Z$, they read

$$\kappa \rightarrow \kappa' = \kappa + d_\kappa \Lambda, \tag{3.4}$$

$$K \rightarrow K' = K + [K, \Lambda]. \tag{3.5}$$

Consider the d -form $\Omega(K, \star K)$, where $\star K$ is the Hodge dual of K and Ω is an invariant metric on $\mathfrak{h} \ltimes \mathfrak{h}^*$. The transformation law (3.3) for K , the observation that any g can be written as $g = e^T e^Z$, and the invariance condition (2.4) imply that $\Omega(K, \star K)$ remains unchanged under gauge transformations. The functional

$$\begin{aligned} S_{\text{YM}} &= \frac{1}{8\pi^2} \int_{M_d} \Omega(K, \star K) \\ &= \frac{1}{16\pi^2} \int_{M_d} \sqrt{\gamma} d^d x \Omega(K^{\mu\nu}, K_{\mu\nu}) \end{aligned} \tag{3.6}$$

is thus gauge invariant and can be taken as the classical action of $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$ Yang–Mills theory. Variation of S_{YM} with respect to κ gives for the field equation

$$d_\kappa \star K = 0. \tag{3.7}$$

For $d \geq 4$, it is also possible to consider the gauge invariant four-form $\Omega(K, K)$. Since $\Omega(K, K)$ does not require a metric, it can be regarded as the Lagrangian of a topological field theory in four dimensions, the classical action being

$$S_P = \frac{1}{8\pi^2} \int_{M_4} \Omega(K, K). \tag{3.8}$$

The form $\Omega(K, K)$ is the first Pontrjagin class of the principal bundle over M_d with structure group $G_{\mathfrak{h} \ltimes \mathfrak{h}^*}$, and the exterior derivative of a Chern–Simons three-form. That is, $\Omega(K, K) = d\mathcal{L}_{\text{CS}}(\kappa)$ with

$$\mathcal{L}_{\text{CS}}(\kappa) = \Omega(\kappa, d\kappa + \frac{2}{3} \kappa \wedge \kappa). \tag{3.9}$$

In analogy with the case of semisimple Lie algebras, one may formulate Chern–Simons field theory on a three-dimensional manifold M_3 with the classical action

$$S_{\text{CS}} = \frac{1}{8\pi^2} \int_{M_3} \mathcal{L}_{\text{CS}}(\kappa). \tag{3.10}$$

The connection κ and the curvature K can be expanded in the Lie algebra basis $\{T_i, Z^j\}$ as

$$\kappa = \alpha + \beta, \quad \alpha := \alpha^i T_i, \quad \beta := \beta_i Z^i, \tag{3.11}$$

$$K = F + B, \quad F := F^i T_i, \quad B := B_i Z^i, \tag{3.12}$$

where α^i and β_i are one-forms on M_d , and F^i and B_i are two-forms. Substitution in Eq. (3.1) gives

$$F = d\alpha + \frac{1}{2} [\alpha, \alpha] \Leftrightarrow F^i = d\alpha^i + \frac{1}{2} f_{jk}^i \alpha^j \wedge \alpha^k, \tag{3.13}$$

$$B = d\beta + [\alpha, \beta] \Leftrightarrow B_i = d\beta_i + f_{ij}^k \alpha^j \wedge \beta_k. \tag{3.14}$$

In infinitesimal form, gauge transformations read

$$\alpha \rightarrow \alpha' = \alpha + dT + [\alpha, T], \tag{3.15}$$

$$\beta \rightarrow \beta' = \beta + dZ + [\alpha, Z] + [\beta, T], \tag{3.16}$$

whereas for the field strength they become

$$F \rightarrow F' = F + [F, T], \tag{3.17}$$

$$B \rightarrow B' = B + [B, T] + [F, Z]. \tag{3.18}$$

The Bianchi identity $d_\kappa K = 0$ unfolds in the two identities

$$dF + [\alpha, F] = 0, \tag{3.19}$$

$$dB + [\alpha, B] + [\beta, F] = 0, \tag{3.20}$$

and the field equation $d_\kappa \star K = 0$ splits in

$$d \star F + [\alpha, \star F] = 0, \tag{3.21}$$

$$d \star B + [\alpha, \star B] + [\beta, \star F] = 0. \tag{3.22}$$

There are a few observations that, despite their simplicity, are worth making. Firstly, the curvature F has the same dependence on α as results from gauging the algebra \mathfrak{h} . It is B that mixes α with β . Secondly, the Lagrangian $\Omega(K, \star K)$ has a kinetic term for all the field components α^i and β_i of the gauge field κ . Note in this regard that, for ω degenerate, $\omega(F, \star F)$ does not define a Yang–Mills Lagrangian since it does not contain a kinetic term for all the α^i . Thirdly, the field strength B , its Bianchi identity (3.20) and its field equation (3.22) are linear in β . And lastly, the field equations (3.21) and (3.22) do not depend on ω .

The Pontrjagin and Chern–Simons forms read

$$\Omega(K, K) = \omega(F, F) + 2 \Omega(F, B) \tag{3.23}$$

and

$$\mathcal{L}_{\text{CS}}(\kappa) = \mathcal{L}_{\text{CS}}(\alpha) + 2 \Omega(\beta, F) + d\Omega(\beta, \alpha). \tag{3.24}$$

The first term on the right hand side in Eq. (3.24) is the Chern–Simons three-form for α computed with the invariant

bilinear form ω ,

$$\mathcal{L}_{CS}(\alpha) = \omega \left(\alpha, d\alpha + \frac{2}{3} \alpha \wedge \alpha \right). \tag{3.25}$$

For \mathfrak{h} the Lorentz algebra in three dimensions, the metric Ω has the form in Eq. (2.5) and $\mathcal{L}_{CS}(\kappa)$ in Eq. (3.24) gives, for $\omega_{ij} = 0$, the Chern–Simons Lagrangian of three-dimensional gravity [1, 2], modulo an exact form.

