# New formula for a resonant scattering near an inelastic threshold

L. Leśniak

The Henryk Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, 31-342 Kraków, Poland

**Abstract.** We show that the Flatté formula is not adequate to interpret precision data on a resonance production near an inelastic threshold. A unitary parameterization, satisfying generalized Watson's theorem for the production amplitudes, is proposed to replace the Flatté parameterization in the phenomenological analyses of the experimental data.

**Keywords:** Multichannel scattering, approximations, analytic properties of S matrix, meson-meson interactions

PACS: 11.80.Gw, 11.80.Fw, 11.55.Bq, 13.75.Lb

# INTRODUCTION

In 1976 S. M. Flatté analysed the  $\pi\eta$  and the KK coupled channel systems and proposed the following parameterization of the *S*-wave production amplitudes  $A_i$ :

$$A_i \sim \frac{M_R \sqrt{\Gamma_0 \Gamma_i}}{M_R^2 - E^2 - iM_R (\Gamma_1 + \Gamma_2)}, \quad i = 1, 2.$$

$$\tag{1}$$

Here E is the effective mass (c.m. energy),  $M_R$  is a resonance mass (the  $a_0(980)$  mass in this particular case), the first channel width

$$\Gamma_1 = g_1 k_1, \quad k_1 = \frac{1}{2E} \sqrt{[E^2 - (m_\eta + m_\pi)^2][E^2 - (m_\eta - m_\pi)^2]},$$
 (2)

 $k_1$  being the pion or eta c.m. momentum. Above the K $\overline{K}$  threshold the second channel width  $\Gamma_2 = g_2 k_2$ , where  $k_2 = \sqrt{\frac{E^2}{4} - m_K^2}$  is the kaon c.m. momentum. Below the threshold  $\Gamma_2 = ig_2 p_2$ , where  $p_2 = \sqrt{m_K^2 - \frac{E^2}{4}}$ . At the threshold energy  $E_0 = 2m_K$ ,  $q = k_1(E_0)$  and  $\Gamma_0 = g_1 q$ . The Flatté production amplitudes (1) depend on three real parameters: the resonance mass  $M_R$  and the two coupling constants  $g_1$  and  $g_2$ . Some discussions related to the Flatté parameterization can be found in Refs. [2-4].

At first we consider an elastic scattering amplitude in the second channel. Without a coupling to the first channel it can be written as

$$T_{22} = \frac{\sin \delta_2}{k_2} e^{i\delta_2} \equiv \frac{1}{k_2 \cot \delta_2 - ik_2},$$
(3)

where  $\delta_2$  is the channel two phase shift. Near the KK threshold, for  $k_2$  close to 0, one gets the effective range expansion:

$$k_2 \cot \delta_2 \approx \frac{1}{a} + \frac{1}{2} r k_2^2, \tag{4}$$

where *a* is the scattering length and *r* is the effective range. Both *a* and *r* are real. In presence of a coupling to the first channel the  $T_{22}$  amplitude (3) can to be written as

$$T_{22} = \frac{1}{2ik_2}(\eta e^{2i\delta_2} - 1), \tag{5}$$

where  $\eta$  denotes the inelasticity parameter. The inelastic coupling near the threshold can effectively be taken into account by a modification of the effective range expansion:

$$T_{22} = \frac{1}{\frac{1}{\frac{1}{A} - i \, k_2 + \frac{1}{2} \, R \, k_2^2}}.$$
(6)

Here A denotes the *complex* scattering length and R is the *complex* effective range. Thus for a description of the elastic scattering in the second channel one needs four real parameters. In the Flatté formula, however, we have only three parameters , so it is evident that this parameterization is not sufficient to describe a system of the two coupled channels.

We introduce a new formula for the denominator W of the production amplitudes above an inelastic threshold:

$$A_i \sim \frac{1}{W(E)}, \quad W(E) = M_R^2 - E^2 - iM_R g_1 q - iM_R g_2 k_2 + N k_2^2, \tag{7}$$

where *N* is a new complex constant. Below the threshold one should replace  $k_2$  by  $ip_2$ . Since in the KK channel  $E^2 = E_0^2 + 4k_2^2$ , the denominator W(E) in (7) can be directly related to the denominator of  $T_{22}$  in Eq. (6):

$$\frac{W(E)}{M_R g_2} = \frac{1}{A} - ik_2 + \frac{1}{2}Rk_2^2,$$
(8)

where we find the inverse of the scattering length

$$Re(\frac{1}{A}) = \frac{M_R^2 - E_0^2}{M_R g_2}, \quad Im(\frac{1}{A}) = -\frac{g_1}{g_2} q,$$
 (9)

and the effective range

$$R = \frac{2N-8}{M_R g_2}.$$
(10)

In the Flatté approximation N = 0, hence  $ReR = \frac{-8}{M_{Rg_2}}$  and ImR = 0. The zero value of the imaginary part of the effective range is an essential limitation of the Flatté formula.

