Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2015, Article ID 680970, 13 pages
http://dx.doi.org/10.1155/2015/680970

Research Article

Stability of Real Parametric Polynomial Discrete Dynamical Systems

Fermin Franco-Medrano[1,2] and Francisco J. Solis[1]

[1] *Applied Mathematics, CIMAT, 36240 Guanajuato, GTO, Mexico*
[2] *Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan*

Correspondence should be addressed to Francisco J. Solis; solis@cimat.mx

Received 23 November 2014; Revised 22 January 2015; Accepted 23 January 2015

Academic Editor: Zhan Zhou

We extend and improve the existing characterization of the dynamics of general quadratic real polynomial maps with coefficients that depend on a single parameter λ and generalize this characterization to cubic real polynomial maps, in a consistent theory that is further generalized to real mth degree real polynomial maps. In essence, we give conditions for the stability of the fixed points of any real polynomial map with real fixed points. In order to do this, we have introduced the concept of *canonical polynomial maps* which are topologically conjugate to any polynomial map of the same degree with real fixed points. The stability of the fixed points of canonical polynomial maps has been found to depend solely on a special function termed *Product Position Function* for a given fixed point. The values of this product position determine the stability of the fixed point in question, when it bifurcates and even when chaos arises, as it passes through what we have termed *stability bands*. The exact boundary values of these stability bands are yet to be calculated for regions of type greater than one for polynomials of degree higher than three.

1. Introduction

The theory of discrete dynamical systems with iteration functions given by polynomials is an intensive research subject where a wide variety of discrete models have been proposed to describe and to analyze different mechanisms in various areas of science. For example, in Biology and more specifically in Population Dynamics there are many simple models that are used to study the asymptotic behavior of some species that live in isolated generations; see, for instance, [1–7].

Although the dynamics of parametric polynomial discrete systems are very complex their bifurcation diagrams have proved to be a very useful visual tool. A new method for constructing a rich class of bifurcation diagrams for unimodal maps was presented in [8], where the behavior of quadratic maps was analyzed when the dependence of their coefficients was given by continuous functions of a parameter. Conditions on the coefficients of the quadratic maps were given in order to obtain regular reversal maps. Our first goal is to restate the results for more complex systems (cubic) than the quadratic systems analyzed in [8] and to state the results in the frame of a new formulation that would allow for generalization. Our second goal is to generalize the existing results on real quadratic maps for arbitrary real polynomial maps within a framework that allows us to understand the dynamics for a larger set of discrete systems. It is important to remark that our results are analytical and depend only on the parametric derivative of the system evaluated at equilibrium points. There are diverse results based on other approaches such as the linearized stability due to Lyapunov; see, for instance, [9, 10]. In our opinion, our approach is natural for polynomial iteration functions whereas the linearized stability can be used for more complex discrete systems with iteration functions such as piecewise functions. It is also important to notice that there is diverse numerical software specialized in the numerical continuation and bifurcation study of continuous and discrete parameterized dynamical systems, such as Auto [11] and MatCont [12].

Before attempting to obtain general results for polynomial discrete systems, we want to motivate them with those for a nontrivial system. To do this, we propose in Section 2 analyzing the stability of a general cubic discrete dynamical system. Then in Section 3, we use a general framework in order to analyze the stability for real polynomial discrete dynamical systems by using some of the ideas introduced in the previous section. In order to illustrate the obtained results several examples are included in Section 4. Finally, conclusions are given in Section 5.

2. Cubic Discrete Dynamical Systems

To motivate our results, we will start with a general cubic discrete system since it is in this case, besides linear and quadratic systems, when explicit calculations can be achieved. Then, for such system, we will define two particular forms, namely, the *Linear Factors Form* and the *Canonical Form*. We will show that these two forms are actually topologically conjugate, which in turn means that the property of chaos is preserved between the maps, which allows us to determine stability properties for any cubic map with real fixed points by analyzing only the Canonical Cubic Map.

Consider the cubic discrete dynamical system given in its General Form by $y_{n+1} = f_3(y_n;\lambda)$, where the iteration function is given by the following definition.

Definition 1 (general cubic map). The *general cubic map (GCM)* is defined by

$$f_3(y) := y + P_{f_3}(y), \tag{1}$$

where

$$P_{f_3}(y) = \alpha + \beta y + \gamma y^2 + \delta y^3 \tag{2}$$

is called the *fixed points polynomial* of f_3. All the coefficients α, β, γ, and δ are functions of the parameter λ.

It is evident that any cubic map can be put in this form by adjusting the corresponding values of the coefficients in the fixed points polynomial. By the fundamental theorem of algebra, we know that (2) has three roots, by which the GCM has three fixed points. The roots of P_{f_3} are then $y_0 = S + T - (1/3)\tilde{\gamma}$, $y_1 = -(1/2)(S+T)-(1/3)\tilde{\gamma}+(1/2)i\sqrt{3}(S-T)$, and $y_2 = -(1/2)(S+T)-(1/3)\tilde{\gamma}-(1/2)i\sqrt{3}(S-T)$, where $\tilde{\alpha} = \alpha/\delta$, $\tilde{\beta} = \beta/\delta$, $\tilde{\gamma} = \gamma/\delta$, $Q = (3\tilde{\beta}-\tilde{\gamma}^2)/9$, $R = (9\tilde{\beta}\tilde{\gamma}-27\tilde{\alpha}-2\tilde{\gamma}^3)/54$, $D = Q^3 + R^2$, $S = \sqrt[3]{R+\sqrt{D}}$, and $T = \sqrt[3]{R-\sqrt{D}}$. The coefficients of the fixed points polynomial (2) and its roots are related by $y_0+y_1+y_2 = -\tilde{\gamma}$, $y_0y_1+y_1y_2+y_2y_0 = \tilde{\beta}$, and $y_0y_1y_2 = -\tilde{\alpha}$. D is called the discriminant and we have three cases.

(i) If $D > 0$ then one fixed point is real and the other two are complex conjugates.

(ii) If $D = 0$ then the three fixed points are real with at least two of them equal.

(iii) If $D < 0$ then all fixed points are real and distinct.

It is the last two cases (real fixed points) that will interest us most for the time being. Suppose in particular that $D < 0$. Then, we can write

$$\begin{aligned} y_0 &= 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) - \frac{1}{3}\tilde{\beta}, \\ y_1 &= 2\sqrt{-Q}\cos\left(\frac{\theta+\pi}{3}\right) - \frac{1}{3}\tilde{\beta}, \\ y_2 &= 2\sqrt{-Q}\cos\left(\frac{\theta+2\pi}{3}\right) - \frac{1}{3}\tilde{\beta}, \end{aligned} \tag{3}$$

where $\cos\theta = R/\sqrt{-Q^3}$. Using the previous notation we have the following definition.

Definition 2 (linear factors form of the cubic map). Let f_3 be a general cubic map with three fixed points, y_0, y_1, and $y_2 \in \mathbb{C}$. One can write f_3 as

$$\begin{aligned} h_3(y) &= y + M(y-y_0)(y-y_1)(y-y_2) \\ &= y + s\widetilde{M}(y-y_0)(y-y_1)(y-y_2), \end{aligned} \tag{4}$$

where all M, y_0, y_1, and y_2 are functions of the parameter λ, $s = \operatorname{sign}(M)$, $\widetilde{M} = |M|$; one calls h_3 the *Linear Factors Form of the cubic map (LFFCM)*.