4 Semidirect instantons: general analysis

Let us turn our attention to self–antiself dual instantons on \mathbf{R}^4 . They are described by connections κ_S that solve the equation $\star K = \pm K$, where the positive sign corresponds to self-duality and the negative sign to anti-selfduality. For such connections, the field equation reduces to the Bianchi identity; thus it is trivially satisfied, and $S_{YM}[\kappa_S] = S_P[\kappa_S]$. Since the Pontrjagin index $S_P[\kappa_S]$ is a homotopy invariant and homotopy invariants are the same as for $G_{\mathfrak{h}}$ gauge theory, one has

$$\begin{aligned} S_{YM} [G_{\mathfrak{h} \times \mathfrak{h}^*}; \kappa_S] &= \pm S_P [G_{\mathfrak{h} \times \mathfrak{h}^*}; \kappa_S] \\ &= \pm S_P [G_{\mathfrak{h}}; \alpha_S] = S_{YM} [G_{\mathfrak{h}}; \alpha_S]. \end{aligned} \tag{4.1}$$

Finiteness of the Yang–Mills action on the rightmost side of this equation requires the curvature \mathfrak{h} -component F_S to approach zero at the three-sphere S^3_∞ at infinity. This in turn demands α_S to approach a pure gauge configuration. That is, $\alpha_S \rightarrow h^{-1}dh$ at S^3_∞ for some h in $G_{\mathfrak{h}}$. Note that no boundary condition for β_S is needed. These arguments can be made more explicit by noting that $S_P[\kappa]$ is the integral over S^3_∞ of the Chern–Simons three-form $\mathcal{L}_{CS}(\kappa)$ in Eq. (3.24). For a connection $\kappa = (\alpha, \beta)$ that approaches $(\alpha_\infty = h^{-1}dh, \beta_\infty)$ at S^3_∞ , with β_∞ arbitrary, Eq. (3.24) and $F_\infty = 0$ imply that $S_P[G_{\mathfrak{h} \times \mathfrak{h}^*}; \kappa] = S_P[G_{\mathfrak{h}}; \alpha]$.

All in all, the instanton charge, call it N , and the boundary conditions for a self–antiself dual $G_{\mathfrak{h} \times \mathfrak{h}^*}$ instanton $\kappa_S = (\alpha_S, \beta_S)$ are specified by those of the embedded $G_{\mathfrak{h}}$ instanton,

$$N = S_P [G_{\mathfrak{h} \times \mathfrak{h}^*}; \alpha_S, \beta_S] = \frac{1}{8\pi^2} \int_{\mathbf{R}^4} \omega(F_S, F_S). \tag{4.2}$$

This implies in particular that β_S does not contribute to the instanton charge,

$$\frac{1}{8\pi^2} \int_{\mathbf{R}^4} \Omega(F_S, B_S) = 0. \tag{4.3}$$

The self–antiself duality equation $\star K = \pm K$ splits in

$$\star F = \pm F \Leftrightarrow \star \left(d\alpha + \frac{1}{2} [\alpha, \alpha] \right) = \pm \left(d\alpha + \frac{1}{2} [\alpha, \alpha] \right), \tag{4.4}$$

$$\star B = \pm B \Leftrightarrow \star \left(d\beta + [\alpha, \beta] \right) = \pm \left(d\beta + [\alpha, \beta] \right). \tag{4.5}$$

Equation (4.4) and the boundary condition $\alpha \rightarrow h^{-1}dh$ set a differential problem for α , whose solutions are the self–

antiself dual $G_{\mathfrak{h}}$ instantons. For every solution α_S , Eq. (4.5) becomes an homogeneous linear differential problem for β , with solution β_S . In what follows we present a method to find the most general solution β_S for a given α_S .

Take \mathfrak{h} to be simple and ω_{ij} in Eq. (2.5) a metric on \mathfrak{h} . This is the case of all self–antiself dual $G_{\mathfrak{h}}$ instantons known to date [14–24]. Introduce generators $Z_i = \omega_{ij} Z^j$. The commutation relations for $\{T_i, Z_j\}$ and the metric Ω take the form (2.9) and (2.11), with $s = 0$. Since any gauge field $\kappa' = (\alpha', \beta')$ obtained from a solution $\kappa_S = (\alpha_S, \beta_S)$ by a $G_{\mathfrak{h} \times \mathfrak{h}^*}$ gauge transformation is trivially a solution, we restrict our attention to gauge nonequivalent solutions. The space of all such solutions with instanton charge N is the moduli space $\mathcal{M}_N(G_{\mathfrak{h} \times \mathfrak{h}^*})$.

Standard arguments [25, 26] show that if κ_S is a solution to the self–antiself duality equation, $\kappa' = \kappa_S + \delta\kappa$ is a gauge nonequivalent solution if $\delta\kappa$ satisfies the equation

$$d_{\kappa_S} \delta\kappa = \star d_{\kappa_S} \delta\kappa \tag{4.6}$$

and the gauge-fixing condition

$$d_{\kappa_S} \star \delta\kappa = 0. \tag{4.7}$$

Any infinitesimal local gauge transformation $d_{\kappa_S} \Lambda = d\Lambda + [\kappa_S, \Lambda]$, with Λ in $\mathfrak{h} \times \mathfrak{h}^*$, solves Eq. (4.6). Its solutions $\delta\kappa$ may then include a transformation of this type. The point is that for κ' and κ_S to be gauge nonequivalent, $\delta\kappa$ cannot just be an infinitesimal gauge transformation, and this is what Eq. (4.7) takes care of.