# Elastic scattering in the first channel and a transition between channels

In close analogy to Eq. (5), the elastic scattering amplitude in the first channel depends on the phase shift  $\delta_1$ :

$$T_{11} = \frac{1}{2ik_1} (\eta e^{2i\delta_1} - 1) \cdot$$
(11)

At the K $\overline{K}$  threshold  $\eta = 1$ ,  $\delta_1(q) \equiv \delta_0$  and  $T_{11}(E_0) = \frac{\sin \delta_0}{q} e^{i\delta_0}$ . Using the unitarity property of the scattering amplitudes we can derive a new formula for  $T_{11}$  above the K $\overline{K}$  threshold:

$$T_{11} = \frac{e^{i\delta_0}}{k_1} \frac{\sin \delta_0 + i \operatorname{Im} \left( e^{-i\delta_0} A \right) k_2 - \frac{1}{2} \operatorname{Im} \left( e^{-i\delta_0} A R \right) k_2^2}{1 - i A k_2 + \frac{1}{2} A R k_2^2} .$$
(12)

There are five independent parameters in  $T_{11}$ : Re A, Im A, Re R, Im R and  $\delta_0$ . Below the KK threshold  $k_2 \rightarrow ip_2$ . In the Flatté limit  $\delta_0$  equals to the phase of the complex scattering length A and the second numerator of  $T_{11}$  in Eq. (12) becomes constant  $(\sin \delta_0)$ .

A general form of the transition amplitude from the first to the second channel is the following:

$$T_{12} = \frac{1}{2\sqrt{k_1 k_2}} \sqrt{1 - \eta^2} e^{i(\delta_1 + \delta_2)}.$$
 (13)

In the new parameterization near the threshold  $T_{12}$  reads:

$$T_{12} = \frac{1}{\sqrt{k_1}} e^{i\delta_0} \frac{\sqrt{ImA - \frac{1}{2}} |A|^2 ImR k_2^2}{1 - iA k_2 + \frac{1}{2} AR k_2^2}.$$
 (14)

Let us remark that if Im A = Im R = 0 then  $T_{12} = 0$  (no transition between channels). In the Flatté limit Im R = 0 and the numerator of  $T_{12}$  is a constant independent on  $k_2$ .

#### Poles of the scattering amplitudes

All the three amplitudes, given by Eqs. (6), (12) and (14), have a common denominator

$$D(k_2) = 1 - iA k_2 + \frac{1}{2}A R k_2^2$$
 (15)

The amplitude poles coincide with the zeroes of  $D(k_2)$  located at  $z_1$  and  $z_2$ :

$$z_{1,2} = \frac{i}{R} \pm \sqrt{-\frac{1}{R^2} - \frac{2}{AR}}.$$
 (16)

From these equations we obtain the following relations for the scattering length *A* and the effective range *R*:

$$A = -i(\frac{1}{z_1} + \frac{1}{z_2}), \qquad \qquad R = \frac{2i}{z_1 + z_2}.$$
(17)

In the Flatté approximation  $Re z_1 = -Re z_2$ . This constraint has an important impact on the values of the complex energy poles  $E_{1,2} = \sqrt{E_0^2 + 4z_{1,2}^2}$ .

# **NEW FORMULA FOR THE PRODUCTION AMPLITUDES**

Parameterization of the production amplitudes  $A_i$  can be done in terms of the linear combination of the amplitudes  $T_{ij}(k_2)$ :

$$A_1 = f_1 T_{11} + f_2 T_{12}, \quad A_2 = f_1 T_{12} + f_2 T_{22}.$$
(18)

Here  $f_1$ ,  $f_2$  are real functions of energy (or momentum  $k_2$ ) and the two-channel scattering amplitudes in a new approach are written as  $T_{ij}(k_2) = \frac{N_{ij}(k_2)}{D(k_2)}$ , where the numerators  $N_{ij}$  can be directly obtained from Eqs. (12), (6) and (14) by using Eq. (15). Then

$$A_1 = \frac{B_1(k_2)}{D(k_2)}, \quad A_2 = \frac{B_2(k_2)}{D(k_2)}, \tag{19}$$

where

$$B_1 = f_1(k_2)N_{11}(k_2) + f_2(k_2)N_{12}(k_2), \quad B_2 = f_1(k_2)N_{12}(k_2) + f_2(k_2)N_{22}(k_2).$$
(20)

A possible approximation of  $f_i(k_2)$  near the inelastic threshold is:

$$f_1(k_2) \approx \alpha_1 + \beta_1 k_2^2, \quad f_2(k_2) \approx \alpha_2 + \beta_2 k_2^2;$$
 (21)

 $\alpha_1, \alpha_2$  are normalization constants and  $\beta_1, \beta_2$  are real coefficients.