Now we will apply a linear transformation to (4) so that one fixed point is mapped to zero and the "amplitude" coefficient of the Linear Factors term is unity; this can be done since f_3 is cubic and at least one of the fixed points is real, so we can always map this fixed point to zero. The linear transformation can be chosen by each one of the following transformations:

$$\begin{gathered} T_0(x) = y_0 \pm \frac{x}{\sqrt{M}}, \qquad T_1(x) = y_1 \pm \frac{x}{\sqrt{M}}, \\ T_2(x) = y_2 \pm \frac{x}{\sqrt{M}}, \end{gathered} \tag{5}$$

by taking $y = T_k(x)$, $k \in \{0,1,2\}$. Without loss of generality, we will use T_0 with the plus sign and call it simply T, so that we get the following.

Definition 3 (canonical cubic map). The *Canonical Cubic Map (CCM)* is defined by

$$g_3(x;\lambda) = x + sx(x - x_1(\lambda))(x - x_2(\lambda)), \tag{6}$$

where it has been stressed out that both fixed points x_1 and x_2 depend upon the parameter λ.

So if $M > 0$ then

$$g_3(x) = x + x(x-x_1)(x-x_2), \tag{7}$$

and if $M < 0$

$$g_3(x) = x - x(x-x_1)(x-x_2). \tag{8}$$

The relationship between the roots of the Linear Factors Form of the cubic map and the Canonical Cubic Map (CCM) is given by the following.

Corollary 4. *The fixed points of the Linear Factors Form of the cubic map and the Canonical Cubic Map are related by*

$$x_1(\lambda) = \sqrt{M(\lambda)}\left[y_1(\lambda) - y_0(\lambda)\right],$$
$$x_2(\lambda) = \sqrt{M(\lambda)}\left[y_2(\lambda) - y_0(\lambda)\right]. \quad (9)$$

We have then reduced the parametric dependence to only two functions of the parameter λ: x_1 and x_2. Notice T is a homeomorphism between the *domains* of both maps; this will help us in Section 3 to prove that the Linear Factors Form and the Canonical Form of polynomial maps are actually topologically conjugate, which in turn means that the stability and chaos properties are preserved between the maps, which allows us to determine stability properties for any cubic map by analyzing only the CCM.

2.1. Stability for the Canonical Cubic Map. Let us determine the stability of the periodic points of the CCM. This analysis will suffice for any cubic map with real fixed points, by means of topological conjugacy. However, we can only explicitly give this for the fixed points. We already know, by construction, that the fixed points of the CCM are $x_0 = 0$, x_1, and x_2. While the first is constant, the other two fixed points are set to be functions of the parameter λ. By evaluating in g_3', we get the eigenvalue functions. For $x_0 = 0$ we have $\phi_0(\lambda) = g_3'(0) = sx_1(\lambda)x_2(\lambda) + 1$. So the stability condition for this fixed point is

$$-2 < sx_1x_2 < 0. \quad (10)$$

We can draw some conclusions from this. In order for zero to be a stable (attracting) fixed point one must have the following.

Lemma 5. *The following are sufficient conditions for the asymptotic stability of the zero fixed point of the Canonical Cubic Map:*

(i) *in magnitude, $|x_1||x_2| < 2$;*

(ii) *if $M > 0$, x_1 and x_2 must have different signs; or*

(iii) *if $M < 0$, x_1 and x_2 must have the same sign.*

Notice that the stability condition (10) states that the product of the relative positions from the other two fixed points to the zero fixed point must be within the range $(-2, 0)$, for positive M [or $(0, 2)$ for negative M], for the zero fixed point to be asymptotically stable.

The case of $x_k = 0$, $k \in \{1, 2\}$, is not included in the discussion here since this would represent repeated fixed points (multiplicity), which will be discussed in Section 2.3 below; likewise, in the remainder of this section we will avoid dealing with multiplicity of the fixed points. Now, for x_1, its eigenvalue function is $\phi_1(\lambda) = g_3'(x_1(\lambda)) = 1 + sx_1(\lambda)(x_1(\lambda) - x_2(\lambda))$, so that the stability condition for this fixed point is

$$-2 < sx_1(x_1 - x_2) < 0. \quad (11)$$

This fact gives us the following.

Lemma 6. *The following are sufficient conditions for the asymptotic stability of the x_1 fixed point.*

If $M > 0$ then

(i) *x_1 and x_2 must have the same sign;*

(ii) $|x_1| < |x_2| < |x_1 + 2/x_1|$.

On the other hand, if $M < 0$,

(i) $|x_2| < |x_1|$;

(ii) *if $|x_1| \geq \sqrt{2}$, then $|x_1 - 2/x_1| < |x_2| < |x_1|$; or*

(iii) *if $0 < x_1 < \sqrt{2}$, then $x_1 - 2/x_1 < x_2 < x_1$; or*

(iv) *if $-\sqrt{2} < x_1 < 0$, then $x_1 < x_2 < x_1 - 2/x_1$.*

Again, notice that the stability condition (11) for x_1 can be translated as that the product of the relative positions between the other two fixed points and x_1 must be within the range $(-2, 0)$ for positive M [or $(0, 2)$ for negative M]. Also notice that when $0 < |x_1| < \sqrt{2}$ the bound $x_1 - 2/x_1$ may be negative even if $x_1 > 0$ or positive even if $x_1 < 0$, therefore the usefulness of the distinction. For x_2 we have analogous results since it is indistinguishable from x_1 in the present formulation.

We will later generalize these "stability conditions" to functions of the parameter which are different for each fixed point, but of whose value depends on the stability of not only the fixed points, but also higher period periodic points, through period doubling bifurcations. From the stability conditions for the three fixed points we have proved the following.

Corollary 7. *A cubic polynomial map with three different real roots can only have a single attracting fixed point.*

Proof. Compare the stability conditions for the three fixed points. □

Also we have proved the following theorem.

Theorem 8. *Then sufficient conditions for the stability of a fixed point of the Canonical Cubic Map are as follows.*

If $M > 0$,

(i) *the product of the relative positions between each unstable fixed point and the stable one must be negative, which means one position is positive and the other negative, which leads us to the following;*

(ii) *the fixed point that lies between the other two will be stable, while the outer fixed points will be unstable, as long as the following holds;*

(iii) *the product of the relative positions between each unstable fixed point and the stable one must be greater than −2.*

And if $M < 0$,

(i) *the product of the relative positions between each unstable fixed point and the stable one must be positive, which means that both relative positions are positive, which leads us to the following;*

Table 1: Bifurcation values for the canonical cubic map.

k	c_k
1	2
2	3.0 ± 0.005
3	3.236 ± 0.002
4	3.288 ± 0.0005
5	3.29925 ± 0.00025
$\vdots$	$\vdots$
∞	$\sim 3.30228 \pm 5 \times 10^{-6}$

(ii) *either zero or the outer fixed point will be stable, while the other two fixed points will be unstable, as long as the following holds;*

(iii) *the product of the relative positions between each unstable fixed point and the stable one must be less than 2.*

2.2. Higher Period Periodic Points. Although, as previously stated, in general, we cannot calculate the values of the periodic points of period 2 or higher, we can calculate for which values of the stability conditions above the fixed points undergo period doubling bifurcations. We will see in Section 3 that these stability conditions can actually be generalized to something called the "Product Position Function," which depends on the parameter and is different for each fixed point. An asymptotic parameterization of the fixed points allowed us to determine the bifurcation values, c_k, of the fixed points of the CCM up to some precision. The values obtained are shown in Table 1. When the stability conditions of each fixed point cross these values, bifurcations take place. An estimation of the bifurcation value for the onset of chaos through a period doubling cascade has been calculated as $c_\infty \sim 3.30228 \pm 5 \times 10^{-6}$.