Expand $\delta\kappa$ in the basis $\{T_i, Z_j\}$ as

$$\delta\kappa = \delta\alpha + \delta\beta, \quad \delta\alpha := \delta\alpha^i T_i, \quad \delta\beta := \delta\beta^i Z_i, \tag{4.8}$$

and substitute these expansions in Eqs. (4.6) and (4.7). This gives for $\delta\alpha$ and $\delta\beta$ the equations

$$\star \left(d\delta\alpha + [\alpha_S, \delta\alpha] \right) = \pm \left(d\delta\alpha + [\alpha_S, \delta\alpha] \right), \tag{4.9}$$

$$d\star\delta\alpha + [\alpha_S, \star\delta\alpha] = 0, \tag{4.10}$$

and

$$\begin{aligned} \star \left(d\delta\beta + [\alpha_S, \delta\beta] + [\beta_S, \delta\alpha] \right) &= \pm \left(d\delta\beta + [\alpha_S, \delta\beta] \right. \\ &\quad \left. + [\beta_S, \delta\alpha] \right), \end{aligned} \tag{4.11}$$

$$d\star\delta\beta + [\alpha_S, \star\delta\beta] + [\beta_S, \star\delta\alpha] = 0. \tag{4.12}$$

The solutions $\delta\kappa_S = (\delta\alpha_S, \delta\beta_S)$ to these equations describe gauge nonequivalent displacements in the moduli space $\mathcal{M}_N(G_{\mathfrak{h} \times \mathfrak{h}^*})$. We will use standard terminology and refer to them as zero modes (since they are the zero modes of a linear differential operator).

In Eqs. (4.4), (4.9), and (4.10) one recognizes the problem of charge N self–antiself dual $G_{\mathfrak{h}}$ instantons and their zero modes. Given its solution $\{\alpha_S, \delta\alpha_S\}$, we want to solve Eqs. (4.5), (4.11), and (4.12) for β and $\delta\beta$. Let us first understand the solution to the $G_{\mathfrak{h}}$ problem. A solution α_S to Eq. (4.4) depends on a set of free parameters $\{u^a\}$ that

describe instanton degrees of freedom and that occur in the differential problem as integration constants [19–26]. In the ADHM approach, the $\{u^a\}$ appear as free parameters in the quaternion matrices in terms of which α_S is constructed. Using the fact that partial derivatives $\partial/\partial u^a$ commute with the exterior differential d and noting the Jacobi identity for the generators $\{T_i\}$ of \mathfrak{h} , it is trivial to check that (i) derivatives $\partial\alpha_S/\partial u^a$ of α_S along u^a and (ii) rotations $[\alpha_S, T_i]$ of α_S about T_i solve the moduli equation (4.9). The problem is that they may not satisfy the gauge-fixing condition (4.10). To correct this, one includes infinitesimal local $G_{\mathfrak{h}}$ transformations and writes for the zero modes

$$\delta_{(a)}\alpha_S = \frac{\partial\alpha_S}{\partial u^a} + dt_{(a)} + [\alpha_S, t_{(a)}], \tag{4.13}$$

$$\delta_{(i)}\alpha_S = [\alpha_S, T_i] + dt_{(i)} + [\alpha_S, t_{(i)}], \tag{4.14}$$

where $t_{(a)} = t_{(a)}^j T_j$ and $t_{(i)} = t_{(i)}^j T_j$ are \mathfrak{h} -valued functions that must be chosen so that Eq. (4.10) holds. The zero modes $\delta_{(a)}\alpha_S$ and $\delta_{(i)}\alpha_S$ give the gauge nonequivalent deformations of α_S . Introducing angles τ^i for the rotations around T_i , one may take $\{u^a, \tau^i\}$ as local coordinates on the moduli space of charge N self–antiself dual $G_{\mathfrak{h}}$ instantons $\mathcal{M}_N(G_{\mathfrak{h}})$.

4.1 The connection

We now turn to Eq. (4.5). Writing $\beta = \beta^i Z_i$ and noting the commutation relations $[T_i, T_j] = f_{ij}^k T_k$ and $[T_i, Z_j] = f_{ij}^k Z_k$, Eq. (4.5) gives for β^i the same equation as the moduli equation (4.9) gives for the components $\delta\alpha^i$ of $\delta\alpha$. The latter is solved by derivatives $\partial\alpha_S/\partial u^a$ and rotations $[\alpha_S, T_i]$. Hence, modulo gauge transformations, the most general solution for β is a linear combination

$$\beta_S = \sum_a \tilde{u}^a \frac{\partial\alpha_S^i}{\partial u^a} Z_i + \tilde{\tau}^i [\alpha_S, Z_i] \tag{4.15}$$

with arbitrary coefficients \tilde{u}^a and $\tilde{\tau}^i$. Upon substitution in Eq. (3.14), the \mathfrak{h}^* -component of the curvature becomes

$$B_S = \sum_a \tilde{u}^a \frac{\partial F_S^i}{\partial u^a} Z_i + \tilde{\tau}^i [F_S, Z_i]. \tag{4.16}$$

This is trivially self–antiself dual and does not contribute to the instanton charge. To check the latter, use $\Omega(F, [F, Z_i]) = 0$ for any two-form F , so that

$$\int_{\mathbf{R}^4} \Omega(F_S, B_S) = \frac{1}{2} \sum_a \tilde{u}^a \frac{\partial}{\partial u^a} \int_{\mathbf{R}^4} \omega(F_S, F_S). \tag{4.17}$$

Since $\int \omega(F_S, F_S)$ is a constant, equal to $8\pi^2 N$, with N the charge of the $G_{\mathfrak{h}}$ instanton specified by α_S , the derivatives on the right hand side vanish and Eq. (4.3) is reproduced.

Once we have (α_S, β_S) , we look for the solutions $\delta\beta$ to Eqs. (4.11) and (4.12). There are two types of solutions: those with $\delta\alpha = \delta\alpha_S \neq 0$, and those with $\delta\alpha = 0$.

4.2 Zero modes with $\delta\alpha \neq 0$

A perturbation $\alpha_S \rightarrow \alpha_S + \delta\alpha_S$ produces a change $\beta_S \rightarrow \beta_S + \delta\beta_S$ given by

$$\delta\beta_S = \sum_b \tilde{u}^b \frac{\partial \delta\alpha_S^j}{\partial u^b} Z_j + \tilde{\tau}^j [\delta\alpha_S, Z_j]. \tag{4.18}$$

Employing the fact that $\delta\alpha_S$ satisfies Eqs. (4.9) and (4.10), it is a matter of simple algebra to check that $\delta\beta_S$ solves the moduli equation (4.11) and the gauge-fixing condition (4.10). Hence, to every $G_{\mathfrak{h}}$ zero mode $\delta\alpha_S$ there corresponds a $G_{\mathfrak{h} \times \mathfrak{h}^*}$ zero mode $(\delta\alpha_S, \delta\beta_S)$.