## Watson's theorem and its generalization above the inelastic threshold

Below the inelastic threshold Watson's theorem is satisfied by the production amplitude  $A_1$ :

$$Im A_1 = k_1 T_{11} A_1^* . (22)$$

From this equation one infers that the phase of  $A_1$  is equal to the phase of  $T_{11}$  which in turn equals to the phase shift  $\delta_1$ .

A generalization to the two coupled channels can be done as follows:

$$Im A_1 = k_1 T_{11} A_1^* + k_2 T_{12} A_2^*, (23)$$

$$Im A_2 = k_2 T_{22} A_2^* + k_1 T_{21} A_1^*.$$
(24)

In a matrix notation one can define:

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$



**FIGURE 1.** Squares of the amplitude moduli versus the c.m. energy. The solid lines correspond to the new parameterization, given by Eqs. (6) and (12), the dashed lines - to the Flatté formula.

and write the matrix form of the generalized Watson theorem as  $Im A = T k A^*$ . This is equivalent to  $A = S A^*$ , where S denotes the S- matrix. Its elements are related to the scattering matrix elements  $T_{ij}$  (i, j = 1, 2) by

$$S_{ij} = \delta_{ij} + 2 i \sqrt{k_i k_j} T_{ij} .$$

#### NUMERICAL EXAMPLE: A CASE OF THE $a_0(980)$ RESONANCE

The  $a_0(980)$  resonance is situated close to the KK threshold. It decays predominantly to the  $\pi\eta$  channel in which two mesons interact in the *S*-wave, isospin one state. A coupled channel formalism for the separable meson-meson interactions in two or three channels has been developed in [5]. Then in Ref. [6] it was applied to study the  $a_0$  resonances in the  $\pi\eta$  and the KK channels. The model parameters were fixed using the data of the Crystal Barrel and of the E-852 Collaborations. The following threshold parameters have been presently calculated: Re A = 0.17 fm, Im A = 0.41 fm, Re R = -11.32 fm, and Im R = -3.18 fm. Let us stress here that the imaginary part of the effective range cannot be neglected. In Fig. 1 we see important differences between the amplitude intensities calculated in two cases: 1. for Im R = -3.18 fm and 2. Im R = 0 fm (Flatté's limit). All curves are normalized to 1 at the KK threshold but already at the distance of 50 MeV to the left and to the right of the maximum the relative deviations between the two cases reach as much as 100 %.

In Fig. 2 the pole positions of the amplitudes are shown in the complex momentum  $k_2$  and in the complex energy planes. One can notice a large shift of  $Re E_1$  between the new result and the Flatté value. It exceeds 10 MeV and is larger than the present experimental energy resolution of many experiments. Thus, using the Flatté formula in the data analysis can lead to an important distortion of the particle spectra and to large theoretical errors of the threshold parameters. In particular, this may influence resonance masses and widths presented by the Particle Data Group in the Review of Particle Physics. The phases of the scattering amplitudes are also different in the two cases.



**FIGURE 2.** Pole positions in the  $K\overline{K}$  complex energy plane (left panel) and in the complex momentum plane (right panel)

# CONCLUSIONS

- 1. The Flatté formula is not sufficiently accurate to be used in analyses of the newest data on the resonance production near inelastic thresholds. Its application can lead to a substantial distortion of the effective mass distributions and to a displacement of the resonance pole positions.
- 2. A simple unitary parameterization, satisfying a generalized Watson theorem for the production amplitudes, is proposed. It enables one to determine crucial measurable particle interaction parameters, like the complex scattering length and the complex effective range. It is shown that a near threshold resonance should be characterized by two distinct complex poles.
- 3. A generalization of the new parameterization to the coupled particle systems other than  $\pi\eta$  and KK is straightforward.
- 4. New formula can be applied in numerous analyses of present and future experiments (for example: Belle, BaBar, CLEO, BES, KLOE, COSY, Tevatron, LHC, CLAS at JLab, Panda etc.). They can also serve to reanalyse older experiments with an aim to improve our knowledge of hadron spectroscopy and of reaction mechanisms.

## REFERENCES

- 1. S. M. Flatté, Phys. Lett., B63, 224 (1976).
- V. Baru, J. Heidenbauer, C. Hahnhart, Yu. Kalashnikova, A. Kudryavtsev, Phys. Lett., B586, 53 (2004).
- 3. B. Kerbikov, Phys. Lett., **B596**, 200 (2004).
- 4. V. Baru, J. Heidenbauer, C. Hahnhart, A. Kudryavtsev, U.-G. Meissner, Eur. Phys. J., A23, 523 (2005)
- 5. L. Leśniak, Acta Physica Polonica, **B27**, 1835 (1996).
- 6. Agnieszka Furman and Leonard Leśniak, Phys. Lett., **B538**, 266 (2002).