From these values, we can construct the analogue of the stability bands of the CQM for the CCM.

Definition 9 (stability bands of the CCM). Let $x_1, x_2 : \mathscr{A} \subseteq \mathbb{R} \to \mathbb{R}$ be the two nonzero fixed points of the family of cubic maps g_3, as given by Definition 3, and let $\{c_k\}_{k\in\mathbb{N}}$ be the sequence of bifurcation values of Table 1. The open interval

$$(-c_{k+1}, -c_k), \quad \lambda \in \mathscr{A} \tag{12}$$

is called the kth *stability band* of the CCM.

Notice, however, that in contrast with the stability bands of the Canonical Quadratic Map, the stability bands of the CCM cannot be plotted along the fixed points plots, at least not directly as just defined, but rather they must be represented in a separate plot for the stability conditions, as we will see in the examples of Section 4.

2.3. Multiplicity of the Fixed Points. When multiplicity of the fixed points takes place in the CCM, without loss of generality, g_3 can take the following forms:

$$g_3(x) = \begin{cases} x + sx^2(x - x_1), & \text{if } x_2 = x_0 = 0, x_1 \neq 0 \\ x + sx(x - x_1)^2, & \text{if } x_1 = x_2 \neq 0 \\ x + sx^3, & \text{if } x_1 = x_2 = 0, \end{cases} \tag{13}$$

with corresponding derivatives

$$g_3'(x) = \begin{cases} 1 + 2sx(x - x_1) + sx^2, & \text{if } x_2 = x_0 = 0, x_1 \neq 0 \\ 1 + sx(x - x_1)^2 + 2sx(x - x_1), & \text{if } x_1 = x_2 \neq 0 \\ 1 + 3sx^2, & \text{if } x_1 = x_2 = 0, \end{cases} \tag{14}$$

and therefore, $g_3'(x_k) = 1$, $k \in \{0, 1, 2\}$, for all three cases, so that we deal with nonhyperbolic fixed points.

Proposition 10. *The stability of the fixed points of the CCM when they present multiplicity is, without loss of generality, as follows.*

(1) *If $x_2 = x_0 = 0$, $x_1 \neq 0$, the zero fixed point is an unstable fixed point with multiplicity of two;*

  (i) *if $M > 0$ and*

    (a) *if $x_1 > 0$ it is semiasymptotically stable from the right,*

    (b) *if $x_1 < 0$ it is semiasymptotically stable from the left;*

  (ii) *or if $M < 0$ and*

    (a) *if $x_1 > 0$ it is semiasymptotically stable from the left,*

    (b) *if $x_1 < 0$ it is semiasymptotically stable from the right.*

(2) *If $x_1 = x_2 \neq 0$, this fixed point has multiplicity of two and it is unstable; moreover*

  (i) *if $M > 0$ and*

    (a) *if $x_1 > 0$ it is semiasymptotically stable from the left,*

    (b) *if $x_1 < 0$ it is semiasymptotically stable from the right;*

  (ii) *or if $M < 0$ and*

    (a) *if $x_1 > 0$ it is semiasymptotically stable from the right,*

    (b) *if $x_1 < 0$ it is semiasymptotically stable from the left.*

(3) *If $x_0 = x_1 = x_2 = 0$, the zero fixed point has multiplicity of three;*

  (i) *if $M > 0$, it is unstable;*

  (ii) *if $M < 0$, it is asymptotically stable.*

Proof. Notice that

$$g_3''(x) = \begin{cases} 2s(x - x_1) + 4sx, & \text{if } x_2 = x_0 = 0, x_1 \neq 0 \\ 4s(x - x_1) + 2sx, & \text{if } x_1 = x_2 \neq 0 \\ 6sx, & \text{if } x_1 = x_2 = 0, \end{cases}$$
$$g_3'''(x) = 6s \neq 0, \tag{15}$$

for all cases. Using the stability (ST) and semistability (SST) theorems for nonhyperbolic points [13, pp. 4-5], therefore

(1) if $x_2 = x_0 = 0$, $x_1 \neq 0$, the zero fixed point has multiplicity of two and we have that $g_3'(0) = 1$, $g_3''(0) = -2sx_1 \neq 0$, and $g_3'''(0) = 6s \neq 0$; therefore, by ST, the zero fixed point is an *unstable* fixed point. Applying SST we get the particular cases of semistability;

(2) if $x_1 = x_2 \neq 0$, this fixed point has multiplicity of two and we have that $g_3'(x_1) = 1$, $g_3''(x_1) = 2sx_1 \neq 0$, and $g_3'''(0) = 6s \neq 0$; therefore, by ST, this fixed point is *unstable*; moreover, the semistability cases are inferred from SST again;

(3) if $x_0 = x_1 = x_2 = 0$, the zero fixed point has multiplicity of three and we have that $g_3'(0) = 1$, $g_3''(0) = 0$, and $g_3'''(0) = 6s \neq 0$; therefore, by ST, if $M > 0$ the zero fixed point is *unstable* and if $M < 0$ it is *asymptotically stable*.

□

3. Polynomial Discrete Dynamical Systems

Consider again a one-dimensional discrete dynamical system defined by

$$y_{n+1} = f(y_n; \lambda), \tag{16}$$

where f is a polynomial in one real variable y with real fixed points and whose coefficients depend smoothly on the real parameter λ. Depending on the form of f we have defined previously the General, Linear Factors, and Canonical Forms of the cubic maps. Next, we will define precisely the General and Canonical Maps of a mth degree polynomial map.

Definition 11 (general polynomial map). The General Polynomial Map of mth degree (GPM-m) is defined by

$$f_m(y) := y + (-1)^{m-1} P_{f_m}(y), \tag{17}$$

where

$$P_{f_m}(y) := (-1)^{m-1} \sum_{i=0}^{m} a_i y^i. \tag{18}$$

P_{f_m} is called the *fixed points polynomial* associated with f_m.

It is known that any mth degree real polynomial in one variable can be put into the General Form by means of adjusting the value of the a_1 coefficient properly in the fixed points polynomial. This is the broadest class of real polynomials of finite degree. The roots of P_{f_m} are the fixed points of f_m. In the case of m odd, the fundamental theorem of algebra guarantees the existence of at least one real fixed point. Let $y_i \in \mathbb{C}$, $i \in \{0, 1, \ldots, m-1\}$, be the m roots of P_{f_m}; then $(y - y_i)$ is a factor of P_{f_m} by the factor theorem; therefore we can rewrite P_{f_m} as

$$\begin{aligned} P_{f_m}(y) &= M(y - y_0)\cdots(y - y_{m-1}) \\ &= M\prod_{j=0}^{m-1}(y - y_j), \quad M \in \mathbb{R}, \end{aligned} \tag{19}$$

and then define the following.