Using the expressions for $\delta\alpha_S$ in Eqs. (4.13) and (4.14), $\delta\beta_S$ can be recast as

$$\delta_{(a)}\beta_S = \frac{\partial\beta_S^i}{\partial u^a} Z_i + dz_{(a)} + [\alpha_S, z_{(a)}] + [\beta_S, t_{(a)}], \tag{4.19}$$

$$\delta_{(i)}\beta_S = [\beta_S, T_i] + dz_{(i)} + [\alpha_S, z_{(i)}] + [\beta_S, t_{(i)}]. \tag{4.20}$$

Here $z_{(a)}$ and $z_{(i)}$ are the \mathfrak{h}^* -valued functions

$$z_{(a)} = \sum_b \tilde{u}^b \frac{\partial t_{(a)}^j}{\partial u^b} Z_j + \tilde{\tau}^j [t_{(a)}, Z_j], \tag{4.21}$$

$$z_{(i)} = \sum_b \tilde{u}^b \frac{\partial}{\partial u^b} (T_i + t_{(i)}^j Z_j) + \tilde{\tau}^j [t_{(i)} + T_i, Z_j], \tag{4.22}$$

and $t_{(a)}$ and $t_{(i)}$ are the same functions as occur in the zero modes $\delta_{(a)}\alpha_S$ and $\delta_{(i)}\alpha_S$. The deformations $\delta_{(a)}\beta_S$, $\delta_{(i)}\beta_S$ in Eqs. (4.19) and (4.20) exhibit the pattern of a parametric derivative $\partial\beta_S/\partial u^a$, rotation $[\beta_S, T_i]$, followed by an infinitesimal gauge transformation. Furthermore, $\delta\alpha_S$ and $\delta\beta_S$ can be combined in

$$\delta_{(a)}\kappa_S = \frac{\partial\kappa_S}{\partial u^a} + d\Lambda_{(a)} + [\kappa_S, \Lambda_{(a)}], \tag{4.23}$$

$$\delta_{(i)}\kappa_S = [\kappa_S, T_i] + d\Lambda_{(i)} + [\kappa_S, \Lambda_{(i)}], \tag{4.24}$$

where $\Lambda_{(a)} = t_{(a)} + z_{(a)}$ and $\Lambda_{(i)} = t_{(i)} + z_{(i)}$.

4.3 Zero modes with $\delta\alpha = 0$

For $\delta\alpha = 0$, the moduli equation (4.11) and the gauge-fixing condition (4.12) for $\delta\beta^i$ reduce to those for the zero modes $\delta\alpha^i$ of the self–antiself dual $G_{\mathfrak{h}}$ instanton α_S . It then trivially follows that there are $\dim_N(G_{\mathfrak{h}})$ additional zero modes $\delta\kappa_S = (\delta\alpha_S, \delta\beta_S)$ with

$$\begin{aligned} \delta_{(\bar{a})}\alpha_S &= 0, \\ \delta_{(\bar{a})}\beta_S &= \delta_{(a)}\alpha_S^j Z_j = \frac{\partial\beta_S}{\partial\tilde{u}^a} + dt_{(a)}^j Z_j + [\alpha_S, t_{(a)}^j Z_j], \end{aligned} \tag{4.25}$$

$$\begin{aligned} \delta_{(\bar{i})}\alpha_S &= 0, \\ \delta_{(\bar{j})}\beta_S &= \delta_{(i)}\alpha_S^j Z_j = \frac{\partial\beta_S}{\partial\tilde{\tau}^i} + dt_{(i)}^j Z_j + [\alpha_S, t_{(i)}^j Z_j]. \end{aligned} \tag{4.26}$$

These have the same structure of all zero modes, partial derivatives with respect to moduli parameters, \tilde{u}^a and $\tilde{\tau}^i$ in this case, followed by infinitesimal gauge transformations.

To summarize, the gauge field (α_S, β_S) , with α_S the connection of a charge N self–antiself dual $G_{\mathfrak{h}}$ instanton and β_S as in Eq. (4.15), specifies a self–antiself dual $G_{\mathfrak{h}\times\mathfrak{h}^*}$ instanton with the same charge. The dimension of its moduli space $\mathcal{M}_N(G_{\mathfrak{h}\times\mathfrak{h}^*})$ is twice the dimension of $\mathcal{M}_N(G_{\mathfrak{h}})$. As local coordinates on $\mathcal{M}_N(G_{\mathfrak{h}\times\mathfrak{h}^*})$, one may take $\{u^a, \tau^j, \tilde{u}^a, \tilde{\tau}^j\}$, where u^a and τ^j are local coordinates on $\mathcal{M}_N(G_{\mathfrak{h}})$, and \tilde{u}^a and $\tilde{\tau}^j$ are kind of dual coordinates. If the zero modes of the $G_{\mathfrak{h}}$ instanton α_S are given by Eqs. (4.13) and (4.14), the zero modes of the (α_S, β_S) instanton take the form in Eqs. (4.13)–(4.14), (4.19)–(4.20), and (4.25)–(4.26). We may call these instantons cotangent $T^*G_{\mathfrak{h}}$, or semidirect $G_{\mathfrak{h}} \times G_{\mathfrak{h}^*}$ instantons.

The moduli space $\mathcal{M}_N(G_{\mathfrak{h}\times\mathfrak{h}^*})$ inherits a natural metric from the field theory defined by the overlap of deformations $\delta\kappa = (\delta\alpha, \delta\beta)$. If U and V stand for two arbitrary moduli coordinates, the moduli space metric coefficients are given by

$$G_{UV} = \frac{1}{8\pi^2} \int_{\mathbf{R}^4} \Omega (\delta_{(U)}\kappa, \star\delta_{(V)}\kappa). \tag{4.27}$$

Denote by H the metric on $\mathcal{M}_N(G_{\mathfrak{h}})$, with components

$$H_{pq} = \frac{1}{8\pi^2} \int_{\mathbf{R}^4} \omega (\delta_{(p)}\alpha, \star\delta_{(q)}\alpha). \tag{4.28}$$

Using $\Omega(T_i, T_j) = \Omega(T_i, Z_j) = \omega(T_i, T_j)$ and the results in this section for the zero modes, one has

$$G_{UV} = \frac{p}{\tilde{p}} \begin{pmatrix} q & \tilde{q} \\ H_{pq} + \Delta_{pq} & H_{pq} \\ H_{pq} & 0 \end{pmatrix}. \tag{4.29}$$

where Δ_{pq} stands for

$$\Delta_{pq} = \frac{1}{8\pi^2} \int_{\mathbf{R}^4} [\omega (\delta_{(p)}\alpha, \star\delta_{(q)}\beta) + (p \leftrightarrow q)]. \tag{4.30}$$

The $\mathfrak{h}\mathfrak{h}$ -coefficient G_{pq} is the sum of H_{pq} and a contribution Δ_{pq} that arises from the \mathfrak{h}^* -components $\delta_{(p,q)}\beta$ of the deformations along the moduli space directions p and q .

In the next section we explicitly realize this construction for $\mathfrak{h} = \mathfrak{su}(2)$ and instanton charge one.

5 The semidirect extension BPST instanton and its moduli

On \mathbf{R}^4 take coordinates $x^\mu = (x^1, x^2, x^3, x^4)$ and Euclidean metric $\delta_{\mu\nu}$. Set $\mathfrak{h} = \mathfrak{su}(2)$, with basis $[T_i, T_j] = \epsilon_{ijk} T_k$. The most general invariant bilinear form ω that can be defined on $\mathfrak{su}(2)$ is $\omega_{ij} = \omega_0\delta_{ij}$, with ω_0 an arbitrary constant that is conventionally set equal to $1/2g^2$.

The classical double $\mathfrak{su}(2) \times \mathfrak{su}(2)^*$ has commutators

$$\begin{aligned} [T_i, T_j] &= \epsilon_{ijk} T_k, & [T_i, Z_j] &= \epsilon_{ijk} Z_k, \\ [Z_i, Z_j] &= 0, & i &= 1, 2, 3, \end{aligned} \tag{5.1}$$

and the most general metric Ω on it reads

$$\Omega = \frac{T_i}{Z_i} \frac{1}{2g^2} \begin{pmatrix} T_j & Z_j \\ \delta_{ij} & \delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix}. \tag{5.2}$$

This is of the form (2.5), or more precisely, of the form (2.11) with $s = 0$. In the basis $\{T_i, Z_j\}$ the connection κ has components α^i and β^j , and the curvature K has components F^i and B^j , given by

$$F^i = d\alpha^i + \frac{1}{2} \epsilon^{ijk} \alpha^j \wedge \alpha^k, \quad B^j = d\beta^j + \epsilon^{ijk} \alpha^j \wedge \beta^k. \tag{5.3}$$

In what follows we restrict ourselves to the positive sign in the equation $\star K = \pm K$. This corresponds to selfdual instantons and, with the metric convention (5.2), positive instanton charge. The negative sign, antiself dual instantons with negative instanton charge, is analogously treated. The group $G_{\mathfrak{su}(2) \times \mathfrak{su}(2)^*}$ is the cotangent bundle $T^*SU(2)$, isomorphic to the semidirect product $SU(2) \times \mathbf{R}^3$.

Equation $\star F^i = F^i$, with F^i as in Eq. (5.3), is solved by $SU(2)$ selfdual instantons. Take as solution the BPST instanton [14], whose connection α_S^i and curvature F_S^i are given in singular gauge by

$$\alpha_S^i = \frac{2\rho^2}{r_a^2(r_a^2 + \rho^2)} \tilde{\eta}^i{}_{\mu\nu} (x - a)^\nu dx^\mu \tag{5.4}$$

and

$$\begin{aligned} F_S^i &= \frac{2\rho^2}{r_a^2(r_a^2 + \rho^2)^2} \\ &\times \left[4\tilde{\eta}^i{}_{\mu\nu} (x - a)^\nu (x - a)_\nu - \tilde{\eta}^i{}_{\mu\nu} r_a^2 \right] dx^\mu \wedge dx^\nu. \end{aligned} \tag{5.5}$$

Here ρ is an arbitrary constant, r_a is the radius of the three-sphere

$$r_a^2 = (x - a)^\mu (x - a)_\mu \tag{5.6}$$

centered at any point a^μ in \mathbf{R}^4 , and $\bar{\eta}^i_{\mu\nu}$ are the 't Hooft symbols [27]

$$\bar{\eta}^i_{\mu\nu} = -\bar{\eta}^i_{\nu\mu}, \quad \bar{\eta}^i_{4j} = \delta_{ij}, \quad \bar{\eta}^i_{jk} = \epsilon_{ijk}, \tag{5.7}$$

whose properties are collected in the appendix. The BPST connection has instanton charge one in units of $1/g^2$,

$$S_P[SU(2); \alpha_S] = \frac{1}{16\pi^2 g^2} \int_{\mathbf{R}^4} F_S^i \wedge F_S^i = \frac{1}{g^2}. \tag{5.8}$$

The moduli space of the BPST instanton [21–26] is an eight-dimensional manifold on which one may take as global coordinates the instanton size ρ , the four coordinates a^μ of the instanton center, and three angles τ^i , which account for rotations about the generators $\{T_i\}$ of $\mathfrak{su}(2)$. The deformations along these moduli directions are [25]

$$\delta_{(\rho)} \alpha_S = \frac{\partial \alpha_S}{\partial \rho}, \tag{5.9}$$

$$\delta_{(a^\mu)} \alpha_S = \frac{\partial \alpha_S}{\partial a^\mu} + d\alpha_{\mu S} + [\alpha_S, \alpha_{\mu S}] = -F_{\mu\nu S} dx^\nu, \tag{5.10}$$

$$\delta_{(\tau^i)} \alpha_S = [\alpha_S, T_i] + dt T_i + [\alpha_S, t T_i], \tag{5.11}$$

where t is the function

$$t(r_a) = -\frac{\rho^2}{r_a^2 + \rho^2}. \tag{5.12}$$

5.1 The semidirect BPST instanton and its zero modes

The results in Sect. 4 imply that, for $\alpha = \alpha_S$, the most general solution to the equation $\star B^i = B^i$ is, modulo gauge transformations,

$$\beta_S^i = \tilde{\rho} \frac{\partial \alpha_S^i}{\partial \rho} + \tilde{a}^\mu \frac{\partial \alpha_S^i}{\partial a^\mu} + \epsilon^{ikj} \alpha_S^k \tilde{\tau}^j, \tag{5.13}$$

where $\tilde{\rho}$, \tilde{a}^μ and $\tilde{\tau}^j$ are free parameters. The curvature B^i then becomes

$$B_S^i = \left(\tilde{\rho} \frac{\partial}{\partial \rho} + \tilde{a}^\mu \frac{\partial}{\partial a^\mu} \right) F_S^i + \epsilon^{ikj} F_S^k \tilde{\tau}^j. \tag{5.14}$$