Definition 12 (linear factors form). Let f_m be the GPM-m and y_j, $j \in \{0, \ldots, m-1\}$, its m fixed points. Then one can write

$$\begin{aligned} f_m(y) &= y + (-1)^{m-1} M\prod_{i=0}^{m-1}(y - y_i), \\ &= y + (-1)^{m-1}\operatorname{sgn}(M)|M|\prod_{i=0}^{m-1}(y - y_i), \\ &= y + (-1)^{m-1} s\widetilde{M}\prod_{i=0}^{m-1}(y - y_i), \end{aligned} \tag{20}$$

with the definitions of s and $\widetilde{M}$ used in Section 2. One calls this the *Linear Factors Form* of f_m.

We can directly verify that for $m = 3$ we obtain the corresponding Linear Factors Form of the cubic maps. Once we know the m fixed points of a map f_m, it is straightforward to write its Linear Factors Form. The motivation behind the $(-1)^{m-1}$ factor is that we want that, for purely aesthetic reasons, if $M > 0$, the fixed points are real, and $0 < y_0 < y_1 < \cdots < y_{m-1}$, we have that $f_m'(0) \geq 0$, which is thus accomplished.

We will now restrict this set of polynomials to those whose fixed points polynomials have only real roots, that is, maps with real fixed points only, though not necessarily distinct. Let us make this precise by the following.

Definition 13 (canonical polynomials set). The Canonical Polynomials Set, denoted by $P_C[y]$, is

$$P_C[y] := \left\{ f \in \mathbb{R}[y] \mid P_f \text{ has only real roots} \right\}, \tag{21}$$

where $\mathbb{R}[y]$ is the set of polynomials with real coefficients on the variable y. Likewise, $P_C^m[y]$ denotes $P_C[y] \bigcap \mathbb{R}_m[y]$, where $\mathbb{R}_m[y]$ is the set of polynomials of degree m with real coefficients on the variable y.

The set P_C has been our main work ground for the analysis in this work and, as it turns out, its elements can be put in a much nicer form, easier to understand. We can further reduce the complexity of this set of maps by means of the transformation

$$y = T_m(x) := s\widetilde{M}^{-1/(m-1)} x + y_0, \tag{22}$$

where y_0 is a real fixed point of the map in its linear product form. Notice that T_m is linear; therefore it has an inverse

$$x = T_m^{-1}(y) = s\widetilde{M}^{1/(m-1)}(y - y_0). \tag{23}$$

T_m is in fact a homeomorphism. We will drop the subscript m when referring to the transformation for no specific degree. Applying this transformation to y we reach the following.

Definition 14 (canonical polynomial map). The canonical polynomial map of mth degree (CPM-m) is

$$g_m(x) := x + (-1)^{m-1} s^m x \prod_{i=1}^{m-1} (x - x_i), \quad m \geq 2, \tag{24}$$

where

$$x_i = s\widetilde{M}^{1/(m-1)}(y_i - y_0), \tag{25}$$

and y_j are the m fixed points of the corresponding Linear Factors Form map of mth degree, (at least) y_0 being real.

It is clear from the definition that $x_0 = 0$ always. Notice also that the x_i result from evaluating T_m^{-1} in the corresponding y_i. We can easily prove that not only does the canonical map result from applying T, but also the canonical map is in fact T-conjugate to the Linear Factors Form.

Proposition 15. *Let f_m and T_m and g_m be as defined above, having f_m at least one real fixed point; let y_0 be this real fixed point, without loss of generality. Then f_m is T_m-conjugate to g_m.*

Proof. It is clear that T_m is a homeomorphism since it is linear. Then, we must only prove that $T_m \circ f_m = g_m \circ T_m$; that is, $f_m(T_m(x)) = T_m(g_m(x))$. We then have

$$\begin{aligned}
&f_m(T_m(x)) \\
&\quad = f_m\left(s\widetilde{M}^{-1/(m-1)}x + y_0\right) \\
&\quad = s\widetilde{M}^{-1/(m-1)}x + y_0 \\
&\qquad + (-1)^{m-1} s\widetilde{M} \prod_{i=0}^{m-1} \left(s\widetilde{M}^{-1/(m-1)}x + y_0 - y_i\right) \\
&\quad = s\widetilde{M}^{-1/(m-1)}x + (-1)^{m-1} s^2 \widetilde{M}\widetilde{M}^{-1/(m-1)}x \\
&\qquad \cdot \prod_{i=1}^{m-1} \left(s\widetilde{M}^{-1/(m-1)}x + y_0 - y_i\right) + y_0 \\
&\quad = s\widetilde{M}^{-1/(m-1)}x + (-1)^{m-1} s^{m-1} \widetilde{M}^{-1/(m-1)}x \\
&\qquad \cdot \prod_{i=1}^{m-1} \left[x - s\widetilde{M}^{1/(m-1)}(y_i - y_0)\right] + y_0 \\
&\quad = s\widetilde{M}^{-1/(m-1)}x + (-1)^{m-1} s^{m-1} M^{-1/(m-1)}x \\
&\qquad \cdot \prod_{i=1}^{m-1} (x - x_i) + y_0 \\
&\quad = s\widetilde{M}^{-1/(m-1)} \left[x + (-1)^{m-1} s^m x \prod_{i=1}^{m-1} (x - x_i)\right] + y_0 \\
&\quad = T_m(g_m(x)),
\end{aligned} \tag{26}$$

where we have used $s^2 = 1$ and $s = s^{-1}$. □

This turns out to be very useful, since we know that topological conjugacy is an equivalence relation that *preserves the property of chaos*. This means that the analysis of stability and chaos (i.e., the "dynamics") of real polynomial maps with real fixed points is reduced to the study of the canonical polynomial maps defined above, since we can always take any polynomial in $P_C[x]$ to its Canonical Form by means of T, determine the stability properties, and then go back to the original polynomial. A commutative diagram of the conjugacy is in Figure 5.

3.1. Stability and Chaos in the Canonical Map of Degree m. The derivative of g_m, recalling $x_0 = 0$ to simplify notation, is

$$g_m'(x) = 1 + (-1)^{m-1} s^m \sum_{j=0}^{m-1} \prod_{i=0, i\neq j}^{m-1} (x - x_i). \tag{27}$$

Evaluating (27) in the fixed point x_k we get the eigenvalue function for each x_k:

$$\begin{aligned}
\phi_k(\lambda) &= g_m'(x_k(\lambda)) \\
&= 1 + (-1)^{m-1} s^m \prod_{i=0, i\neq k}^{m-1} (x_k(\lambda) - x_i(\lambda)) \\
&= 1 + s^m \prod_{i=0, i\neq k}^{m-1} (x_i(\lambda) - x_k(\lambda)).
\end{aligned} \tag{28}$$

Then, the asymptotic stability condition $|g_m'(x_k)| < 1$ implies that

$$-2 < s^m \prod_{i=0, i\neq k}^{m-1} (x_i - x_k) < 0. \tag{29}$$

From (29) we can recover all the stability conditions for the fixed points of the Canonical Quadratic Map and cubic map. The above leads us to the following.

Definition 16 (Product Position Function). Let g_m be the canonical polynomial map of mth degree and $x_0 = 0$, and let $x_1, \ldots, x_{m-1}$be its m fixed points, all of which depend upon the parameter λ. Let x_k be a real fixed point among the latter. Then

$$D_{m,k}(\lambda) := s^m \prod_{i=0,i\neq k}^{m-1} \left(x_i(\lambda) - x_k(\lambda)\right), \quad k \in \{0,\ldots,m-1\}, \quad m \geq 2, \tag{30}$$

is called the *Product Position Function (PPF)* of x_k.