The $\mathfrak{su}(2) \times \mathfrak{su}(2)^*$ connection (α_S, β_S) specifies a charge one $SU(2) \times \mathbf{R}^3$ instanton that we call a semidirect or cotangent BPST instanton. It depends on 16 moduli parameters, ρ , a^μ , τ^j , $\tilde{\rho}$, \tilde{a}^μ , and $\tilde{\tau}^j$. The derivatives entering β_S^i and B_S^i are trivially calculated from the expression of α_S .

The $\mathfrak{su}(2)$ -components of the zero modes along the moduli directions ρ , a^μ and τ^i are those in Eqs. (5.9)–(5.11). Upon

substitution in Eqs. (4.19) and (4.20), we obtain for their $\mathfrak{su}(2)^*$ -companions

$$\delta_{(\rho)} \beta = \frac{\partial \beta_S}{\partial \rho}, \tag{5.15}$$

$$\delta_{(a^\mu)} \beta = \frac{\partial \beta_S}{\partial a^\mu} + d\beta_{\mu S} + [\alpha_S, \beta_{\mu S}] + [\beta_S, \alpha_{\mu S}] = -B_{\mu\nu S} dx^\nu, \tag{5.16}$$

$$\delta_{(\tau^i)} \beta = [\beta_S, T_i] + dz_{(\tau^i)} + [\alpha_S, z_{(\tau^i)}] + [\beta_S, t T_i], \tag{5.17}$$

where $z_{(\tau^i)}$ is a function of x^μ given by

$$z_{(\tau^i)}(x) = -\frac{2\rho}{(r_a^2 + \rho^2)^2} \left[\tilde{\rho} r_a^2 + \rho \tilde{a}^\lambda (x - a)_\lambda \right] Z_i + \frac{r_a^2}{r_a^2 + \rho^2} \tilde{\tau}^j [T_i, Z_j]. \tag{5.18}$$

As a cross check, one may directly verify, by a long but simple calculation, that $\delta_{(\rho, a^\mu, \tau^i)} \beta_S$ indeed satisfy the moduli equation (4.11) and the gauge-fixing condition (4.12). We remark that $\delta_{(\rho, a^\mu, \tau^i)} \beta_S$ follow from Eqs. (4.19) and (4.20) and that no additional gauge transformation has been introduced so as to ensure that the gauge-fixing condition holds.

The zero modes associated to the moduli coordinates $\tilde{\rho}$, \tilde{a}^μ and $\tilde{\tau}^i$ are given by Eqs. (4.25)–(4.26), which in our case take the form

$$\delta_{(\tilde{\rho})} \alpha = 0, \quad \delta_{(\tilde{\rho})} \beta = \frac{\partial \alpha_S^i}{\partial \rho} Z_i, \tag{5.19}$$

$$\delta_{(\tilde{a}^\mu)} \alpha = 0, \quad \delta_{(\tilde{a}^\mu)} \beta = -F_{\mu\nu S} Z_i dx^\nu, \tag{5.20}$$

$$\delta_{(\tilde{\tau}^i)} \alpha = 0, \quad \delta_{(\tilde{\tau}^i)} \beta = dt Z_i + [\alpha_S, (1 + t) Z_i], \tag{5.21}$$

with t as in Eq. (5.12).

5.2 The moduli space metric

The expressions for the zero modes above and some calculations lead to the moduli space metric

$$G_{UV} = \begin{matrix} & \rho & a^\nu & \tau^j & \tilde{\rho} & \tilde{a}^\nu & \tilde{\tau}^j \\ \begin{matrix} \rho \\ a^\mu \\ \tau^i \\ \tilde{\rho} \\ \tilde{a}^\mu \\ \tilde{\tau}^i \end{matrix} & \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & \delta_{\mu\nu} & 0 & 0 & \delta_{\mu\nu} & 0 \\ 0 & 0 & \frac{1}{2} \rho (\rho + 2\tilde{\rho}) \delta_{ij} & 0 & 0 & \frac{1}{2} \rho^2 \delta_{ij} \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_{\mu\nu} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \rho^2 \delta_{ij} & 0 & 0 & 0 \end{pmatrix} \end{matrix}. \tag{5.22}$$

The change of coordinates

$$\begin{matrix} \sigma \\ \tilde{\sigma} \end{matrix} = \begin{matrix} \tilde{\rho} - \rho r_- \\ \tilde{\rho} - \rho r_+ \end{matrix}, \quad \begin{matrix} b^\mu \\ \tilde{b}^\mu \end{matrix} = \begin{matrix} \tilde{a}^\mu - a^\mu r_- \\ \tilde{a}^\mu - a^\mu r_+ \end{matrix}, \quad \begin{matrix} \theta^i \\ \tilde{\theta}^i \end{matrix} = \begin{matrix} \tau^i - \tilde{\tau}^i s_+ \\ \tau^i - \tilde{\tau}^i s_- \end{matrix}, \tag{5.23}$$

with r_{\pm} and s_{\pm} given by

$$r_{\pm} = \frac{2}{1 \pm \sqrt{5}} \quad s_{\pm} = \frac{2\rho}{\rho + 2\bar{\rho} \pm \sqrt{4\rho^2 + (\rho + 2\bar{\rho})^2}}, \quad (5.24)$$

brings the metric to the diagonal form

$$dL^2 = \frac{1}{2g^2} \left[d\sigma^2 + db^\mu db^\mu + f d\theta^i d\theta^i - d\tilde{\sigma}^2 - d\tilde{b}^\mu d\tilde{b}^\mu - \tilde{f} d\tilde{\theta}^i d\tilde{\theta}^i \right], \quad (5.25)$$

where f and \tilde{f} are positive functions of σ and $\tilde{\sigma}$. This shows that the metric has signature (8, 8).

The field theory is invariant under translations and $SO(4)$ rotations in \mathbf{R}^4 , and under $SU(2) \times \mathbf{R}^3$ gauge transformations. These symmetries go into isometries of the moduli metric. Indeed, \mathbf{R}^4 translations give rise to translations in b^μ and \tilde{b}^μ , generated by $\partial/\partial b^\mu$ and $\partial/\partial \tilde{b}^\mu$. Rotations become $SO(4) \cong SU(2)_+ \times SU(2)_-$ rotations in b^μ and \tilde{b}^μ , generated by

$$\chi_{\pm}^i = \frac{1}{2} \left[\epsilon^{ijk} b^j \frac{\partial}{\partial b^k} \pm \left(b^i \frac{\partial}{\partial b^4} - b^4 \frac{\partial}{\partial b^i} \right) \right] \quad (5.26)$$

and $\tilde{\chi}_{\pm}^i$, obtained from Eq. (5.26) by replacing b^μ with \tilde{b}^μ . Finally gauge transformations become translations in τ^i and $\tilde{\tau}^i$ generated by $\partial/\partial \tau^i$ and $\partial/\partial \tilde{\tau}^i$. Note that, in the conventional BPST instanton, one has translational and rotational invariance in a^μ . The first one is an isometry here, but the second one is not, due to the occurrence of the term $da^\mu d\tilde{a}^\mu$ in the moduli metric.

5.3 Complex structures

Let us show that the moduli space $\mathcal{M}_1(SU(2) \times \mathbf{R}^3)$ is a hyper-Kähler manifold. We do this by finding three complex structures $J^i = \frac{1}{2} (J^i)_{UV} dU \wedge dV$, with components $(J^i)_{UV}$, such that

$$(J^i)^U{}_W (J^j)^W{}_V = -\delta^{ij} \delta^U{}_V + \epsilon^{ijk} (J^k)^U{}_V. \quad (5.27)$$

As in the BPST case, one expects the moduli space to inherit its complex structures from those of \mathbf{R}^4 , which can be written as $-\tilde{\eta}^i{}_{\mu\nu} dx^\mu \wedge dx^\nu$. This suggests the ansatz

$$(J^i)_{UV} = -\frac{1}{8\pi^2} \int_{\mathbf{R}^4} d^4x \tilde{\eta}^i{}_{\mu\nu} \Omega(\delta_{(U)}\kappa^\mu, \delta_{(V)}\kappa^\nu). \quad (5.28)$$

Using the expressions for the zero modes and some algebra and integration, one has

$$(J^i)_{UV} = \begin{matrix} & \rho & a^\nu & \tau^k & \bar{\rho} & \bar{a}^\nu & \bar{\tau}^k \\ \begin{matrix} \rho \\ a^\mu \\ \tau^j \\ \bar{\rho} \\ a^\mu \\ \tau^j \end{matrix} & \begin{pmatrix} 0 & 0 & (\rho + \bar{\rho})\delta^i{}_k & 0 & 0 & \rho\delta^i{}_k \\ 0 & \tilde{\eta}^i{}_{\mu\nu} & 0 & 0 & \tilde{\eta}^i{}_{\mu\nu} & 0 \\ -(\rho + \bar{\rho})\delta^i{}_j & 0 & \frac{1}{2}\rho(\rho + 2\bar{\rho})\epsilon^i{}_{jk} & -\rho\delta^i{}_j & 0 & \frac{1}{2}\rho^2\epsilon^i{}_{jk} \\ 0 & 0 & \rho\delta^i{}_k & 0 & 0 & 0 \\ 0 & \tilde{\eta}^i{}_{\mu\nu} & 0 & 0 & 0 & 0 \\ -\rho\delta^i{}_j & 0 & \frac{1}{2}\rho^2\epsilon^i{}_{jk} & 0 & 0 & 0 \end{pmatrix} \end{matrix}. \quad (5.29)$$

Noting that $(J^i)^U{}_V = G^{UV}(J^i)_{WV}$, with G^{UV} the inverse of G_{UV} in (5.22), it is straightforward to check that the two-forms J^i in Eq. (5.29) indeed satisfy Eq. (5.27); hence they are complex structures. It is worth remarking that the moduli space is hyper-Kähler, despite not being a Riemannian manifold. It looks like hyper-Kählerity is “transmitted” to $\mathcal{M}_1(SU(2) \times \mathbf{R}^3)$ via its Riemannian submanifolds.

We finish by studying the compatibility of the isometries of the moduli metric with the complex structures. Recall that for an isometry generated by a Killing vector ξ to be compatible with a tensor A , the Lie derivative $\mathcal{L}_\xi A$ of A along ξ must vanish. For an isometry given in a chart $\{u^a\}$ by $u^a \rightarrow u'^a = u^a + \epsilon \xi^a(u)$, we use for the Lie derivative the convention $\mathcal{L}_\xi A = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [A'(u) - A(u)]$. With this convention, one may check that the isometries generated by $\xi = \partial_{b^\mu}, \partial_{\tilde{b}^\mu}, \chi_+^i, \partial_{\tau^i}$ and $\partial_{\tilde{\tau}^i}$ are compatible with the complex structures J^i . However, for $\xi = \chi_-^i$, one has $\mathcal{L}_{\chi_-^i} J^j = \epsilon^{ijk} J^k$. The complex structures are thus rotated by $SU(2)_-$ rotations, but they remain unchanged by the other isometries.