The definition is motivated by the fact that $D_{m,k}$ is a product of the positions relative to x_k of each of the other $m-1$ fixed points and that this quantity is fundamental in determining the stability of the fixed points. These positions are positive when $x_i > x_k$ and negative when $x_i < x_k$. We have stressed the dependence on the parameter λ in the definition of $D_{m,k}$ so that its character as a function is clear, stemming from the corresponding dependence on λ of the fixed points. In this way, the stability condition for the kth fixed point is reduced to

$$-2 < D_{m,k}(\lambda) < 0. \tag{31}$$

Since $D_{m,k}$ must be negative in order for x_k to be stable as a sufficient condition and an *odd* number of factors $(x_i - x_k)$ must be negative for the product in $D_{m,k}$ to be negative, it follows that if m is even or $M > 0$, an odd number of negative factors $(x_i - x_k)$ is a necessary condition for the hyperbolic fixed point x_k to be stable; that is, if $M > 0$, an *odd* number of fixed points must lie below x_k and, consequently, an even or zero (resp., odd) number of fixed points must lie above x_k if m is even (resp., odd). By similar arguments, we can prove the following.

Proposition 17 (necessary conditions for the stability of x_k). *Let g_m, $D_{m,k}$ be defined as above and let x_k be a hyperbolic real fixed point of g_m. The following are necessary conditions for x_k to be an asymptotically stable fixed point:*

(i) *if $M > 0$ or m is even, an odd number of fixed points must have values lower than x_k; or*

(ii) *if m is odd and $M < 0$, zero or an even number of fixed points must have values lower than x_k.*

We must remark that the above conditions are *not* sufficient for a fixed point to be an attractor. The sufficient condition, however, is stated as follows.

Theorem 18 (sufficient condition for the stability of x_k). *Let g_m, $D_{m,k}$ be defined as above and let x_k be a hyperbolic real fixed point of g_m. Then, a necessary and sufficient condition for x_k to be an attractor is that*

$$-2 < D_{m,k}(\lambda) < 0. \tag{32}$$

Below the value of -2 there are other "stability bands" that lead to further period doubling bifurcations of the fixed points as they are crossed, but they must be calculated numerically and, as we have seen, depend on the degree m of the polynomial.

Let us remark that for those values of λ in (32) where $D_{m,k}(\lambda) = -2$ or $D_{m,k}(\lambda) = 0$ there are some stability conditions that require higher parametric derivatives; see, for instance, [14, 15].

4. Examples

Here we will deal with specific parameterizations for the fixed points x_1 and x_2 in order to clarify the above findings and to demonstrate how we can construct bifurcation diagrams with specific predetermined properties with cubic maps. We will consider $M > 0$ unless otherwise stated explicitly.

Example 1. First, consider linear parameterizations for both x_1 and x_2 as

$$x_1(\lambda) = -\lambda, \qquad x_2(\lambda) = \lambda. \tag{33}$$

The result is plotted in the lower panel of Figure 1, where we see that the middle fixed point is $x_0 = 0$ always, so we expect this to be the only stable fixed point, until the separation between this and the other points breaks the stability condition and the period doubling bifurcation cascade sets on. The corresponding stability conditions are shown in the middle panel of Figure 1, where we confirm that the curve for x_0 is the only one within the stability band $(-2, 0)$ until $\lambda \approx 1.45$, where the curve crosses the barrier of -2 getting into the stability band $(-3, -2)$, causing x_0 to bifurcate. The corresponding bifurcation diagram is shown in the upper panel of Figure 1, where we confirm the statement stated above.

Example 2. Now we will explore the full range of stability regions by making a linearly varying fixed point pass through the regions defined by the constant x_0 and a constant x_1. We define then

$$x_1(\lambda) = 2, \qquad x_2(\lambda) = 6\lambda + 1. \tag{34}$$

We then obtain the plot of the lower panel of Figure 2, for the selected range of interest of the parameter λ. In the middle panel of the same figure we can see the stability curves for the fixed points, where we see that initially, from left to right, all fixed points are unstable and then, progressively, x_0, x_2, and x_1 become stable, the latter one losing stability for still greater values of λ. The corresponding bifurcation diagram is shown in the upper panel, where we can see how first the stable fixed point is $x_0 = 0$, since it is the middle one, but begins in the chaotic region and goes "reversal" towards being stable; then, as x_2 crosses through zero, it becomes the middle stable fixed point and when it in turn crosses the constant x_2, this latter one becomes the stable fixed point, again loosing stability when x_2 crosses the stability band for x_1.

Example 3 (quartic maps). Using Definition 12 for $n = 4$, we have that $f_4(y) = y - M(y-y_0)(y-y_1)(y-y_2)(y-y_3)$. Suppose f_4 has at least one real fixed point. Without loss of generality, suppose this fixed point is y_0. Then $T_4(x) = s\widetilde{M}^{-1/3}x + y_0$. Making the substitution $y = T_4(x)$ we can verify that we get $g_4(x) = x - x(x-x_1)(x-x_2)(x-x_3)$, where $x_i = s\widetilde{M}^{1/3}(y_i - y_0)$,

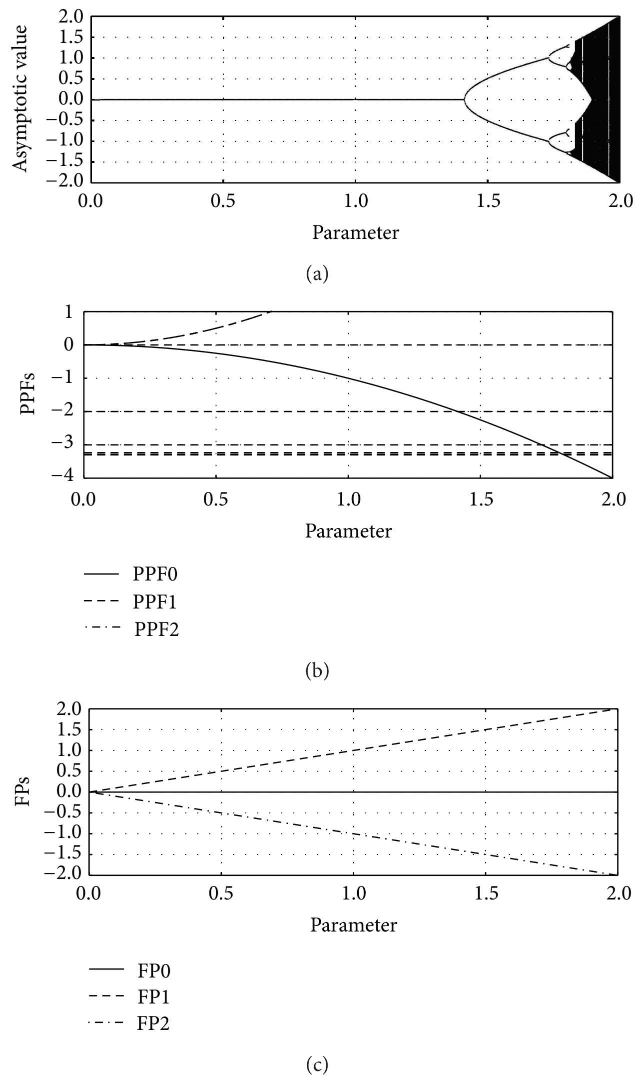

Figure 1: Bifurcation diagram (a), Product Position Functions and stability bands (b), and fixed points (c) for the linear parameterization of the fixed points example of the CCM.