6 Outlook

In this paper we have proposed a method to obtain the self-antiself dual solutions for a gauge group $G_{\mathfrak{h} \times \mathfrak{h}^*}$ from those for $G_{\mathfrak{h}}$. This hints to using Medina and Revoy’s theorem [8] to find structure results for the self-antiself dual instantons of the Lie groups with metric Lie algebras. One may advance a few ideas on the subject. According to the theorem, it would suffice to consider three cases: (1) simple Lie algebras, (2) Abelian algebras, and (3) double extensions of a metric Lie algebra by either a simple or a one-dimensional Lie algebra.

Simple real Lie algebras are the Lie algebras of real simple Lie groups, whose instantons would be regarded as the basic objects in terms of which state structure results. Next on the list is the Abelian Lie algebra. This case is trivial, since on \mathbf{R}^4 there are no Abelian instantons. One is left with the Lie groups of double extensions.

The double extension $\mathfrak{d}(\mathfrak{m}, \mathfrak{h})$ of a metric Lie algebra \mathfrak{m} by a Lie algebra \mathfrak{h} is obtained [8, 9] by forming the classical double $\mathfrak{h} \times \mathfrak{h}^*$ and, then, by acting with \mathfrak{m} on \mathfrak{h} via antisymmetric derivations. Since \mathfrak{m} needs to be metric, three possibilities must be considered for \mathfrak{m} . The first one is that \mathfrak{m} is a simple

real Lie algebra. In this case [9], the algebra of antisymmetric derivations of \mathfrak{m} is \mathfrak{m} itself and the double extension is isomorphic to the direct product $\mathfrak{m} \times (\mathfrak{m} \ltimes \mathfrak{m}^*)$. The corresponding Lie group is then the direct product $G_{\mathfrak{m}} \times G_{\mathfrak{m} \ltimes \mathfrak{m}^*}$ and its instantons are determined in terms of the $G_{\mathfrak{m}}$ instantons using the construction presented here. The second possibility is that \mathfrak{m} is Abelian, of dimension m . Being Abelian, any nondegenerate, symmetric bilinear form on \mathfrak{m} is a metric, and this can always be brought to a diagonal form with all its eigenvalues equal to either $+1$ or -1 . If there are p positive and q negative eigenvalues, the algebra \mathfrak{h} of antisymmetric derivations is any subalgebra of $\mathfrak{so}(p, q)$. In this case, by extending the arguments at the beginning of Section 4, it can be shown that the third homotopy group of $G_{\mathfrak{d}(\mathfrak{m}, \mathfrak{h})}$ is equal to the third homotopy group of $G_{\mathfrak{h}}$. This motivates studying the self-antiseif dual solutions of such theories in detail. The third option, that \mathfrak{m} is a double extension, takes us back to the starting point.

One would also like to include matter fields in the analysis. Their coupling to an $\mathfrak{h} \ltimes \mathfrak{h}^*$ gauge field requires additional matter fields components, which introduce additional field equations that may lead to new nontrivial configurations.

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Appendix

The 't Hooft symbols, defined in Eq. (5.7), satisfy the algebraic identities [27]

$$\bar{\eta}^i{}_{\mu\nu} \bar{\eta}^i{}_{\gamma\tau} = \delta_{\mu\gamma} \delta_{\nu\tau} - \delta_{\mu\tau} \delta_{\nu\gamma} - \epsilon_{\mu\nu\gamma\tau}, \tag{A.1}$$

$$\bar{\eta}^i{}_{\mu\nu} \bar{\eta}^j{}_{\mu\tau} = \delta^{ij} \delta_{\nu\tau} + \epsilon^{ijk} \bar{\eta}^k{}_{\nu\tau}, \tag{A.2}$$

$$\epsilon_{\mu\nu\sigma\tau} \bar{\eta}^i{}_{\tau\sigma} = \bar{\eta}^i{}_{\mu\nu} \delta_{\sigma\gamma} + \bar{\eta}^i{}_{\nu\sigma} \delta_{\mu\gamma} + \bar{\eta}^i{}_{\sigma\mu} \delta_{\nu\sigma}, \tag{A.3}$$

$$\epsilon^{ijk} \bar{\eta}^j{}_{\mu\nu} \bar{\eta}^k{}_{\gamma\tau} = \delta_{\mu\gamma} \bar{\eta}^i{}_{\nu\tau} - \delta_{\mu\tau} \bar{\eta}^i{}_{\nu\gamma} - \delta_{\nu\gamma} \bar{\eta}^i{}_{\mu\tau} + \delta_{\nu\tau} \bar{\eta}^i{}_{\mu\gamma}. \tag{A.4}$$

These have been widely used in the computations of Sect. 5. The one-forms

$$\bar{\chi}^i = \frac{1}{r_a^2} \bar{\eta}^i{}_{\mu\nu} (x - a)^\nu dx^\mu \tag{A.5}$$

are Maurer–Cartan forms for $SU(2) \cong S_3$. Letting the radius r_a vary, one obtains the frame $\bar{\mathcal{F}} = \{\bar{e}^i = r_a \bar{\chi}^i, \bar{e}^4 = -dr_a\}$, which has the same orientation as $\{dx^\mu\}$. We could have

worked in regular gauge, in which the BPST connection reads

$$\alpha_{s,\text{reg}}^i = \frac{2}{r_a^2 + \rho^2} \eta^i{}_{\mu\nu} (x - a)^\nu dx^\mu, \tag{A.6}$$

with the 't Hooft symbols $\eta^i{}_{\mu\nu}$ given in terms of $\bar{\eta}^i{}_{\mu\nu}$ by

$$\eta^i{}_{j4} = -\bar{\eta}^i{}_{j4} = \delta_{ij}, \quad \eta^i{}_{jk} = \bar{\eta}^i{}_{jk} = \epsilon_{ijk}. \tag{A.7}$$

Maurer–Cartan one-forms can also be defined now,

$$\chi^i = \frac{1}{r_a^2} \eta^i{}_{\mu\nu} (x - a)^\nu dx^\mu. \tag{A.8}$$

Together with dr_a , they form a frame $\mathcal{F} = \{e^i = r_a \chi^i, e^4 = dr_a\}$ with the same orientation as $\{dx^\mu\}$. All the calculations in Sect. 5 can be analogously performed in this gauge.

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