$i \in \{0, 1, 2, 3\}$. The stability of a real fixed point x_k is given by the Product Position Function $D_{4,k}(\lambda) = \prod_{i=0, i\neq k}^{3}(x_i(\lambda) - x_k(\lambda))$, whose value must remain between minus two and zero in order for x_k to be asymptotically stable; that is, if all fixed points are real,

$$\begin{aligned} -2 &< D_{4,0}(\lambda) = x_1 x_2 x_3 < 0, \\ -2 &< D_{4,1}(\lambda) = -x_1(x_2 - x_1)(x_3 - x_1) < 0, \\ -2 &< D_{4,2}(\lambda) = -x_2(x_1 - x_2)(x_3 - x_2) < 0, \\ -2 &< D_{4,3}(\lambda) = -x_3(x_1 - x_3)(x_2 - x_3) < 0, \end{aligned} \tag{35}$$

for 0, x_1, x_2, and x_3 to be asymptotically stable fixed points, respectively. For example, let

$$x_1(\lambda) = \lambda, \quad x_2(\lambda) = -\lambda, \quad x_3(\lambda) = 2\lambda. \tag{36}$$

The plots of these fixed points with their corresponding parametric dependence on λ are shown in Figure 3(c). In light of Proposition 17 we expect only x_0 and x_3 to be able to be asymptotically stable fixed points in any given range of λ. As Figure 3(b) shows, precisely x_0 and x_3 are the fixed points whose product distances cross the stability band $(-2, 0)$ in the range of λ being plotted. As we recall, the product distance functions are the "stability conditions" of the fixed points. As long as the product distances remain within the stability interval, the fixed points are attractors, as we can verify in Figure 3(a); also in this last panel, we can see the two attracting fixed points at the beginning of the plotted range; then, first x_3 loses its stability and gives rise to the period doubling bifurcations cascade which leads to chaotic behavior; later, zero also loses its stability and also gives rise to period doubling and chaos.

Example 4. The logistic map, $L_\lambda(x) = \lambda x(1 - x)$, is the most immediate and obvious example application [16–19]. This map undergoes a series of period doubling bifurcations

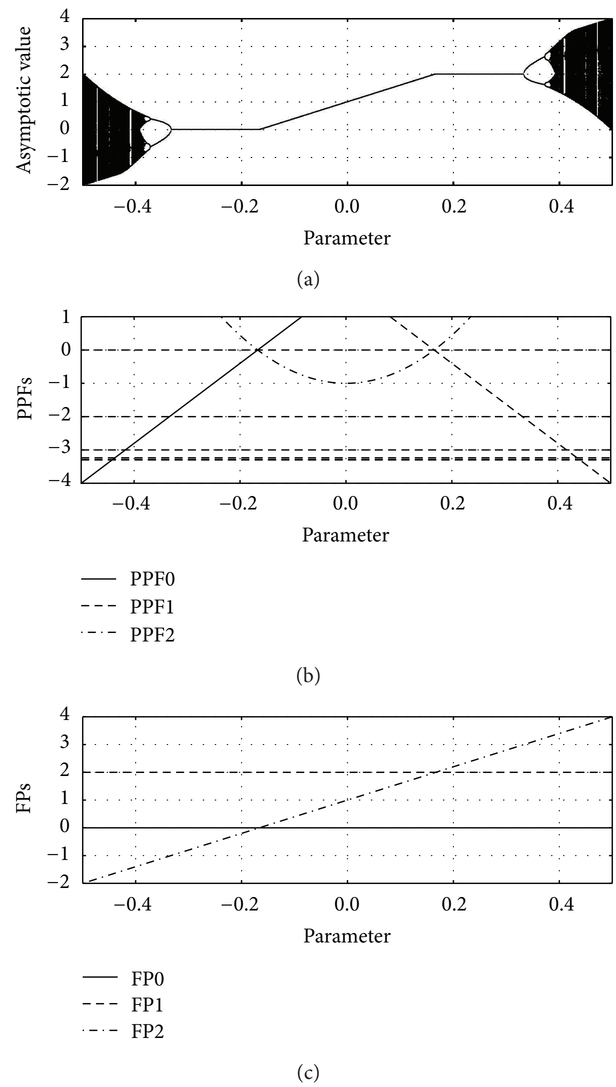

Figure 2: Bifurcation diagram (a), Product Position Functions and stability bands (b), and fixed points (c) for the constant and linear mixed parameterizations of the fixed points example of the CCM.

starting at the value of $\lambda = 3$, ultimately achieving a chaotic nature at $\lambda \approx 3.570$ [18, p. 47]. The fixed points of the logistic map are $y_0 = 0$ and $y_1 = (\lambda - 1)/\lambda$ [18, p. 43]. The corresponding Linear Factors Form of the logistic map is then $h_\lambda(x) = x - \lambda x(x - ((\lambda - 1)/\lambda))$, where we can identify the functions of the parameters M, y_0, and y_1 from its definitions [13] as $s = +1$, $\widetilde{M}(\lambda) = \lambda$, $y_0(\lambda) = 0$, and $y_1(\lambda) = (\lambda - 1)/\lambda$. The corresponding nonzero fixed point of the canonical logistic map, $x_1(\lambda) = s\widetilde{M}(y_1 - y_0)$, is then simply [13] $x_1(\lambda) = \lambda - 1$, from which we can state that the *canonical logistic map* takes the explicit form $g_\lambda = x - x(x - \lambda + 1)$. In order to determine the stability properties of these fixed points, both zero and nonzero, in the canonical logistic map, it is then sufficient, as we have proved in Section 3, to observe the behavior of the Product Position Functions (PPFs) of these fixed points; namely, $D_{g,0}(\lambda) = x_1 - 0 = x_1 = \lambda - 1$ and $D_{g,1}(\lambda) = 0 - x_1 = -x_1 = 1 - \lambda$. By determining when these PPFs cross the stability bands whose boundaries are shown in Table 2 [13] we can readily determine when these fixed points are stable or unstable, when they bifurcate, and when they reach any 2^n attracting periodic orbit for any n, up to crossing the b_∞ band. This whole process is depicted in Figure 4. In particular, we can see from Table 2, and again in Figure 4, that when $-1 < x_1 < 0$, the zero fixed point is attracting since its PPF lies within the first stability band and then exchanges stability at $x_1 = 0$, when this last FP becomes stable and proceeds to a period doubling cascade upon its PPF, $-x_1$, crossing the bands defined by the bifurcation values $-b_1$, $-b_2$, and so forth until reaching $x_1 = b_\infty \approx 2.569941$. This last value agrees quite well and improves upon the approximation reported in [18] of 3.570 for the logistic map, since with the calculations of the present work $\lambda_\infty = 1 + b_\infty \approx 3.569941 \pm 5 \times 10^{-7}$. Finally, since these maps are topologically conjugate and it is known that the logistic map is chaotic starting with $\lambda = 4$ [18], we conclude that the Canonical Quadratic Map must be so starting from $x_1 = 3$, which we may denote by b_c, our final "bifurcation" value.

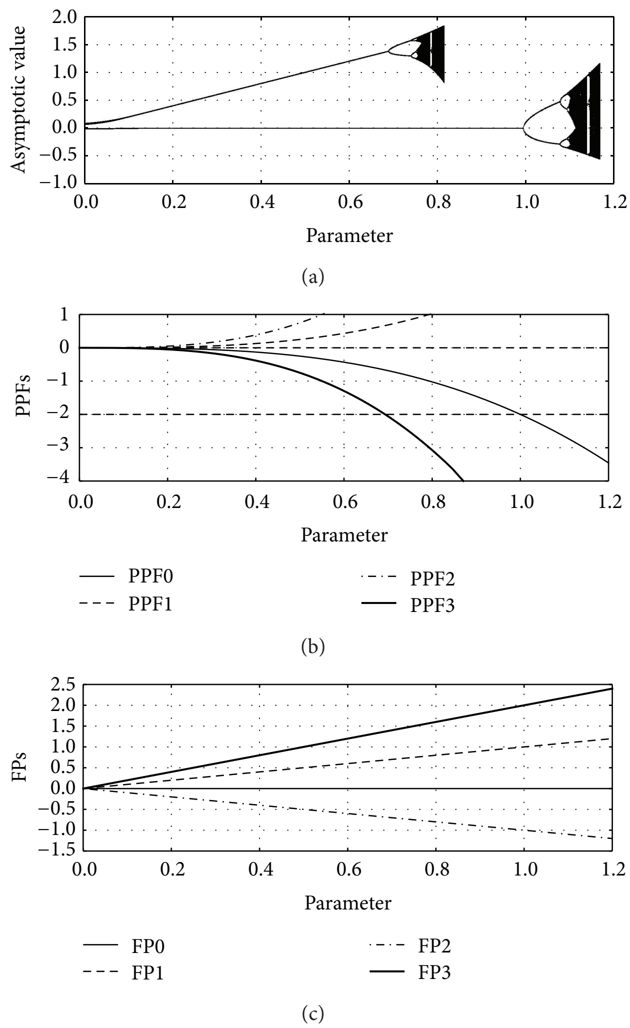

Figure 3: Bifurcation diagram (a), Product Position Functions and first stability band (b), and fixed points (c) of the quartic polynomial map example.

Table 2: Bifurcation values for the CQM. Reproduced from [13].

k	b_k
1	2
2	$\sqrt{6}$
3	2.5440 ± 0.0005
4	2.5642 ± 0.0002
5	$2.56871 \pm 4 \times 10^{-5}$
6	$2.56966 \pm 1 \times 10^{-5}$
7	$2.569881 \pm 5 \times 10^{-6}$
$\vdots$	$\vdots$
∞	$\sim 2.569941 \pm 5 \times 10^{-7}$

Example 5 (harvesting strategies). The connection of the logistic map with population models is old and well known. In [19] there are a few examples of second degree polynomials used as recurrence functions for modeling "harvesting," or hunting, strategies of animal populations. The main idea is that the animal populations grow whenever there are food and resources in the environment which, by account of its finite resources, has a certain "carrying capacity"; this leads to a maximum population which this environment can hold, the population growing according to the logistic model. The population may then be "harvested," or hunted, at a certain rate yet to be specified and, depending on this rate, it is not hard to imagine that the final fate of the population of animals may be (i) extinction, if the rate is too high; (ii) steady population below the carrying capacity, if the rate is "just right"; or (iii) steady population at its maximum value dictated by the carrying capacity of the environment. A final fourth possibility—perhaps more rare—is to (iv) bring more individuals of the species from outside the system under analysis and introduce them to it, therefore making it possible for the population to surpass in number the carrying capacity of the system, but only to return naturally to the maximum value after a finite number "time-steps." To examine this in detail consider the system defined by the recurrence relation $\Delta y \equiv y_{n+1} - y_n = r(1 - y_n)y_n$. Here the population growth in any

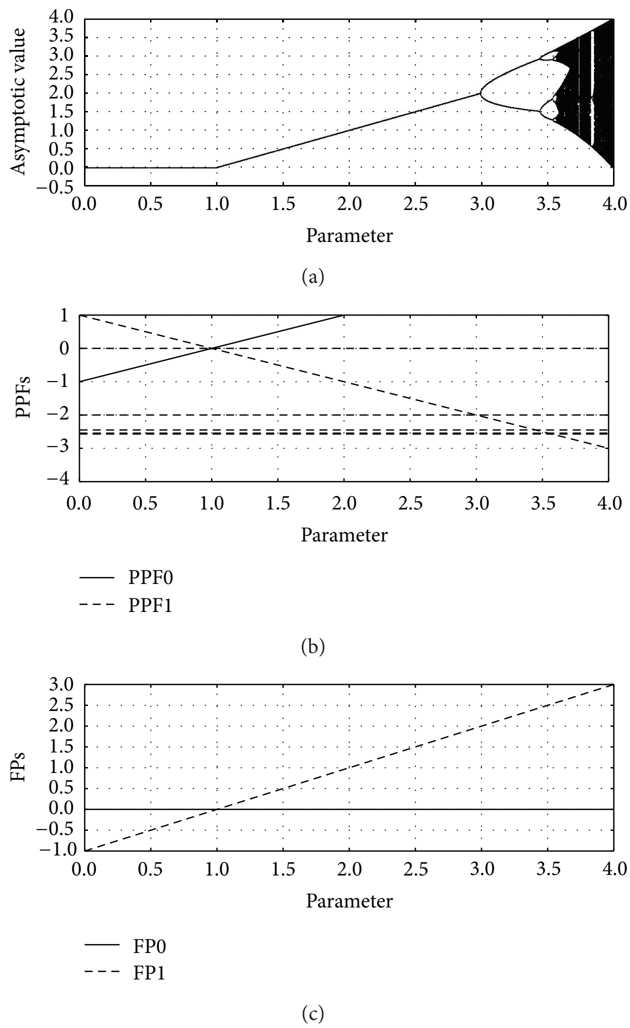

Figure 4: Bifurcation diagram (a), Product Position Functions (b), and fixed points (c) for the canonical logistic map.

$$\begin{tikzcd}
y \arrow[r, equal] & y \arrow[r, "f_m"] \arrow[d, "T^{-1}"] & f_m(y) \arrow[r, equal] & f_m(y) \arrow[d, "T^{-1}"] \\
x \arrow[r, equal] \arrow[u, "T"] & x \arrow[r, "g_m"'] & g_m(x) \arrow[r, equal] \arrow[u, "T"] & g_m(x)
\end{tikzcd}$$

Figure 5

given period Δy is proportional to both the initial population y_n and the difference between this and the maximum population, normalized to the value of 1. The proportion constant is the growth rate r. After some simplification, we can rewrite system as $y_{n+1} = (1 + r)y_n - ry_n^2$. We have yet to add the "harvesting term(s)," which we might do in several ways. If we consider a fixed rate (say each period we harvest a proportion b of the population (in terms of the maximum)), then $f(y) = (1 + r)y - ry^2 - b$ is the corresponding map of this discrete dynamical system. From here we see that the fixed points polynomial (FPP) of f is $P_f(y) = ry^2 - ry + b$, whose FPs we can determine to be $y_\pm = (1 \pm \sqrt{1 - 4b/r})/2$, from which we immediately calculate the Linear Factors Form (LFF) to be $f(y) = y - ry(y - y_+)(y - y_-)$. From this form it is also straightforward to determine that in the corresponding Canonical Form we have $x_1(b) = r(y_+ - y_-) = \sqrt{r(r - 4b)}$. Both systems are related then by the linear transformations $y_n = y_- + x_n/r$ and $x_n = r(y_n - y_-)$. With this choice, the zero fixed point in the canonical map corresponds to y_- and the nonzero fixed point to y_+. To analyze the stability of this system, consider r to be fixed and given and consider b to be the parameter of this family of systems. The one immediate conclusion is that, for $x_1 \in \mathbb{R}$, we necessarily have that $x_1 \geq 0$. Now, real x_1 implies $b \leq r/4$. Over the $r/4$ value x_1 becomes complex which does not give any fixed points (but would mean "overharvesting"). Since "harvesting" cannot be negative, it is clear $b = 0$ corresponds to the maximum value of $x_1 = r$ and that $x_1 = 0$ when $b = r/4$. Remember now that to analyze the stability of x_1 we consider its PPF; $D_1 = -x_1$. Analogously, $D_0 = x_1$. Since the maximum value of x_1 is r, this being the "unconstrained growth rate," $0 < r < 1$, we have that $-r < D_1 < 0$, therefore putting the x_1 in the stability range between $b_0 = 0$ and $-b_1 = -2$ determined by the first stability band. This range is never left in any situation with physical meaning so we conclude that the nonzero fixed point, that is, y_+ in the original system, is always the only asymptotically stable fixed point and the zero fixed point, that is, y_- in the original system, is always unstable. The only case to analyze with care is $b = 0$ since then the two fixed points collide, but the semistability theorem [18] guarantees that in this case $x_1 = 0$ is semistable from the right.

In conclusion,

(1) when $b = r/4$ the population faces extinction asymptotically. Over this value extinction is achieved in a finite number of steps, there not being any more fixed points;

(2) when $0 < b < r/4$, $x_n \rightarrow r$ and the population tends to $y_+ = 0.5(1 + \sqrt{1 - 4b/r})$;

(3) when $b = 0$, that is, no harvesting, x_n still tends to $x_1 = 0$ from the right and, correspondingly, the population tends asymptotically to $y_+ = 1$.

5. Conclusions

We can summarize the findings of this work as having successfully given conditions for the stability of the fixed points of any real polynomial map with real fixed points and that depends on a single parameter. In order to do this we have defined "canonical polynomial maps" which are topologically conjugate to any polynomial map of the same degree with real fixed points. Then, the stability of the fixed points of the canonical polynomial maps has been found to depend solely on a special function called "product position" of a given fixed point. The values of this product position determine the stability of the fixed point and when it bifurcates to give rise to attracting periodic orbits of period 2^n for all n and even when chaos arises through the period doubling cascade, as it passes through different "stability bands," although the exact values and widths of these stability bands are yet to be calculated for regions of type greater than one for higher order polynomials. The latter must be done numerically. The proposed methodology allows us to create discrete dynamical systems with some prescribed bifurcation diagram. Ultimately it is desired to obtain extensive tables of the bifurcation values for higher order polynomials. The power and simplicity of the proposed methodology will best be appreciated with 3rd or higher degree polynomials and when the implications for the Taylor polynomial of any nonlinear map are understood.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors wish to thank the support of the Mexican National Council of Science and Technology (CONACYT) and the Centro de Investigación en Matemáticas (CIMAT) for their financial support for the present research.

References

[1] H. V. Kojouharov and B. M. Chen, "Nonstandard methods for advection-diffusion-reaction equations," in *Applications of Nonstandard Finite Difference Schemes*, pp. 55–108, World Science Publisher, River Edge, NJ, USA, 2000.

[2] S. Sinha and S. Parthasarathy, "Unusual dynamics of extinction in a simple ecological model," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 93, no. 4, pp. 1504–1508, 1996.

[3] G. Chen and X. Dong, "From chaos to order—perspectives and methodologies in controlling chaotic nonlinear dynamical systems," *International Journal of Bifurcation and Chaos*, vol. 3, no. 6, pp. 1363–1409, 1993.

[4] R. M. May, "Biological populations with non-overlapping generations: stable points, stable cycles, and chaos," *Science*, vol. 186, no. 4164, pp. 645–647, 1974.

[5] R. M. May, "Simple mathematical models with very complicated dynamics," *Nature*, vol. 261, no. 5560, pp. 459–467, 1976.

[6] T. Hüls, "A model function for polynomial rates in discrete dynamical systems," *Applied Mathematics Letters*, vol. 17, no. 1, pp. 1–5, 2004.

[7] J. C. Panetta, "A mathematical model of drug resistance: heterogeneous tumors," *Mathematical Biosciences*, vol. 147, no. 1, pp. 41–61, 1998.

[8] F. J. Solis and L. Jódar, "Quadratic regular reversal maps," *Discrete Dynamics in Nature and Society*, vol. 2004, no. 2, pp. 315–323, 2004.

[9] B. Zhang, S. Xu, and Y. Zou, "Improved stability criterion and its applications in delayed controller design for discrete-time systems," *Automatica*, vol. 44, no. 11, pp. 2963–2967, 2008.

[10] Q. Zhu and L. Guo, "Stable adaptive neurocontrol for nonlinear discrete-time systems," *IEEE Transactions on Neural Networks*, vol. 15, no. 3, pp. 653–662, 2004.

[11] E. Doedel and J. Kernevez, "AUTO: software for continuation and bifurcation problems in ordinary differential equations," Tech. Rep., California Institute of Technology, 1986.

[12] A. Dhooge, W. Govaerts, Y. A. Kuznetsov, H. G. Meijer, and B. Sautois, "New features of the software MatCont for bifurcation analysis of dynamical systems," *Mathematical and Computer Modelling of Dynamical Systems: Methods, Tools and Applications in Engineering and Related Sciences*, vol. 14, no. 2, pp. 147–175, 2008.

[13] F. Franco-Medrano, *Stability and chaos in real polynomial map [M.S. thesis]*, Centro de Investigación en Matemáticas A.C. (CIMAT), Guanajuato, Mexico, 2013.

[14] F. M. Dannan, S. N. Elaydi, and V. Ponomarenko, "Stability of hyperbolic and nonhyperbolic fixed points of one-dimensional maps," *Journal of Difference Equations and Applications*, vol. 9, no. 5, pp. 449–457, 2003.

[15] V. Ponomarenko, "Faà di Bruno's formula and nonhyperbolic fixed points of one-dimensional maps," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 29, pp. 1543–1549, 2004.

[16] R. A. Holmgren, *A First Course in Discrete Dynamical Systems*, Springer, New York, NY, USA, 1994.

[17] R. Devaney, *A First Course in Chaotic Dynamical Systems*, Perseus Books Publishing, L.L.C., 1992.

[18] S. N. Elaydi, *Discrete Chaos: with Applications in Science and Engineering*, Chapman & Hall, Boca Raton, Fla, USA, 2nd edition, 2008.

[19] J. T. Sandefur, *Discrete Dynamical Systems: Theory and Applications*, Oxford University Press, Oxford, UK, 1990.